replacing the nonlinear load by a 1 Ω resistor. The dominant impulse response poles and residues of the circuit are then calculated using AWE technique. For this circuit, two sets of poles and residues are required: from input to output and from a current source connected parallel to load to output. The first order response is then obtained using the first set of pole-residue by exciting the circuit with the actual input. Subsequently, the input is killed and the higher order responses are calculated by exciting the circuit with a current source connected parallel to load and using the second set of pole-residue. The order is increased until the algorithm converges. The output waveforms, obtained with the proposed method and HSPICE, are given in Fig. 2 for comparison. The first, third, fifth, and eleventh order outputs are shown in Fig. 3.

Example 2: The second example, which is shown in Fig. 4, has been taken from [16]. The nonlinear elements are defined as: $I_a = 0.001V_a^3$, $I_b = V_b/750 + 0.002V_b^3$, and $I_{out} = 0.001V_{out}^3$ The applied input voltage waveform for this circuit is 4.5 ns pulse with 1.5 ns rise and fall times. The amplitude of the input pulse is 5 volts. The output waveform obtained using the proposed method is compared with HSPICE result in Fig. 5.

IV. Conclusion

A new method has been proposed for the transient analysis of circuits with relatively few and mildly nonlinear terminations. In this approach, the method of Volterra-series analysis of the nonlinear elements is combined with AWE-based techniques for the linear part of the circuit. The method is noniterative and corresponds to recursive analysis of a linear circuit with different excitations. Therefore, it has no convergence problem. Since it is based on AWE technique, it uses a very small number of LU decompositions with respect to the traditional methods.

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On Lyapunov Control of the Duffing Equation

Henk Nijmeijer and Harry Berghuis

Abstract—**In this brief, we develop feedback control strategies for a chaotic dynamic system such as the Duffing equation. Our controllers are of the so-called Lyapunov-type and are inspired by robot manipulator feedback controls. The different controllers we propose include observer-based controllers that even can cope with parametric uncertainties of the original system. Some simulation examples support the developed methods.**

I. Introduction

Recently, an increasing interest has been developed in controling chaotic nonlinear systems as arising in physics and engineering; from the various relevant references we mention [4]–[8], [12], [13], [15], and references therein. A very essential element in the control of chaos is that, in many cases, the ultimate goal of control is to decrease random effects and to stabilize the system at an equilibrium point, or more general, about a given reference trajectory. In such cases, one is in fact naturally led to reduce or even more completely annihilate the chaotic dynamics that an uncontrolled system may exhibit. Depending on the specific desired behavior of the system, several methods for controlling chaotic systems have been proposed, see, e.g., [6], [13]. Among the methods given there, a prominent role is played by the so-called *Lyapunov-type methods*. At the same time and earlier, various authors have investigated stabilizing control schemes for second-order mechanical systems, as in particular robot manipulators. Let us give a sample of relevant references [1], [10], [14], [17], [19], noting that also this field is strongly progressing at the moment. It should be noted that also in this context Lyapunov-type methods are very popular and useful.

The purpose of this paper is essentially to develop a controller-observer scheme for controlling a chaotic second-order system such as

Manuscript received September 8, 1994; revised February 27, 1995. This paper was recommended by Associate Editor G. Chen.

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IEEE Log Number 9413209.

1057–7122/95$04.00 © 1995 IEEE

the Duffing equation; see also [5] where a controller without observer has been derived. Our controller-observer analysis is inspired by the aforementioned papers on robot control. Recall that Duffing's equation describes a specific nonlinear circuit or a pendulum moving in a viscous medium and is in controlled form given by

$$\ddot{x} + p\dot{x} + p_1 x + p_2 x^3 = u(t) + q\cos(\omega t) \tag{1.1}$$

or, setting $x_1 \equiv x$, $x_2 \equiv \dot{x}$, as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -px_2 - p_1 x_1 - p_2 x_1^3 + u(t) + q\cos(\omega t) \end{cases} \tag{1.2}$$

where p, p_1, p_2, q, and ω are constants, and $u(\cdot)$ is the *physical control input*. Depending on the choice of these constants, it is known that solutions of (1.2) exhibit periodic, almost periodic, and chaotic behavior; see [5]. The general problem that we will study throughout this note is whether we are able to find a suitable output-feedback controller

$$u = \phi(x_1, \hat{x}_2, x_d, t) \tag{1.3}$$

such that for the closed loop (1.2), (1.3) the solution $x(t) = x_1(t)$ asymptotically converges to a desired trajectory $x_d(t), t \geq 0$. Here, $\hat{x}_2$ is obtained via an observer for the "velocity" x_2, and $x_d(t)$ may represent *any* smooth time-function, including fixed points or periodic orbits.

