Quantization of differential calculi on universal enveloping algebras

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A method to construct differential calculi on quantized universal enveloping algebras is discussed. These differential calculi are obtained by quantizing calculi on "classical" enveloping algebras provided with appropriate co-Poisson structures. The procedure is demonstrated by applying it to the standard quantizations of the Heisenberg algebra and the algebra $gl(2)$. © *1996 American Institute of Physics.* [S0022-2488(96)04206-5]

I. INTRODUCTION

Noncommutative differential geometry, currently a field of active research, deals with differential calculus on algebras which are generally noncommutative. There are a few basic principles to construct such a noncommutative theory. One can replace the commutative function algebra on a space by some noncommutative algebra and try to generalize the basic concepts of the traditional case to this more abstract situation. Then there is the approach, already standard in algebraic geometry, to encode the structure of the underlying space into the function algebra defined on the space, which in turn is deformed. This is the approach customary in quantum group theory. These ideas led Connes (see, e.g., Ref. 1) and his collaborators to create "noncommutative geometry." Here, the commutative function algebra is replaced by some noncommutative C^*-algebra.

In quantum group theory it was Woronowicz,[2] who first developed the theory of differential calculus on quantum groups, giving a very interesting example of noncommutative differential geometry. This rather abstract theory has been reformulated in more concrete terms by Wess and Zumino.[3] A substantial number of very interesting papers, proposing other approaches, elucidating various aspects, studying concrete examples or dealing with applications have been written since. See, e.g., Refs. 4–7.

In this paper we discuss a method to construct a differential calculus on a quantized universal enveloping algebra $U_h(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. We follow the idea of Faddeev and his school[8] that all objects of a quantized theory should appear naturally as a result of quantization of appropriate Poisson structures. Accordingly, our starting point should be a differential calculus on $U(\mathfrak{g})$. These differential calculi are provided and studied in Ref. 9. Moreover, they turn out to be Hopf algebras and actually such a differential calculus turns out to be the universal enveloping algebra of a color Lie superalgebra, see Ref. 10. Consequently, our starting point is this enveloping algebra $U(L)$ equipped with an appropriate co-Poisson bracket δ. Its restriction to L, notation δ_L, defines a color Lie bisuperalgebra structure which may be obtained by extending the cocommutator $\delta_{\mathfrak{g}}$ of $\mathfrak{g}$.

The procedure is illustrated by two examples. We apply it to the standard quantizations of the enveloping algebra of the Heisenberg algebra and the algebra $gl(2)$.

Matrix quantum groups can be embedded as Hopf algebra in a quantization of the enveloping algebra of the dual Lie algebra, see Ref. 11. This indicates that our construction can be used to obtain differential calculi on quantum groups. Work on this is in progress; we will report on this in the near future.

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II. THE QUANTIZATION METHOD

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the field of complex numbers C. We present a procedure to construct differential calculi on quantized universal enveloping algebras of $\mathfrak{g}$ by quantizing differential calculi on the classical universal enveloping algebra $U(\mathfrak{g})$. This method connects the following two concepts related to $U(\mathfrak{g})$: quantization and differential calculus. In the classical limit the inter-relation between these concepts is expressed by a compatibility condition between the cocommutator that determines the Lie bialgebra structure on $\mathfrak{g}$ and the differential d. In order to explain the origin of this compatibility condition we shortly recall some notions related to differential calculi and quantization.

A quantization of the Lie algebra $\mathfrak{g}$ is a Hopf algebra deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$. Usually, $U_h(\mathfrak{g})$ is called a quantized universal enveloping algebra. The map δ defined by

$$\delta(x)=\frac{\Delta_h(x)-\Delta_h^{\mathrm{op}}(x)}{h} \bmod h \qquad \delta: U(\mathfrak{g})\mapsto U(\mathfrak{g})\otimes U(\mathfrak{g}) \tag{2.1}$$

is a co-Poisson bracket on $U(\mathfrak{g})$. In this formula Δ_h represents the comultiplication of $U_h(\mathfrak{g})$ and Δ_h^{op} the opposite comultiplication given by $\Delta_h^{\mathrm{op}} = \sigma \circ \Delta_h$, where σ is the ordinary flip operator on the tensor product. The restriction of δ to $\mathfrak{g}$, which will be denoted by $\delta_{\mathfrak{g}}$, defines a Lie bialgebra $(\mathfrak{g},\delta_{\mathfrak{g}})$. This means that $\delta_{\mathfrak{g}}:\mathfrak{g}\mapsto\mathfrak{g}\otimes\mathfrak{g}$ is a 1-cocycle and $\delta_{\mathfrak{g}}^*:\mathfrak{g}^*\mapsto\mathfrak{g}^*\otimes\mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$. The Lie bialgebra $(\mathfrak{g},\delta_{\mathfrak{g}})$ is called the classical limit of the quantization $U_h(\mathfrak{g})$ and $\delta_{\mathfrak{g}}$ is called the cocommutator. For more details on this we refer to Ref. 12.

