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Chebyshev Approximation by *H*-Polynomials: A Numerical Method

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1. INTRODUCTION

In previous papers, [8, 11] H-polynomials or "Horner-like" polynomials were introduced and some theoretical aspects were considered, especially with regard to Chebyshev approximation by these functions. The subject of this paper is a numerical method for computing locally best approximations which depends on an alternative representation of the classes of polynomials. In Section 2 we recall the definition and some of the main properties of H-polynomials. In Section 3 we derive the announced alternative representation leading to the algorithm described in Section 4. Finally, in Section 5 we give some examples.

2. H-POLYNOMIALS

DEFINITION. An H-polynomial is a function $z_n(x)$ of the real variable x and the real parameters $a_0, \ldots, a_n$, generated by the following rules. Let $j = j(k)$ be a function with the properties:

$$j, k \text{ are integers}, \qquad 1 \leqslant k \leqslant n, \quad 1 \leqslant j \leqslant \max\{1, k-1\}. \tag{2.1}$$

$z_n(x)$ is recursively defined by $z_0(x) = a_0$,

$$\left.\begin{aligned} z_k(x) &= z_{k-1}(x)x + a_k && \text{if } j(k) = 1 \\ &= \pm z_{k-1}(x)\, z_{j(k)}(x) + a_k && \text{if } j(k) > 1 \end{aligned}\right\} \quad (k = 1, \ldots, n). \tag{2.2}$$

In the case $j(k) \equiv 1$, we get as $z_n(x)$ the polynomial

$$p_n(x) = a_0 x^n + \cdots + a_{n-1}x + a_n,$$

generated by the Horner-algorithm. In [8] it was shown that the H-polynomials generated by different functions $j(k)$ are essentially different. Thus, for $n \geqslant 1$ we have $(n-1)!$ different H-polynomials. Given x, the evaluation of $z_n(x)$ requires exactly as many multiplications and additions as the evaluation of a polynomial of degree n by Horner's algorithm. Thus we have $(n-1)!$ different classes of polynomials, with respect to computing time equivalent to the class of all polynomials of degree less or equal to n. This makes it likely that for a number of standard functions we can find computer approximations of a higher accuracy than yielded by ordinary polynomial approximations, requiring the same computing time and similar storage as the latter. In [11] it was shown that a best approximation does not always exist. Thus we have a nonlinear approximation problem of a rather general type. Conditions for a given function to be a locally best approximation are given in [7, 12].

3. An Alternative Representation of H-Polynomials

Let $n \geqslant 1$ be a fixed natural number, $j(k)$ a given function satisfying (2.1), and $z_k(x)$ be given by (2.2). Then we can write

$$z_k(x) = \sum_{\nu=0}^{g(k)} f_\nu^k(a_0, \ldots, a_k)\, x^{g(k)-\nu}, \qquad k = 0, \ldots, n, \tag{3.1}$$

where the f_ν^k are polynomials in the variables $a_0, \ldots, a_k$. For the degree $g(k)$ of $z_k(x)$ we get from (2.2) $g(0) = 0$,

$$\begin{aligned} g(k) &= g(k-1) + 1 && \text{if } j(k) = 1 \\ &= g(k-1) + g(j(k)) && \text{if } j(k) > 1 \end{aligned} \qquad (k = 1, \ldots, n). \tag{3.2}$$

We can regard $z_k(x)$ as a manifold M_k of dimension $k+1$ in the $(g(k)+1)$-dimensional vectorspace of all polynomials of degree $g(k)$ or less. (3.1) is a parameter representation of M_k. We wonder if we can describe M_k in the following way:

$$M_k = \left\{ z_k(x) = \sum_{\nu=0}^{g(k)} d_\nu x^{g(k)-\nu} \;\middle|\; c_j(d_0, \ldots, d_{g(k)}) = 0, j = 1, \ldots, g(k) - k \right\}, \tag{3.3}$$

with certain functions c_j. We shall show that such a description is possible for the subset of functions in M_k of degree exactly $g(k)$. To illustrate our theory we consider an example: let $n = 4$ and $j(1) = j(2) = 1, j(3) = j(4) = 2$. Then

$$\begin{aligned}
z_0(x) &= a_0 \\
z_1(x) &= z_0(x)x + a_1 = a_0x + a_1 \\
z_2(x) &= z_1(x)x + a_2 = (a_0x + a_1)x + a_2 \\
z_3(x) &= \sigma_3 z_2(x)\, z_2(x) + a_3 \\
&= \sigma_3(a_0^2x^4 + 2a_0a_1x^3 + (a_1^2 + 2a_0a_2)\, x^2 + 2a_1a_2x + a_2^2 + \sigma_3a_3) \\
z_4(x) &= \sigma_4 z_3(x)\, z_2(x) + a_4 \qquad (3.4)\\
&= \sigma_4[a_0^3x^6 + 3a_0^2a_1x^5 + (3a_0^2a_2 + 3a_1^2a_0)\, x^4 \\
&\quad + (6a_0a_1a_2 + a_1^3)\, x^3 + (3a_1^2a_2 + 3a_0a_2^2 + \sigma_4a_0a_3)\, x^2 \\
&\quad + (3a_1a_2^2 + \sigma_4a_1a_3)x + a_2^3 + \sigma_4a_2a_3 + \sigma_4a_4].
\end{aligned}$$