The manuscript is organized as follows: In Section II, we briefly recall some results on feedback control of second-order (mechanical) systems. Next, in the third section, we translate these results to the control of the Duffing equation (1.2). In Section IV, we discuss some related issues. Section V provides simulations that support our findings. Finally, Section VI contains the conclusions.

II. Feedback Control of Second-Order Systems

Consider the second-order dynamics

$$J\ddot{q} + F\dot{q} = \tau + T(t) \tag{2.1}$$

where $J > 0$, $F \geq 0$, τ is the control, and $T(t)$ is a (known) disturbance. Note that this is exactly the dynamics of a *one* degree-of-freedom robot system [17], where in that case q represents the angular position. Our control objective is to let the system (2.1) follow an arbitrary smooth reference trajectory $q_d(t)$. For this purpose, we select the control input as

$$\tau = J\ddot{q}_d + F\dot{q}_d - T(t) - K_d\dot{e} - K_p e \tag{2.2}$$

where $e \equiv q - q_d$ represents the tracking error, and $K_d > 0, K_p > 0$. This controller consists of three components, namely

1) a position error feedback part $-K_p e$
2) a velocity error feedback part $-K_d\dot{e}$
3) a feedforward part $J\ddot{q}_d + F\dot{q}_d - T(t)$

The feedback terms are required to guarantee that the robot system converges towards $q_d(t)$. Once on this trajectory, the feedforward component keeps the robot moving along it.

The closed-loop consisting of (2.1) and (2.2) is described by the second-order dynamics

$$J\ddot{e} + (F + K_d)\dot{e} + K_p e = 0. \tag{2.3}$$

As can be seen from (2.3), the transient behavior of the error dynamics can be influenced by a suitable choice of the proportional and derivative gain K_p and K_d, respectively. For this system, we have the following result (cf. [19]):

Proposition 2.1: The feedback controller (2.2) guarantees that (2.1) asymptotically converges towards any smooth and bounded reference trajectory $q_d(t)$, i.e.,

$$\lim_{t\to\infty} e(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \dot{e}(t) = 0. \tag{2.4}$$

Proof: Consider the candidate Lyapunov function

$$V_1(e,\dot{e}) = \frac{1}{2}J(\dot{e} + \lambda e)^2 + \frac{1}{2}(K_p + \lambda(F + K_d) - \lambda^2 J)e^2 \tag{2.5}$$

with $\lambda > 0$ constant. A sufficient condition for $V_1(\cdot)$ to be positive definite in $(e, \dot{e})$ is

$$0 < \lambda < J^{-1}K_d. \tag{2.6}$$

Along the closed-loop error dynamics (2.3), the time-derivative of $V_1(e, \dot{e})$ becomes

$$\dot{V}_1(e,\dot{e}) = -(F + K_d - \lambda J)\dot{e}^2 - \lambda K_p e^2. \tag{2.7}$$

Because λ satisfies (2.6), we have that $\dot{V}_1(\cdot)$ is negative definite in the error state $(e, \dot{e})$. Consequently, the closed-loop system (2.3) is asymptotically stable (cf. [9]). □

The Lyapunov function (2.5) is very familiar in robotics literature, e.g., [19]. This function originates from and closely resembles the natural energy contents of the open-loop (robot) dynamics (2.1); see, for instance, [10], [11], [18].