A differential Hopf algebra (see, e.g., Ref. 13) is an N-graded Hopf algebra $\Omega=\Sigma_{p=0}^{\infty}\Omega^p$ equipped with a differential d. This operator d is a graded derivation of degree $+1$ with the property $d^2=0$. Furthermore it satisfies $(d\otimes id+\tau\otimes d)\circ\Delta=\Delta\circ d$ and $\epsilon\circ d=0$, where Δ denotes the comultiplication and ϵ the counit of Ω. The linear map $\tau:\Omega\to\Omega$ has degree zero and satisfies $\tau(a)=(-1)^p a$ for all $a\in\Omega^p$. A differential calculus on $U(g)$ is a differential Hopf algebra Ω with the additional properties $\Omega^0=U(\mathfrak{g})$ and Ω is generated by $\Omega^0\cup d(\Omega^0)$.

In Ref. 10 we showed that a differential calculus on $U(\mathfrak{g})$ of Poincaré–Birkhoff–Witt-type can be described as the universal enveloping algebra of a color Lie superalgebra L which is a natural extension of the Lie algebra $\mathfrak{g}$. For the sake of clarity, we recall the definition of color Lie superalgebra (see Ref. 14). Let G be an abelian semigroup and α a 2-cocycle on G with values in C*. An (α)-color Lie superalgebra is a G-graded algebra L with product [,] satisfying

$$[x,y]=-\alpha(p,q)[y,x] \quad\text{and}\quad \alpha(p,r)[[x,y],z]+\alpha(q,p)[[y,z],x]+\alpha(r,q)[[z,x],y]=0$$

for all $x\in L^p, y\in L^q, z\in L^r$. As in the case of ordinary Lie algebras, one can define the universal enveloping algebra of a color Lie superalgebra and a corresponding Hopf algebraic structure on it. The structure of the above mentioned color Lie superalgebra L, which represents the differential calculus on $U(\mathfrak{g})$, is as follows. L is the N-graded algebra $L=\oplus_{p\in\mathrm{N}}L^p$, where $L^0=\mathfrak{g}=\langle x^1,x^2,\dots,x^n\rangle$, $L^1=\langle\hat{x}^1,\hat{x}^2,..,\hat{x}^n\rangle$, and $L^p=0$ for all $p\geq 2$. The corresponding 2-cocycle α is given by

$$\alpha:\mathrm{N}\times\mathrm{N}\to\mathrm{C}^* \qquad \alpha(p,q)=(-1)^{pq}.$$

The Lie bracket of L is such that its restriction to L^0 is simply the Lie bracket of $\mathfrak{g}$ and the linear map $d:L\mapsto L$ given by $d(x_i)=\hat{x}_i$ and $d(\hat{x}_i)=0$ for all $1\leq i\leq n=\dim(\mathfrak{g})$ is a graded derivation of degree $+1$ on L. So the bracket of L is of the following form:

$$[x_i,x_j]=c_{ij}^k x_k; \quad [x_i,\hat{x}_j]=a_{ij}^k\hat{x}_k; \quad [\hat{x}_i,\hat{x}_j]=0.$$

We use the notation $\hat{x}_i$ to emphasize the fact that these elements have been formally introduced to represent the differentials of the elements x_i. The differential d on $\Omega=U(L)$ is defined as the unique graded derivation extending the operator d on L.

We come to the introduction of a differential calculus on $U_h(\mathfrak{g})$. According to the foregoing, a differential calculus on $U_h(\mathfrak{g})$ is a differential Hopf algebra Ω_h, with differential denoted by d_h, which is generated by $\Omega_h^0 \cup d_h(\Omega_h^0)$, where $\Omega_h^0=U_h(\mathfrak{g})$. On the other hand, as analog to the quantization procedure, it is natural that, by putting h equal to zero, Ω_h reduces to $U(L)$, where L is a color Lie superalgebra extension of $\mathfrak{g}$ of the form described above, and d_h reduces to the differential d of $U(L)$. This implies that (Ω_h, d_h) is a differential Hopf algebra deformation of $(U(L),d)$. We demand this deformation to be homogeneous of degree zero such that the N-grading of $U(L)$ induces the N-grading on $U_h(L)$. In particular $\Omega_h=U_h(L)$ is a Hopf algebra deformation of $U(L)$, or in other words a quantization of the color Lie superalgebra L. The classical limit of this quantization is a color Lie bisuperalgebra (L,δ_L) extending the Lie bialgebra $(\mathfrak{g},\delta_{\mathfrak{g}})$, which is the classical limit of $U_h(\mathfrak{g})$. This gives rise to the following commutative diagram:

