Exact Solution to Localized-Induction-Approximation Equation Modeling Smoke Ring Motion

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We present and discuss a three-parameter class of exact solutions to the localized-induction-approximation equations. These are one-soliton excitations (Bäcklund transforms) of the circular vortex motion. The corresponding generic vortex filament (of infinite or finite length) remains in the interior of the sphere (or torus) moving with a constant velocity. The above solutions generate a new class of exact solutions to the classical one-dimensional continuous Heisenberg ferromagnet model.

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Consider the following nonlinear system:

$$\mathbf{r}_{,t} = \mathbf{r}_{,s} \times \mathbf{r}_{,ss}, \tag{1a}$$

$$\mathbf{r}_{,s} \cdot \mathbf{r}_{,s} = 1, \tag{1b}$$

where $\mathbf{r} = \mathbf{r}(s,t)$ is an E^3-valued function of two real variables s and t, the comma means differentiation, and the multiplication sign (center dot) denotes the skew (scalar) product in E^3. There exist at least three physical applications of the system (1). As an equation of motion of the single vortex filament in the localized-induction approximation[1] (LIA) it can be applid to ordinary fluids[1-8] (I) and with some limitations, to rotating superfluid helium[9] (II) too. Equations (1) can also serve as a "potential" form for the classical one-dimensional continuous (homogeneous) Heisenberg ferromagnet (CCHF) equation[10] (III).

This paper is aimed at a presentation and short discussion of a new class of exact solutions to Eqs. (1). These can be interpreted [in case (I)] as one-soliton excitations (Bäcklund transforms) of the circular vortex motion.

Ad (I) and (II).—In these cases a curve $\mathbf{r} = \mathbf{r}(s,t)$ (t fixed) represents an instantaneous (at rescaled "time" t) shape of some central vorticity line of a very thin single vortex tube (vortex filament). Here s stands for arc-length parameter. Moreover, it is assumed that the vortex filament moves in a perfect unbounded and homogeneous fluid which is at rest at spatial infinity and which is subject to potential external forces. Finally, the essence of the LIA consists of the assumptions that (a) in a neighborhood of order λ of an arbitrary boundary point of the vortex filament there exists a single segment of the filament and the corresponding piece of its central vorticity line can be approximated by a plane curve, and (b) $\lambda \gg$ radius of the vortex filament. All of the assumptions listed above lead to Eqs. (1).

We wish to stress that in the LIA the three-dimensional object (vortex filament) is represented by a one-dimensional object (central vorticity line). From now on the term "vortex filament" means the distinguished central vorticity line.

Ad (III).—The S^2-valued function $\mathbf{S}(s,t) = \mathbf{r}_{,s}(s,t)$ solves the CCHF equation

$$\mathbf{S}_{,t} = \mathbf{S} \times \mathbf{S}_{,ss} \quad (\mathbf{S} \cdot \mathbf{S} = 1). \tag{2}$$

There exists experimental evidence that the system (1) is a reasonable mathematical tool to model phenomena in at least two areas, (I) and (III), of its applicability.[8,11,12] Hasimoto[2] was the first to propose a method to solve Eqs. (1). His elegant method consists of the following. Let us select an arbitrary solution $q(s,t)$ to the nonlinear Schrödinger (NL S) equation

$$iq_{,t} + q_{,ss} + 2|q|^2 q = 0. \tag{3}$$

The complex-valued function $q = |q|\exp(i \arg q)$ defines two real functions $k(s,t) = 2|q(s,t)|$ and $\tau(s,t) = [\arg q(s,t)]_{,s}$. One can show that the functions $k(s,t)$ and $\tau(s,t)$ can be interpreted respectively as a curvature and a torsion of some solution $r = r(s,t)$ to Eqs. (1). According to the fundamental theorem of curves[13] smooth functions $k(s,t) > 0$ and $\tau(s,t)$ define a curve $\mathbf{r} = \mathbf{r}(s,t)$ implicitly and modulo t-dependent rigid motion in E^3. In general this "inverse problem" of curves (to reconstruct a curve from its curvature and torsion) is computationally very difficult. It is also the most serious obstacle[5,7,8] to the direct application of Hasimoto's remarkable method.