The controller (2.2) requires full-state information, i.e., "position" q and "velocity" $\dot{q}$ measurements are necessary in its actual implementation. However, in practice, velocity sensoring equipment is generally not available. To overcome this velocity measurement problem, we modify the controller (2.2) as follows [1]:

$$\textbf{Controller}\left\{\tau = J\ddot{q}_d + F\dot{q}_d - T(t) - K_d\dot{\hat{e}} - K_p\hat{e}\right. \tag{2.8a}$$

$$\textbf{Observer}\begin{cases} \dot{\hat{e}} = w + 2J^{-1}K_d(e - \hat{e}) - J^{-1}Fe \\ \dot{w} = 2J^{-1}K_p(e - \hat{e}) \end{cases}. \tag{2.8b}$$

This *output-feedback* controller (i.e., it only requires knowledge of the position q) consists of two parts: a linear observer part (2.8b) that generates an estimated error state $(\hat{e}, \dot{\hat{e}})$ from the position error e and a controller part (2.8a) that utilizes this estimated error state in the feedback loop. Let us assume that

$$K_p = \lambda K_d \tag{2.9}$$

where $\lambda > 0$ scalar. Then we can prove

Proposition 2.2: Let K_p satisfy (2.9). Under condition (2.6), the closed-loop dynamics (2.1, 8) is asymptotically stable.

Proof: The closed-loop system (2.1, 8) can be written as

$$J\ddot{e} + (F + K_d)\dot{e} + K_p e = K_d\dot{\tilde{e}} + K_p\tilde{e} \tag{2.10a}$$

$$J\ddot{\tilde{e}} + K_d\dot{\tilde{e}} + K_p\tilde{e} = -K_d\dot{e} - K_p e \tag{2.10b}$$

where $\tilde{e} \equiv e - \hat{e}$. Define the Lyapunov function as

$$V(e,\dot{e},\tilde{e},\dot{\tilde{e}}) = V_1(e,\dot{e}) + V_2(\tilde{e},\dot{\tilde{e}})$$

with $V_1(e,\dot{e})$ as in (2.5) and in analogy with this

$$V_2(\tilde{e},\dot{\tilde{e}}) = \frac{1}{2}J(\dot{\tilde{e}} + \lambda\tilde{e})^2 + \frac{1}{2}(K_p + \lambda K_d - \lambda^2 J)\tilde{e}^2. \tag{2.11}$$

Condition (2.6) guarantees that $V(\cdot)$ is positive definite in $(e, \dot{e}, \tilde{e}, \dot{\tilde{e}})$. The time-derivative of $V(\cdot)$ along (2.10) equals

$$\dot{V}(x) = -x^T Q x \tag{2.12}$$

where $x^T = [\lambda e\ \dot{e}\ \lambda\tilde{e}\ \dot{\tilde{e}}]$ and

$$Q = \operatorname{diag}(K_d, F + K_d - \lambda J, K_d, K_d - \lambda J). \tag{2.13}$$

Condition (2.6) is sufficient for $Q > 0$. This completes the proof. □

We stress that Propositions 2.1 and 2.2 are valid for arbitrary bounded reference trajectories $q_d(t)$. This is due to the presence of the feedforward term in the controller. In the next section, it is shown that the previous results can be employed in the context of controlling the Duffing equation.

III. Feedback Control of Duffing's Equation

In [5], the (modified) Duffing equation in controlled form is introduced as

$$\ddot{x} + p\dot{x} + p_1 x + x^3 = u + q\cos(\omega t) \tag{3.1}$$

where p, p_1, q, and ω are constant. In contrast to [5], we do not assume beforehand that $p > 0$. The dynamics of this system is similar to that of (2.1) by putting $J \equiv 1, F \equiv p$, $\tau \equiv u$, and $T(t) \equiv q\cos(\omega t)$. The only differences are the linear and cubic term in x, but these terms do not cause any problems in both the control design and the stability analysis, as will be shown below.

As before, assume we want the system to follow *any* smooth desired trajectory $x_d(t)$. For this purpose, we define the control input as

$$u = \ddot{x}_d + p\dot{x}_d + p_1 x_d + x_d{}^3 - q\cos(\omega t) - K_d\dot{e} - K_p e + \nu \tag{3.2a}$$

$$\nu = 3xx_d e \tag{3.2b}$$

with $e \equiv x - x_d$ and $K_d > 0, K_p > 0$. Essentially, one recognizes in (3.2a) the three parts as in (2.2), namely, proportional and derivative feedback and a feedforward term. Note that ν is introduced in order to deal with the cubic term in (3.1).