$$\begin{array}{ccc} U_h(\mathfrak{g}) & \rightarrow & U_h(L) \\ \downarrow & & \downarrow \\ (g,\delta_{\mathfrak{g}}) & \rightarrow & (L,\delta_L) \end{array}$$

The vertical arrows denote the classical limit and the horizontal ones denote the canonical embeddings of (N-graded) Hopf algebras and color Lie bisuperalgebras.

We denote the comultiplication of $U_h(L)$ by Δ_h. From the definition of a differential Hopf algebra we know that $\Delta_h \circ d_h=(d_h)_\otimes \circ \Delta_h$ with $(d_h)_\otimes = d_h \otimes id + \tau \otimes d_h$. The co-Poisson bracket $\delta: U(L) \mapsto U(L) \otimes U(L)$ is defined as in the classical case described in formula (2.1), with the exception that, in the definition of Δ_h^{op} the operator σ denotes the graded flip operator, which is defined by

$$\sigma(x\otimes y)=(-1)^{pq} y\otimes x \quad \text{for all} \quad x\in U_h(L)^p, y\in U_h(L)^q.$$

One can easily verify that d_h has the property $\sigma\circ(d_h)_\otimes=(d_h)_\otimes \circ\sigma$. From this it follows that

$$\frac{\Delta_h-\Delta_h^{\mathrm{op}}}{h}\circ d_h=(d_h)_\otimes \circ \frac{\Delta_h-\Delta_h^{\mathrm{op}}}{h}$$

and for h equal to zero this reduces to

$$\delta\circ d=d_\otimes\circ\delta,$$

so the differential operator should commute with the co-Poisson bracket δ on $U(L)$. The restriction to L yields the following condition for the cocommutator δ_L:

$$\delta_L \circ d=d_\otimes \circ \delta_L. \tag{2.2}$$

To understand the meaning of this condition, let us assume we have a color Lie superalgebra L equipped with an operator d representing a differential calculus on $U(\mathfrak{g})$. Any Lie bialgebra $(\mathfrak{g},\delta_{\mathfrak{g}})$ gives rise to a unique extension $\delta_L: L \mapsto L\otimes L$ satisfying condition (2.2). We call the differential calculus on $U(\mathfrak{g})$ and the Lie bialgebra $(\mathfrak{g},\delta_{\mathfrak{g}})$ compatible if and only if (L,δ_L) is a color Lie bisuperalgebra. From the preceding reasoning we learn that this is a necessary condition in order to obtain a differential calculus on $U_h(\mathfrak{g})$ starting from the differential calculus on $U(\mathfrak{g})$ given by L. The examples we have studied so far seem to indicate that the condition is also sufficient.

Thus from the discussion above we can subtract the following procedure to construct a differential calculus on a quantized universal enveloping algebra.

(1) Construct a differential calculus on $U(\mathfrak{g})$.
(2) Compute a Lie bialgebra $(\mathfrak{g},\delta_{\mathfrak{g}})$ which is compatible with the constructed differential calculus.
(3) Quantize $(\mathfrak{g},\delta_{\mathfrak{g}})$, i.e., construct a quantized universal enveloping algebra $U_h(\mathfrak{g})$ with classical limit $(\mathfrak{g},\delta_{\mathfrak{g}})$.
(4) Quantize (L,δ_L) where δ_L is the unique extension of $\delta_{\mathfrak{g}}$ given by formula (2.2).

Note that in the last step one can fruitfully use that $U_h(L)$ is an extension of $U_h(\mathfrak{g})$ and that the differential d_h should respect the defining relations of $U_h(L)$. We will illustrate this in more detail in the examples.

Finally, we mention that there is a nice algebraic interpretation for the compatibility condition (2.2). The linear operator d on L is a graded derivation, this means that

$$d([x,y])=[d(x),y)]+(-1)^p[x,d(y)] \quad \text{for } x\in L^p, y\in L,$$

or equivalently $[,]\circ d_{\otimes}=d\circ[,]$, where $[,]$ denotes the Lie bracket of L. Analogously, the linear operator d^* is a graded derivation on L^* with Lie bracket δ_L^* if it satisfies $\delta_L^* \circ (d^*)_{\otimes} = d^* \circ \delta_L^*$. But this is equivalent to Eq. (2.2). So we can express the compatibility condition appropriately by saying that d should be a color Lie bisuperalgebra derivation on (L,δ_L).