Fortunately, the above "inverse problem" in the discussed case becomes "tractable"[7,14] in some geometric approach to soliton systems (approach of soliton surfaces[15]). Namely, the solution $r=r(s,t)$ to Eqs. (1) associated with a given solution $q(s,t)$ to Eq. (3) can be expressed directly in terms of the SU(2)-valued wave function Φ corresponding to the solution $q(s,t)$. Here we make use of the familiar terminology of the inverse method[15,16] of soliton theory in application to the almost classical soliton equation (3). More explicitly, $\Phi=\Phi(s,t;\zeta)$ is defined as an SU(2)-valued solution to the following linear system[16] of the NL S equation (3):

$$\Phi_{,s}=\begin{bmatrix} i\zeta & q \\ -q^* & -i\zeta \end{bmatrix}\Phi, \tag{4a}$$

$$\Phi_{,t}=\begin{bmatrix} -2i\zeta^2+i|q|^2 & -2q\zeta+iq_{,s} \\ 2q^*\zeta+iq^*_{,s} & 2i\zeta^2-i|q|^2 \end{bmatrix}\Phi, \tag{4b}$$

where the asterisk stands for complex conjugation and the so-called spectral parameter ζ is real. The corresponding vortex filament motion is given by

$$\mathbf{r}(s,t)=x(s,t)\mathbf{e}_1+y(s,t)\mathbf{e}_2+z(s,t)\mathbf{e}_3, \tag{5}$$

where the $\mathbf{e}_i$ form an arbitrary right-oriented orthonormal frame in E^3 and the components of (5) are given by the expression

$$\Phi^{-1}(s,t,0)\Phi_{,\zeta}(s,t,0)=-i[x(s,t)\sigma_1+y(s,t)\sigma_2+z(s,t)\sigma_3]. \tag{6}$$

In (6) σ_i are standard Pauli matrices.

The efficiency of the above-described method consists in the fact that for several classes of exact solutions to Eq. (3) (finite-gap solutions,[17] pure soliton solutions,[13,16] and "mixtures" of cnoidal waves and solitons[17]) we know the corresponding wave functions explicitly! For instance, insertion of a one-soliton wave function into formula (6) gives the famous Hasimoto vortex[2,7] (one-soliton excitation of the straight line vortex). Similarly, insertion of a two-soliton wave function into formula (6) gives the scattering of two Hasimoto vortices (two-soliton excitation of the straight line vortex).[7] Finally, we mention that the Kida class[5] of vortex filament motions in LIA can be obtained from (6) on insertion of the wave function corresponding to the cnoidal- (traveling-) wave solution to Eq. (3).[18]

The most simple nontrivial traveling-wave solution to Eq. (3) is the following harmonic plane wave:

$$q=q_0=K\exp(2iK^2t) \tag{7}$$

(K = positive constant). Solving system (4) with $q=q_0$ gives the corresponding Φ while formula (6) leads to the motion of a circular vortex of radius $1/2K$ moving with constant velocity $2K$ along the $\mathbf{e}_3$ axis:

$$\mathbf{r}=\mathbf{r}_0=(2K)^{-1}[\cos(2Ks)\mathbf{e}_1+\sin(2Ks)\mathbf{e}_2+4K^2t\mathbf{e}_3]. \tag{8}$$

Equation (8) is a simplified model of smoke-ring motion.[19] Surely, a more realistic model of the smoke-ring motion has to be a kind of perturbation of the circular vortex motion (8). In the framework of LIA small perturbations of the circular vortex motion have been considered in Ref. 4.

It is interesting that the method described above [Eqs. (5) and (6)] enables one to compute some finite (exact) perturbations of the circular vortex motion in LIA. These are N-soliton excitations (or N-fold Bäcklund transforms[15]) of the circular vortex motion. The resulting formulas are fairly complicated and here we confine ourselves to presentation of one-soliton excitations of the circular vortex motion only:

$$\mathbf{r}=\mathbf{r}_1=\mathbf{r}_0+d\{[-n_1\cos(2Ks)-n_2\sin(2Ks)]\mathbf{e}_1+[-n_1\sin(2Ks)+n_2\cos(2Ks)]\mathbf{e}_2+n_3\mathbf{e}_3\}, \tag{9a}$$

where

$$d=\mathrm{Im}\zeta_1/|\zeta_1|^2 \quad (\zeta_1 \text{ a complex parameter, } \mathrm{Im}\zeta_1>0), \tag{9b}$$

$$n_1=2\,\mathrm{Re}\xi/(1+|\xi|^2), \quad n_2=(|\xi|^2-1)/(1+|\xi|^2), \quad n_3=2\,\mathrm{Im}\xi/(1+|\xi|^2), \tag{9c}$$

$$\xi=K^{-1}\{-i\zeta_1+W\tan[W(s-2\zeta_1t)]\}, \tag{9d}$$

and, finally,

$$W=(K^2+\zeta_1^2)^{1/2}. \tag{9e}$$

We point out that in the framework of the inverse method[15,16] ζ_1 is an additional discrete eigenvalue. Of course, Eqs. (9) constitute a three-(real-)parameter family of exact solutions to Eqs. (1).