Proposition 3.1: The closed-loop system (3.1), (3.2) is asymptotically stable, so

$$\lim_{t\to\infty} e(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \dot{e}(t) = 0 \tag{3.3}$$

if

$$K_d > -p, \quad K_p > -p_1. \tag{3.4}$$

Proposition 3.1 demonstrates that the Duffing equation (3.1) can be forced asymptotically towards arbitrary smooth reference trajectories $x_d(t)$. This is achieved by incorporating in the controller a suitably selected feedforward compensation. A special situation that might be of general interest, see [5], [7], occurs when the desired trajectory $x_d(t)$ represents a (stable or unstable) equilibrium motion of the uncontrolled dynamics (3.1). Hence

$$\ddot{x}_d + p\dot{x}_d + p_1 x_d + x_d{}^3 = q\cos(\omega t). \tag{3.5}$$

As a consequence, the feedforward component in (3.2a) reduces to zero. Then we have

Corollary 3.1: If (3.5) is satisfied, then the controller

$$u = -K_d\dot{e} - K_p e + 3xx_d e \tag{3.6}$$

guarantees that the Duffing equation asymptotically converges towards the equilibrium motion (3.5).

Corollary 3.1 relates our work to that of [5]. In [5], the problem of controlling the Duffing equation (3.1) with $p > 0$ to one of its periodic motions (3.5) is considered. For this purpose, the authors propose controllers of the form

$$u = -K_p e + 3xx_d e \tag{3.7}$$

and prove that these yield asymptotic convergence towards (3.5) when $K_p > -p_1$. If we assume $p > 0$, we may select $K_d = 0$; see (3.4), and consequently, (3.6) reduces to (3.7). However, this means that the error convergence of (3.1, 5, 6) stands or falls with the presence of *damping* (the physical meaning of $p > 0$) in the open-loop system (3.1). More importantly, this open-loop damping plays a major role in the transient performance of the error dynamics, which can hardly be influenced in this way. Hence, it is attractive to inject additional damping in the system via velocity feedback, which motivates the velocity error component $-K_d\dot{e}$ in the controllers (3.2) and (3.6). Consequently, under condition (3.4a) on K_d asymptotic stability of the controlled Duffing equation is guaranteed even when $p \leq 0$. Furthermore, transient characteristics like overshoot and rise-time can naturally be selected by tuning K_d. In addition, via the inclusion of the feedforward term in (3.2), we do not necessarily require the desired motion to be a periodic solution of the Duffing dynamics, in contrast to [5].

Proof of Proposition 3.1 and Corollary 3.1: The closed-loop (3.1), (3.2) is described by

$$\ddot{e} + (p + K_d)\dot{e} + (p_1 + K_p)e + e^3 = 0. \tag{3.8}$$

In correspondence to (2.5), consider the candidate Lyapunov function

$$V_3(e,\dot{e}) = \frac{1}{2}(\dot{e} + \lambda e)^2 + \frac{1}{2}((p_1 + K_p) + \lambda(p + K_d) - \lambda^2)e^2 + \frac{1}{4}e^4 \tag{3.9}$$

with $\lambda > 0$ satisfying

$$0 < \lambda < p + K_d. \tag{3.10}$$

Together with (3.4), this implies that $V_3(\cdot)$ is positive definite. Along (3.8), $\dot{V}_3(e,\dot{e})$ becomes

$$\dot{V}_3(e,\dot{e}) = -(p + K_d - \lambda)\dot{e}^2 - \lambda(p_1 + K_p)e^2 - \lambda e^4 \tag{3.11}$$

which is negative definite if (3.4, 10) are satisfied. Then the proof can be completed along the lines of Section II. □