III. THE HEISENBERG ALGEBRA

As first example, we will consider the Heisenberg algebra H. A basis of H is given by $\{p,q,c\}$ and the Lie product is given by

$$[p,q]=c; \quad [p,c]=0; \quad [q,c]=0.$$

A. The differential calculus on $U(H)$

In Ref. 9 we constructed all differential calculi $\Omega=\Sigma_{p=0}^{\infty}\Omega^p$ of Poincaré–Birkhoff–Witt-type on H. Here, $\Omega^0=U(H)$ and Ω^p denotes the space of p-forms; in particular $\Omega^1=d(\Omega^0)$. As we described in Sec. II, Ω is isomorphic to $U(L)$, where L is a color Lie superalgebra such that $L^0=H$ and $L^1=\hat{H}$. In particular, the differential calculus is completely determined by the map $\rho:H\rightarrow gl(\hat{H})$, which is in fact the commutator in L of elements from L_0 and L_1. (L_1 is a representation of L_0 using the commutator). As basis of $\hat{H}$ we will use $\{\hat{p},\hat{q},\hat{c}\}$, where $d(x)$ denotes the element $\hat{x}$. It turns out that the simplest and most elegant solution is described by

$$\rho(p)(q)=[p,\hat{q}]=\tfrac{1}{2}\hat{c}, \quad \rho(q)(p)=[q,\hat{p}]=-\tfrac{1}{2}\hat{c},$$

and all others equal to zero.

Summarizing we start with the quotient of the free N-graded associative algebra on the alphabet $\{p,q,c,\hat{p},\hat{q},\hat{c}\}$, where $\{p,q,c\}$ are homogeneous of degree 0 and $\{\hat{p},\hat{q},\hat{c}\}$ are homogeneous of degree 1, divided by the ideal I which is generated by the following homogeneous relations:

$$pq-qp=c; \quad pc-cp=0; \quad qc-cq=0$$

$$p\hat{p}=\hat{p}p; \quad p\hat{q}=\hat{q}p+\tfrac{1}{2}\hat{c}; \quad p\hat{c}=\hat{c}p$$

$$q\hat{p}=\hat{p}q-\tfrac{1}{2}\hat{c}; \quad q\hat{q}=\hat{q}q; \quad q\hat{c}=\hat{c}q$$

$$c\hat{p}=\hat{p}c; \quad c\hat{q}=\hat{q}c; \quad c\hat{c}=\hat{c}c$$

$$\hat{p}\hat{q}=-\hat{q}\hat{p}; \quad \hat{p}\hat{c}=-\hat{c}\hat{p}; \quad \hat{q}\hat{c}=-\hat{c}\hat{q}$$

$$\hat{p}\hat{p}=0; \quad \hat{q}\hat{q}=0; \quad \hat{c}\hat{c}=0.$$

The differential d is the unique graded derivation of degree $+1$ satisfying $d(p)=\hat{p}$, $d(q)=\hat{q}$, $d(c)=\hat{c}$ and $d(\hat{p})=0$, $d(\hat{q})=0$, $d(\hat{c})=0$.

B. The compatible cocommutator on *H*

Next we have to construct a Lie bialgebra structure on H, which is compatible with d. This is a matter of straight computation, which we performed using computer algebra. There is a unique solution, given by

$$\delta(c)=0; \quad \delta(p)=p\wedge c; \quad \delta(q)=q\wedge c;$$

and consequently the continuation to $\hat{H}=d(H)$, which is prescribed by $\delta(\hat{x})=(d\otimes id+\tau\otimes d)\circ\delta(x)$, yields

$$\delta(\hat{c})=0; \quad \delta(\hat{p})=\hat{p}\wedge c+p\wedge\hat{c}; \quad \delta(\hat{q})=\hat{q}\wedge c+q\wedge\hat{c}.$$

Note that the restriction of δ to H is a cocommutator of coboundary type with corresponding R-matrix

$$R=p\wedge q. \tag{3.1}$$

One can easily verify that δ itself is not of coboundary type.