Putting $\mathbf{r}_1=\mathbf{r}_0+\mathbf{s}$ one can easily notice that $|\mathbf{s}|=\text{const}=d=\mathrm{Im}\zeta_1/|\zeta_1|^2$. Surely, in general, during time evolution the vortex filament (9) remains in the interior of the sphere of radius $1/2K+d$ moving with the velocity $2K$ of the "carrier" circular vortex. However, in the case $d<1/2K$ the sphere can be replaced by a torus defined by its radii equal to $1/2K$ and d. This observation gives rise to the interpretation of the solution (9) with $d < 1/2K$ as modeling a smoke-ring motion in the framework of LIA.

Depending upon the value of the parameter ζ_1 ($\mathrm{Im}\zeta_1 > 0$) all the solutions (9) can be classified into three classes: (1) class 1, $\mathrm{Re}\zeta_1 \neq 0$; (2) class 2, $\mathrm{Re}\zeta_1 = 0$ and $\mathrm{Im}\zeta_1 > K+$; and (3) class 3, $\mathrm{Re}\zeta_1 = 0$ and $\mathrm{Im}\zeta_1 \leq K$.

Ad class 1.—At a fixed instant of "time" t the vortex filament assumes a fairly curious shape. It "starts" (asymptotically as $s \to -\infty$) its course as a circle $\mathbf{r}_0(s+s_-, +t_-)$ and "ends" (asymptotically as $s \to +\infty$) its course as a circle $\mathbf{r}_0(s+s_+, t+t_+)$ ($s_\pm$ and $t_\pm$ are complicated functions of ζ_1 and K). Moreover, the motion (9) is quasiperiodic: After the time $\pi(2\,\mathrm{Im}\zeta_1)^{-1}\mathrm{Im}\,W^{-1}$ the vortex filament assumes the same shape but is rotated by the angle $2\pi(\mathrm{Im}\zeta_1^{-1})^{-1}\mathrm{Im}(\zeta_1^{-1}W^{-1}K)$. See Fig. 1.

Ad class 2.—Now the motion (9) is periodic and the above discussed circles coincide.

Ad class 3.—This class is distinguished by the following symmetry: A rotation by the angle $2\pi[1-(\mathrm{Im}\zeta_1)^2K^{-2}]^{-1/2}$ around the $\mathbf{e}_3$ axis leaves the vortex filament invariant. Moreover, it is worthwhile to distinguish the subclass 3′: $\zeta_1 = iK[1-(m/n)^2]^{1/2}$ (m and n integers, $0 < m < n$).

Ad subclass 3′.—In this case the vortex filament is a closed curve. It constitutes a locally isometric n-fold covering of the "carrier" circular vortex (8). The corresponding time evolution seems to be quite interesting. Asymptotically ($t \to -\infty$) all n "copies" of (8) coincide. At further stages of the evolution m of them undergo some distinguished transformation: Roughly speaking, each of them makes a full 2π rotation around a local axis tangent to some variable "average" circle. Asymptotically ($t \to +\infty$) all n copies coincide again. It is well known that Eqs. (1) admit an integral of motion: It is the length l of the vortex filament. The "flower opening" process described above is subject to this constraint: $l = n\pi/K$. See Fig. 2.

Now we proceed to discuss some physical aspects of the solution (9). First of all we point out that as far as the ferromagnet application (III) is concerned all the solutions $\mathbf{S} = \mathbf{r}_{,s}$ to the CCHF equation (2) are fully acceptable. For instance, the subclass 3′ generates spin configurations on the ferromagnetic sample of finite length $n\pi/K$.

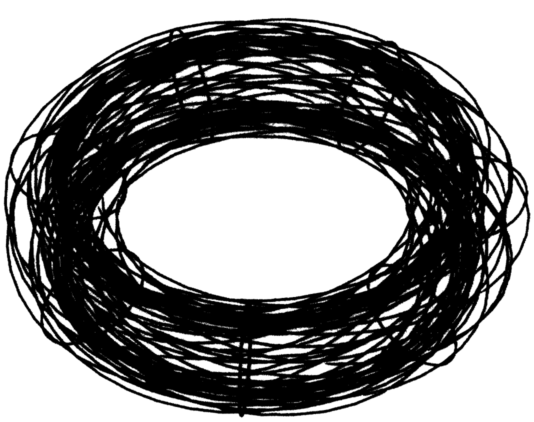

FIG. 1. Torus confinement of vortex filament: Instantaneous shape of solution (9) with $K=0.4$ and $\zeta_1 = 0.2+0.015i$ (class 1).