In order to inject additional damping in the loop, controllers (3.2) and (3.6) require $\dot{x}$, as opposed to (3.7). The need for $\dot{x}$ can be eliminated by a simple observer, without affecting the stability properties of the closed loop. In particular, the output-feedback controller

$$\textbf{Controller}\{u = \ddot{x}_d + p\dot{x}_d + p_1 x_d + x_d{}^3 - q\cos(\omega t) - K_d\dot{\hat{e}} - K_p\hat{e} + 3xx_d e \tag{3.12a}$$

$$\textbf{Observer}\begin{cases}\dot{\hat{e}} = w + 2K_d(e - \hat{e}) - pe \\ \dot{w} = 2K_p(e - \hat{e}) - p_1 e - e^3\end{cases} \tag{3.12b}$$

can be employed, where

$$K_p = \lambda K_d \tag{3.13}$$

and $\lambda > 0$ scalar. Since measuring $\dot{x}$ in the controlled Duffing equation (3.1) might be difficult and noise-sensitive, the controller-observer combination (3.12, 13) seems very attractive.

Proposition 3.2: The closed-loop system (3.1, 12, 13) is asymptotically stable under the conditions (3.4, 13) and

$$0 < \lambda < \min(K_d, p + K_d). \tag{3.14}$$

Corollary 3.2: Assume that (3.4, 13, 14) are satisfied, and that $x_d(t)$ satisfies (3.5). Then

$$u = -K_d\dot{\hat{e}} - K_p\hat{e} + 3xx_d e \tag{3.15}$$

where $(\hat{e}, \dot{\hat{e}})$ as in (3.12b), provides asymptotic error convergence.

Proof: Take the Lyapunov function candidate

$$V(e, \dot{e}, \tilde{e}, \dot{\tilde{e}}) = V_3(e, \dot{e}) + V_4(\tilde{e}, \dot{\tilde{e}}) \tag{3.16}$$

where $V_3(e, \dot{e})$ as in (3.9) and

$$V_4(\tilde{e}, \dot{\tilde{e}}) = \frac{1}{2}(\dot{\tilde{e}} + \lambda \tilde{e})^2 + \frac{1}{2}(K_p + \lambda K_d - \lambda^2)\tilde{e}^2. \tag{3.17}$$

The function $V(\cdot)$ is positive definite under (3.14). This condition together with (3.4) are also sufficient for $\dot{V}(\cdot)$ to be negative definite along (3.1, 12, 13). This completes the proof. □

Proposition 3.2 demonstrates that even in the absence of open-loop damping, i.e., $p \leq 0$, for the Duffing equation asymptotic convergence towards *any* desired trajectory $x_d(t)$ can be guaranteed with an output-feedback type of controller, i.e., by only using x.

IV. Discussion

An important issue is that of robustness of the proposed controllers to parametric uncertainties and bounded disturbances. From Proposition 3.1, it follows that for asymptotic tracking of an arbitrary reference $x_d(t)$ exact knowledge of the system parameters p, p_1, q, and ω is required, cf. (3.2). In practice, however, it may be difficult to determine exactly the parameters of a (chaotic) system (see also [7] about a related point on model uncertainties). So, it would be attractive to implement (3.6) instead of (3.2), even if $x_d(t)$ does not belong to the set of periodic motions (3.5). It is interesting to know that despite this simplification the boundedness of the tracking errors can still be guaranteed under some high-gain condition on K_d. In particular, the tracking errors can be shown to be ultimately uniformly bounded (UUB) or *practically stable*, which implies that the error state tends in final time towards a closed region around zero. The proof is a straightforward extension of the UUB-results for the robot dynamics as given in [16]. By analogy, also the system-parameter-independent output-feedback controller

$$\textbf{Controller}\{u = -K_d\dot{\hat{e}} - K_p\hat{e} \tag{4.1a}$$

$$\textbf{Observer}\begin{cases} \dot{\hat{e}} = w + 2K_d(e - \hat{e}) \\ \dot{w} = 2K_p(e - \hat{e}) \end{cases} \tag{4.1b}$$

yields practical stability of the tracking errors for *arbitrary* smooth and bounded reference motions under a high-gain assumption; see [3]. Notice, however, that high-gain feedback may have practical limitations because of, for instance, noise amplification.