C. Quantization of (H,δ)

In order to quantize the situation above, we note that the R-matrix (3.1) is the standard one. This suggests that we can take for $U_h(H)$ the standard quantization

$$\Delta_h(p)=p\otimes e^{hc}+1\otimes p; \quad \Delta_h(q)=q\otimes 1+e^{-hc}\otimes q; \quad \Delta_h(c)=c\otimes 1+1\otimes c$$

and the only relation in $U_h(H)$ that differs from the relations in $U(H)$ is

$$[p,q]=\frac{\sinh(hc)}{\sinh(h)}.$$

The unit and counit are unchanged. The antipode is rather easy to compute, from Ref. 12 we know that it exists. For example to calculate $S_h(p)$, we consider

$$0=\mu_h\circ(S_h\otimes id)\circ\Delta_h(p)=S_h(p)e^{hc}+p.$$

Hence we find

$$S_h(p)=-pe^{-hc}$$

and similarly

$$S_h(q)=-qe^{hc}; \quad S_h(c)=-c.$$

Although $U_h(H)$ is clearly not cocommutative, one can easily verify that the antipode still satisfies $S_h^2=Id$.

D. Quantization of (L,δ)

From here on we will use $\hat{x}$ to denote the element $d_h(x)$. Due to $\Delta_h\circ d_h=(d_h)_\otimes\circ\Delta_h$, we have $\Delta_h(\hat{c})=\hat{c}\otimes 1+1\otimes\hat{c}$ and

$$\Delta_h(\hat{p})=\hat{p}\otimes e^{hc}+p\otimes h\hat{c}e^{hc}+1\otimes\hat{p};\quad \Delta_h(\hat{q})=\hat{q}\otimes 1+e^{-hc}\otimes\hat{q}-h\hat{c}e^{-hc}\otimes q.$$

It remains to determine the new relations in Ω_h; since $\Delta_h(c)$ and $\Delta_h(\hat{c})$ are unchanged it is natural to require that only the relations $[p,\hat{q}]$, $[\hat{p},q]$, and $[\hat{p},\hat{q}]$ will change.

So, let us assume that $[p,\hat{q}]=\alpha_1$ and $[\hat{p},q]=\alpha_2$. From $[p,q]=[\sinh(hc)]/[\sinh(h)]$ it follows that

$$\alpha_1+\alpha_2=[p,\hat{q}]+[\hat{p},q]=d_h([p,q])=h\hat{c}\,\frac{\cosh(hc)}{\sinh(h)}. \tag{3.2}$$

Further we have

$$\Delta_h(\alpha_1)=\Delta_h(p\hat{q}-\hat{q}p)=(p\otimes e^{hc}+1\otimes p)(\hat{q}\otimes 1-h\hat{c}e^{-hc}\otimes q+e^{-hc}\otimes\hat{q})-(\hat{q}\otimes 1-h\hat{c}e^{-hc}\otimes q$$
$$+e^{-hc}\otimes\hat{q})(p\otimes e^{hc}+1\otimes p)=[p,\hat{q}]\otimes e^{hc}-h\hat{c}e^{-hc}\otimes[p,q]+e^{-hc}\otimes[p,\hat{q}]$$

so,

$$\Delta_h(\alpha_1)=\alpha_1\otimes e^{hc}-h\hat{c}e^{-hc}\otimes\frac{\sinh(hc)}{\sinh(h)}+e^{-hc}\otimes\alpha_1. \tag{3.3}$$

In the same way we find for α_2

$$\Delta_h(\alpha_2)=\alpha_2\otimes e^{hc}+\frac{\sinh(hc)}{\sinh(h)}\otimes h\hat{c}e^{-hc}+e^{-hc}\otimes\alpha_2. \tag{3.4}$$

Equations (3.2), (3.3), and (3.4) suggest to take

$$\alpha_i=\frac{\lambda_i e^{hc}+\mu_i e^{-hc}}{\sinh(h)}\,h\hat{c};\quad (i=1,2).$$

Substitution yields a unique solution

$$[p,\hat{q}]=h\hat{c}\,\frac{e^{-hc}}{2\sinh(h)};\quad [\hat{p},q]=h\hat{c}\,\frac{e^{hc}}{2\sinh(h)}. \tag{3.5}$$

This also dictates the relation between $\hat{p}$ and $\hat{q}$ by differentiation of Eq. (3.5)

$$\hat{p}\hat{q}=-\hat{q}\hat{p}.$$

Finally, from $S_h\circ d_h=d_h\circ S_h$ it follows that the extension of the antipode is given by

$$S_h(\hat{p})=-(\hat{p}-hp\hat{c})e^{-hc};\quad S_h(\hat{q})=-(\hat{q}+hq\hat{c})e^{hc};\quad S_h(\hat{c})=-\hat{c}.$$

It still satisfies $S_h^2=Id$. In order to compute the action of S_h on an arbitrary element of Ω_h, one can use linearity and the antialgebra-morphism property

$$S_h(ab)=(-1)^{rs}S_h(b)S_h(a);\quad (a\in\Omega^r, b\in\Omega^s).$$

E. Summary of results for H

For clearness' sake, we summarize the results of this section in the following theorem.