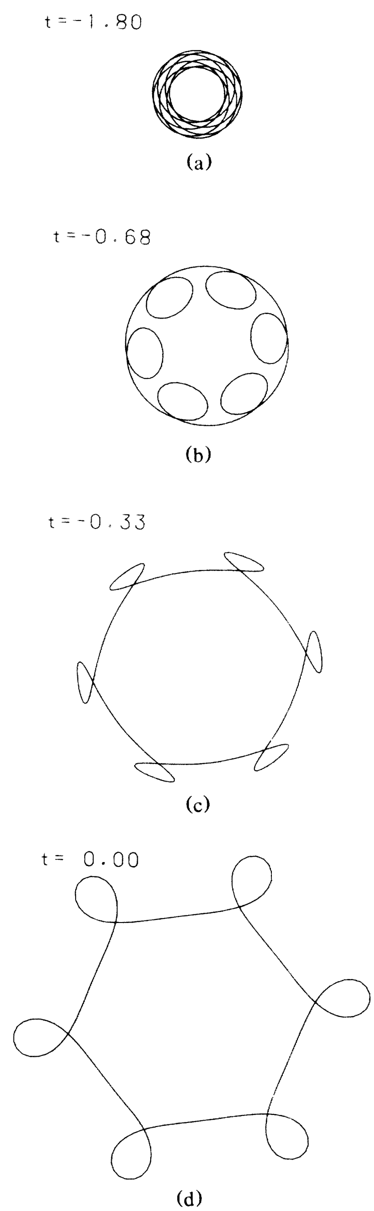

FIG. 2. Time sequence of vortex filament shape with $K=1$ and $\zeta_1 = i[1-(\frac{6}{7})^2]^{1/2}$ (subclass 3′): Front view of "flower opening" process.

On the other hand, since LIA is an approximate theory of the single vortex filament motion it may happen that an instantaneous curve $\mathbf{r}=\mathbf{r}(s,t)$ (t = const) can cross itself (see Fig. 2) and this is not acceptable from the physical point of view.[3,5] Nevertheless, some properties of the solution (9) seem to be quite appealing from a physical standpoint:

(a) The confinement property: The vortex filament remains in the interior of the sphere (torus) which moves with a constant velocity. See Fig. 1.

(b) The oscillation property: For class 1 (2) the motion is quasiperiodic (periodic). It is interesting that the pulsating character of the motion of real smoke rings has been confirmed experimentally.[4,19,20,21]

(c) Kambe and Takao in their paper[4] on theoretical and experimental aspects of smoke-ring motion observed "a pattern of waves developing on the surface of a 'smoke doughnut' which grows after traveling some distance from the orifice. It develops often to almost hexagonal shape, but this does not destroy the smoke ring." It is interesting that this feature of real smoke rings can be also imitated by the solution (9) provided $\mathrm{Re}\zeta_1=0$ and $\mathrm{Im}\zeta_1 \neq K$. Compare, for instance, Fig. 9 of Ref. 4 with Fig. 2 of this work.

Some final remarks are in order.

In the case of N-soliton excitations (N-fold Bäcklund transforms) of the circular vortex motion we have at our disposal a $2N+1$-(real-)parameter family of exact solutions to Eqs. (1); and, making an appropriate choice of these parameters, we are in a position to improve our modeling of the smoke-ring reality.

Some further improvements are also available[18]: N-soliton excitations of the circular vortex motion can be superposed with a wavy excitation corresponding to the cnoidal wave solution to Eq. (3).

The stability problem of the discussed solution (9) is postponed for future investigation. In general, the stability problem of the solutions to the LIA equations (1) is fairly difficult and deserves special attention. See Kida's remarks in Sect. 6 of his paper.[6]

Surely, all the classes of exact solutions to Eqs. (1) mentioned above can be used as "unperturbed" solutions in some future "improved" LIA based on a proper perturbation of Eqs. (1). This is one more reason to compute and classify exact solutions to Eqs. (1) by means of the soliton geometry approach.[15]

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