V. Simulations

To support our results, we simulated with MATLAB™ Duffing's equation (3.1) under output-feedback control (3.12, 13). The Duffing parameters were selected as $p = 0.4$, $p_1 = -1.1$, $q = 2.100$, and $\omega = 1.8$, in which case the Duffing equation displays chaotic behavior [5]. To illustrate that feedback control enables us to completely annihilate the chaotic dynamics and force the system towards an arbitrary desired trajectory, we define the reference motion as

$$x_d(t) = \sin(t), \quad t \geq 0. \tag{5.1}$$

To satisfy (3.4, 14), the controller parameters were chosen to be $K_d = 12.5$ and $\lambda = 4.0$. This choice yields a proportional feedback gain $K_p = \lambda K_d = 50.0$, which corresponds to K_{21} in [5]. The resulting control performance is depicted in Fig. 1, where Fig. 1(a) and 1(b) shows the time-trajectories of $e(t)$ and $\dot{e}(t)$, respectively, and Fig. 1(c) contains the state error trajectory $(e, \dot{e})$. To clearly show the effect of feedback control, the controller is only applied for $t \geq$ 25. After a short transient, the position tracking error e and velocity tracking error $\dot{e}$ converge to zero, and the control objective is attained. This can particularly be seen in Fig. 1(d), which contains the latter part of the state error trajectory.

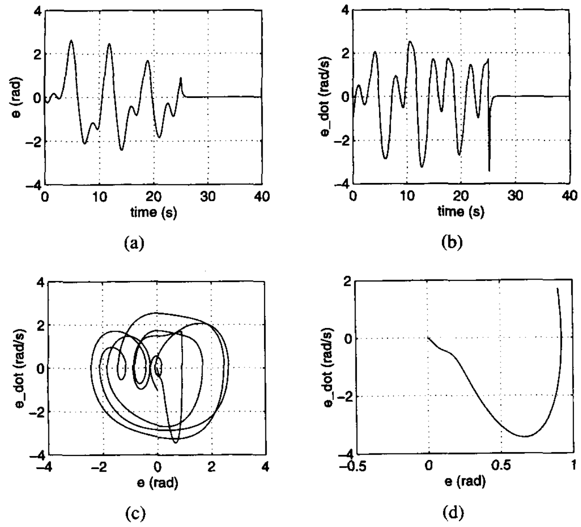


Fig. 1. Duffing's dynamics under output-feedback control (3.12, 13).

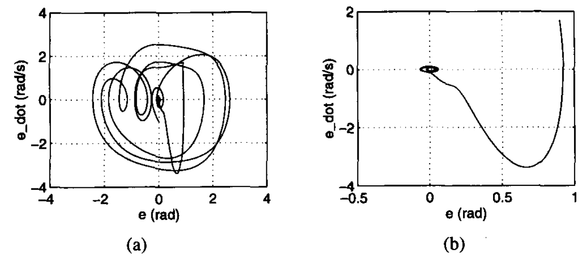
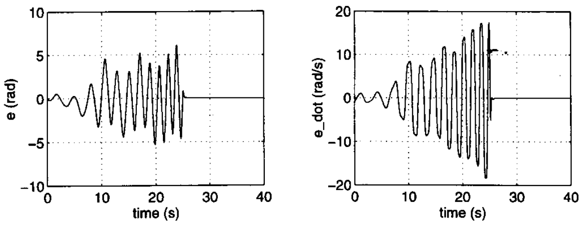


Fig. 2. Duffing's dynamics under robust output-feedback control (4.1).

As discussed in Section IV, when knowledge of p, p_1, q, and ω is lacking, the system-parameter-independent controller (4.1) can be employed. The error-state performance of this controller is shown in Fig. 2, where all gains were selected as before. The results in Fig. 2 indicate that the error state does no longer converge to zero, but that it approaches a bounded region around zero as implied by UUB-stability; see Fig. 2(b). Note that the size of the ultimate error region is small relative to the amplitude of the reference motion (5.1). This implies that the state $(x, \dot{x})$ of the Duffing equation closely follows the reference trajectory $(x_d, \dot{x}_d)$. So, even in the absence of parameter knowledge, we can largely suppress chaotic dynamics by (output) feedback control!