Theorem 1: *The standard quantization of the Heisenberg algebra H, given by*

$$[p,q]=\frac{\sinh(hc)}{\sinh(h)};\quad [p,c]=0;\quad [q,c]=0$$

and

$$\Delta_h(p)=p\otimes e^{hc}+1\otimes p;\quad \Delta_h(q)=q\otimes 1+e^{-hc}\otimes q;\quad \Delta_h(c)=c\otimes 1+1\otimes c$$
$$S_h(p)=-pe^{-hc};\quad S_h(q)=-qe^{hc};\quad S_h(c)=-c$$

admits a differential calculus d_h. *If we denote* $d_h(x)$ *by* $\hat{x}$ *for all elements* x *in* H, *then* $\hat{c}$ *is primitive and*

$$\Delta_h(\hat{p})=\hat{p}\otimes e^{hc}+p\otimes h\hat{c}e^{hc}+1\otimes\hat{p};\quad \Delta_h(\hat{q})=\hat{q}\otimes 1+e^{-hc}\otimes\hat{q}-h\hat{c}e^{-hc}\otimes q$$
$$S_h(\hat{p})=-(\hat{p}-hp\hat{c})e^{-hc};\quad S_h(\hat{q})=-(\hat{q}+hq\hat{c})e^{hc}.$$

The commutation between functions and forms is described by

$$[p,\hat{q}]=h\hat{c}\,\frac{e^{-hc}}{2\sinh(h)};\quad [q,\hat{p}]=-h\hat{c}\,\frac{e^{hc}}{2\sinh(h)}$$

and $[x,\hat{y}]=0$ *for all other choices of elements* x *and* y *from* $\{p,q,c\}$. *Finally, the commutation of forms is determined by the relation* $\hat{x}\hat{y}=-\hat{y}\hat{x}$ *for all* $x,y\in H$.

Finally, we remark that by an obvious and small modification the same result is obtained for the general $2n+1$-dimensional Heisenberg algebra with basis $\{p_i,q_i,c\}_{1\leq i\leq n}$ and Lie product given by

$$[p_i,q_j]=\delta^i_j c;\quad [p_i,c]=0;\quad [q_i,c]=0;\quad [p_i,p_j]=0;\quad [q_i,q_j]=0.$$

IV. THE ALGEBRA *gl*(2)

The next example that we consider is $gl(2)$. We will denote

$$E_+=E_{12};\quad E_-=E_{21};\quad H_+=E_{11};\quad H_-=E_{22}.$$

Hence in the enveloping algebra $U(gl(2))$, we have the following relations:

$$[H_+,H_-]=0;\quad [H_+,E_\pm]=\pm E_\pm;\quad [H_-,E_\pm]=\mp E_\pm;\quad [E_+,E_-]=H_+-H_-.$$

A. The differential calculus on *U*(*gl*(2))

On $U(gl(2))$ we can construct differential calculi of Poincaré–Birkhoff–Witt type; as said in the previous sections, these calculi are completely determined by an appropriate representation $\rho:gl(2)\to\widehat{gl(2}$. For $gl(n)$ there is a natural solution, namely,

$$\rho(x)\hat{y}=\widehat{xy},$$

where xy denotes the product of x and y as $n\times n$ matrices. We will use this ρ in the sequel. Hence we can determine the differential calculus Ω. Apart from the relations above, it satisfies the following relations ($\hat{x}=d(x)$; $x\in gl(2)$):

$$[H_\pm,\hat{E}_\pm]=\hat{E}_\pm;\quad [H_\pm,\hat{E}_\mp]=0;\quad [H_\pm,\hat{H}_\pm]=\hat{H}_\pm;\quad [H_\pm,\hat{H}_\mp]=0$$
$$[E_\pm,\hat{E}_\pm]=0;\quad [E_\pm,\hat{E}_\mp]=\hat{H}_\pm;\quad [E_\pm,\hat{H}_\pm]=0;\quad [E_\pm,\hat{H}_\mp]=\hat{E}_\pm.$$

Note the similar roles that H_+ and H_- play; hence the basis chosen (instead of $H_+ - H_-$ and $H_+ + H_-$) is very natural. One can note that the representation ρ is the direct sum of two 2-dimensional $gl(2)$-representations V_1 and V_2, where $V_1 = \langle \hat{E}_+, \hat{H}_- \rangle$ and $V_2 = \langle \hat{E}_-, \hat{H}_+ \rangle$.

B. The compatible cocommutator on $gl(2)$

Again compatible Lie bialgebra structures can be computed. There is a unique solution if we demand the solution to be homogeneous of degree zero with respect to the natural grading on $gl(2)$ defined by $|H_\pm| = 0$, $|E_\pm| = \pm 1$. This solution is given by

$$\delta(H_\pm) = 0; \quad \delta(E_+) = E_+ \wedge H_-; \quad \delta(E_-) = H_+ \wedge E_-$$

which extends to L as

$$\delta(\hat{H}_\pm) = 0; \quad \delta(\hat{E}_+) = \hat{E}_+ \wedge H_- + E_+ \wedge \hat{H}_-; \quad \delta(\hat{E}_-) = \hat{H}_+ \wedge E_- + H_+ \wedge \hat{E}_-.$$

As in the case of the Heisenberg algebra, δ itself is not of coboundary type. Its restriction to $gl(2)$ is coboundary, the corresponding R-matrix is given by $R = \frac{1}{2}(E_+ \wedge E_- + H_+ \wedge H_-)$.

C. Quantization of $(gl(2), \delta)$

The way of quantizing is similar as in the case of the Heisenberg algebra. Again $\delta(H_\pm) = 0$ suggests to take

$$\Delta_h(H_\pm) = H_\pm \otimes 1 + 1 \otimes H_\pm.$$

Similarly $\delta(E_+) = E_+ \wedge H_-$ and $\delta(E_-) = H_+ \wedge E_-$ suggest

$$\Delta_h(E_+) = E_+ \otimes e^{hH_-} + 1 \otimes E_+; \quad \Delta_h(E_-) = E_- \otimes 1 + e^{hH_+} \otimes E_-.$$

From this it follows that the antipode is given by

$$S_h(H_\pm) = -H_\pm; \quad S_h(E_+) = -E_+ e^{-hH_-}; \quad S_h(E_-) = -e^{-hH_+} E_-.$$

The commutation relation between E_+ and E_- has to be changed, all others in $gl(2)$ remain the same

$$[E_+, E_-] = \frac{e^{hH_+} - e^{hH_-}}{2 \sinh(h)}.$$

D. Quantization of (L, δ)

Extending Δ_h to $\widehat{gl(2)}$ is straightforward, the only complication is that $[H_\pm, \hat{H}_\pm] = \hat{H}_\pm$. Due to this, we have

$$d_h(e^{hH_\pm}) = \hat{H}_\pm e^{hH_\pm}(e^h - 1).$$

So we find

$$\Delta_h(\hat{H}_\pm) = \hat{H}_\pm \otimes 1 + 1 \otimes \hat{H}_\pm$$

and

$$\Delta_h(\hat{E}_+) = \hat{E}_+ \otimes e^{hH_-} + E_+ \otimes \hat{H}_- e^{hH_-}(e^h - 1) + 1 \otimes \hat{E}_+;$$

$$\Delta_h(\hat{E}_-) = \hat{E}_- \otimes 1 + \hat{H}_+ e^{hH_+}(e^h - 1) \otimes E_- + 1 \otimes \hat{E}_-.$$

Now we have to adjust the relations between $gl(2)$ and $\widehat{gl(2)}$, so that Δ_h becomes an algebra-morphism. One can check by direct calculation that we can take all commutators involving either $H_\pm$ or $\hat{H}_\pm$ unchanged. We will require that $[\hat{x},\hat{y}]=0$ remains unchanged for all $x,y\in gl(2)$. Hence our problem is to adjust the commutators $[E_\pm,\hat{E}_\pm]$ and $[E_\pm,\hat{E}_\mp]$.

Let us first consider $[E_+,\hat{E}_+]$. For this we consider $[\hat{E}_+,\hat{E}_+]=0$, so that $\hat{E}_+\hat{E}_+=0$. Applying Δ_h, we find

$$\Delta_h(\hat{E}_+)\Delta_h(\hat{E}_+)=\cdots=(-E_+\hat{E}_+ + e^h\hat{E}_+E_+)\otimes\hat{H}_-e^{2hH_-}(e^h-1)=0.$$

Here, we used that $e^{hH_-}\hat{H}_- = \hat{H}_-e^{hH_-}e^h$, since $[H_-,\hat{H}_-]=\hat{H}_-$. So this forces

$$E_+\hat{E}_+ - e^h\hat{E}_+E_+=0.$$

Similarly we find

$$E_-\hat{E}_- - e^h\hat{E}_-E_-=0.$$

One must check that these relations are compatible with Δ_h, i.e., that $\Delta_h(E_+\hat{E}_+ - e^h\hat{E}_+E_+)=0$. This is indeed the case.