In a third simulation, we select $p = -0.1$, so we have negative damping in the open-loop system. For this choice, the Duffing dynamics grows unstable without feedback control. To stabilize the dynamics (3.1), we need to inject (positive) damping in the system, which can be done with both the controllers (3.2) and (3.12). Fig. 3 gives the tracking error data obtained with the output-feedback controller (3.12), where the controller gain settings were as given above. Before control the phenomenon of instability can clearly be observed, whereas for $t \geq 25$, asymptotic error convergence is attained, as proved in Proposition 3.2. Note that controller (3.7) of [5] does not yield a stable closed-loop system in the absence of open-loop damping, i.e., when $p < 0$.



(a) (b)

Fig. 3. Duffing's dynamics with negative damping under control (3.12, 13).

VI. Conclusion

In this paper, we have described how to design Lyapunov-type controllers to steer a chaotic dynamic system as the Duffing equation towards a given desired trajectory. Our methods are inspired by Lyapunov-type controllers that were recently developed for tracking control of rigid robots. The class of controllers that are discussed include observer-based controllers that may cope with parametric uncertainties and bounded disturbances in the to-be-controlled system. Some simulations illustrate the newly proposed feedback controllers.

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Shift-Variant m-D Systems and Singularities on T^m: Implications for Robust Stability

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Abstract— **This brief addresses the robust asymptotic and BIBO (bounded-input bounded-output) stability of a class of linear shift-variant multidimensional systems. Using a shift-invariant comparison system, necessary and sufficient conditions for the stability of the entire family of systems are derived.**

Nomenclature

$\mathcal{N}_0^m$	The first m-D hyperquadrant.
$\overline{U}^m$	The closed unit polydisk: $\{(\underline{z}) : \lvert z_i\rvert \leq 1, i = 1, \cdots, m\}$.
U^m	The open unit polydisk: $\{(\underline{z}) : \lvert z_i\rvert < 1, i = 1, \cdots, m\}$.
T^m	The distinguished boundary of unit polydisk: $\{(\underline{z}) : \lvert z_i\rvert = 1, i = 1, \cdots, m\}$.
$\underline{n}, \underline{i}, \underline{j}$	The spatial vectors $(n_1, \cdots, n_m)$, $(i_1, \cdots, i_m)$, and $(j_1, \cdots, j_m)$.
$y(\underline{n})$	The output of the m-D system.
$x(\underline{n})$	The input of the m-D system.
$a_{\underline{i}}(\underline{n})$	The shift-varying coefficient of a shifted output in a m-D difference equation. (For example, in a 2-D difference equation, $a_{(3,2)}(n_1, n_2)$ is the coefficient of $y(n_1 - 3, n_2 - 2)$.)
$b_{\underline{j}}(\underline{n})$	The shift-varying coefficient of a shifted input in a m-D difference equation.
N_j	The order of the m-D system in the n_j direction, $j = 1, \cdots, m$.
$\mathcal{I}$	$\{(i_1, \cdots, i_m) : 0 \leq i_k \leq N_k, k = 1, \ldots, m$, and $(i_1, \cdots, i_m) \neq \underline{0}\}$.
$\mathcal{J}$	$\{(j_1, \cdots, j_m) : 0 \leq j_k \leq N_k, k = 1, \ldots, m$, and $(j_1, \cdots, j_m) \neq \underline{0}\}$.

I. Introduction

Results addressing the robust stability problem for 1-D discrete interval polynomials have generated interest in analogous results for the m-D case. Yet even in the work on shift-invariant m-D systems, only a few results address the $m > 2$ case [1], [2]. As for 1-D systems, conditions for the robust stability of shift-variant m-D systems are more restrictive than for the shift-invariant case. Some recent results concerning the robust stability of shift-variant m-D systems can be found in [3]–[5].

Manuscript received December 12, 1994. This work was supported in part by the Office of Naval Research Grant N00014-94-1-0387 and the SAE Foundation. This paper was recommended by Associate Editor G. Chen.

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IEEE Log Number 9413216.