Finally, it remains to obtain $[E_\pm,\hat{E}_\mp]$. A tedious calculation shows that we can choose

$$[E_\pm,\hat{E}_\mp]=\frac{e^h-1}{2\sinh(h)}\hat{H}_\pm e^{hH_\pm}.$$

Finally, the antipode is not difficult to calculate. In fact we have $S_h(\hat{H}_\pm)=-\hat{H}_\pm$ and

$$S_h(\hat{E}_+)=-(\hat{E}_+ + E_+\hat{H}_-(e^{-h}-1))e^{-hH_-};$$

$$S_h(\hat{E}_-)=-e^{-hH_+}\hat{E}_- - \hat{H}_+(e^{-h}-1)e^{-hH_+}E_-$$

as follows from the formula $S_h\circ d_h=d_h\circ S_h$. We remark that the square of the antipode on $U_h(gl(2))$ is given by

$$S_h^2(H_\pm)=H_\pm;\quad S_h^2(E_+)=e^{hH_-}E_+e^{-hH_-}=e^{-h}E_+\quad S_h^2(E_-)=e^{-hH_+}E_-e^{hH_+}=e^hE_-$$

and again due to the commutation of S_h and d_h also

$$S_h^2(\hat{H}_\pm)=\hat{H}_\pm;\quad S_h^2(\hat{E}_\pm)=e^{\mp h}\hat{E}_\pm.$$

Concluding we can say that we completed the quantization. The choice of Δ_h was quite natural, and led to deforming the commutation relations involving only $E_\pm$ and $\hat{E}_\pm$.

E. Summary of results for $gl(2)$

For clearness' sake, we summarize the results of this section in the following theorem.

Theorem 2: *The quantization of the algebra $gl(2)$, given by*

$$[H_+,H_-]=0;\quad [H_+,E_\pm]=\pm E_\pm;\quad [H_-,E_\pm]=\mp E_\pm;\quad [E_+,E_-]=\frac{e^{hH_+}-e^{hH_-}}{2\sinh(h)}$$

and

$$\Delta_h(H_\pm)=H_\pm\otimes 1+1\otimes H_\pm;$$

$$\Delta_h(E_+)=E_+\otimes e^{hH_-}+1\otimes E_+;\quad \Delta_h(E_-)=E_-\otimes 1+e^{hH_+}\otimes E_-$$

with corresponding antipode given by

$$S_h(H_\pm)=-H_\pm;\quad S_h(E_+)=-E_+e^{-hH_-};\quad S_h(E_-)=-e^{-hH_+}E_-$$

admits a differential calculus d_h. If we denote $d_h(x)$ by $\hat{x}$ for all elements x in $gl(2)$, then $\hat{H}_\pm$ is primitive and

$$\Delta_h(\hat{E}_+)=\hat{E}_+\otimes e^{hH_-}+E_+\otimes\hat{H}_-e^{hH_-}(e^h-1)+1\otimes\hat{E}_+;$$

$$\Delta_h(\hat{E}_-)=\hat{E}_-\otimes 1+\hat{H}_+e^{hH_+}(e^h-1)\otimes E_-+1\otimes\hat{E}_-;$$

$$S_h(\hat{E}_+)=-(\hat{E}_++E_+\hat{H}_-(e^{-h}-1))e^{-hH_-},\quad S_h(\hat{E}_-)=-e^{-hH_+}\hat{E}_--\hat{H}_+(e^{-h}-1)e^{-hH_+}E_-.$$

The commutation between functions and forms is described by

$$[H_\pm,\hat{E}_\pm]=\hat{E}_\pm;\quad [H_\pm,\hat{E}_\mp]=0;\quad [H_\pm,\hat{H}_\pm]=\hat{H}_\pm;\quad [H_\pm,\hat{H}_\mp]=0,$$

$$E_\pm\hat{E}_\pm-e^h\hat{E}_\pm E_\pm=0;\quad [E_\pm,\hat{E}_\mp]=\frac{e^h-1}{2\sinh(h)}\hat{H}_\pm e^{hH_\pm};\quad [E_\pm,\hat{H}_\pm]=0;\quad [E_\pm,\hat{H}_\mp]=\hat{E}_\pm.$$

Finally, the commutation of forms is determined by the relation $\hat{x}\hat{y}=-\hat{y}\hat{x}$ for all $x,y\in gl(2)$.

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