Here σ_3 and σ_4 are parameters with values in $\{-1, 1\}$. We have $g(0) = 0$, $g(1) = 1$, $g(2) = 2$, $g(3) = 4$, $g(4) = 6$, and for example

$$f_2^3(a_0, ..., a_3) = \sigma_3(a_1^2 + 2a_0a_2), \qquad f_0^4(a_0, ..., a_4) = \sigma_4a_0^3.$$

Now we prove some properties of the functions f_ν^k in (3.1).

LEMMA. (i) *For $k = 0, ..., n$ we have*

$$f_0^k(a_0, ..., a_k) = \sigma_k a_0^{\mu_k} \qquad (3.5)$$

with $\sigma_k = \pm 1$ and μ_k given by $\mu_0 = 1$,

$$\begin{aligned}
\mu_k &= \mu_{k-1} && \text{if } j(k) = 1 \\
&= \mu_{k-1} + \mu_{j(k)} && \text{if } j(k) > 1.
\end{aligned} \qquad (3.6)$$

(ii) *By $g \in F_i$ we denote that g is a function of at most the variables $a_0, ..., a_i$. Then, for $1 \leqslant i \leqslant k \leqslant n$,*

$$f_{g(i)}^k = \pm N(k, i)\, a_0^{\mu_k - \mu_i} a_i + g_{k,i}, \qquad (3.7)$$

with natural numbers $N(k, i)$ and polynomials $g_{k,i} \in F_{i-1}$. Moreover, $f_\nu^k \in F_{i-1}$ for $\nu < g(i)$.

Proof. The validity of (i) is obvious from (2.2). We prove (ii) by induction on k. For $k = i$ we have $f_{g(i)}^k = f_{g(i)}^i = a_i + g_{i,i}$, $g_{i,i} \in F_{i-1}$, and obviously $f_\nu^k = f_\nu^i \in F_{i-1}$ for $\nu < g(i)$.

Suppose $\ell > i$ and (ii) is valid for $i \leqslant k \leqslant \ell - 1$. We distinguish the cases (α) $j(\ell) = 1$ or $j(\ell) < i$ and (β) $j(\ell) > 1$ and $j(\ell) \geqslant i$.

(α) By induction hypothesis we have

$$z_\ell(x) = \left(\sum_{\nu=0}^{g(\ell-1)} f_\nu^{\ell-1}(a_0, ..., a_{\ell-1})\, x^{g(\ell-1)-\nu} \right) P(x) + a_\ell ,$$

with

$$f_\nu^{\ell-1} \in F_{i-1} \qquad \text{for} \quad \nu < g(i),$$

$$f_{g(i)}^{\ell-1} = \pm N(\ell - 1, i)\, a_0^{\mu_{\ell-1}-\mu_i} a_i + g_{\ell-1,i} , \qquad g_{\ell-1,i} \in F_{i-1} ,$$

and

$$\begin{aligned} P(x) &= x && \text{if } \; j(\ell) = 1 \\ &= \pm z_{j(\ell)}(x) && \text{if } \; j(\ell) > 1. \end{aligned}$$

From $j(\ell) = 1$ or $j(\ell) < i$ we conclude that $P(x)$ is a polynomial whose coefficients are in F_{i-1}. From this and (i) the assertion follows.

(β) By induction hypothesis we have

$$z_\ell(x) = \pm \left(\sum_{\nu=0}^{g(\ell-1)} f_\nu^{\ell-1} x^{g(\ell-1)-\nu} \right) \left(\sum_{\nu=0}^{g(j(\ell))} f_\nu^{j(\ell)} x^{g(j(\ell))-\nu} \right) + a_\ell ,$$

with

$$f_\nu^{\ell-1}, f_\nu^{j(\ell)} \in F_{i-1} \qquad \text{for} \quad \nu < g(i)$$

and

$$f_{g(i)}^{\ell-1} = \pm N(\ell - 1, i)\, a_0^{\mu_{\ell-1}-\mu_i} a_i + g_{\ell-1,i} , \qquad g_{\ell-1,i} \in F_{i-1} ,$$

$$f_{g(i)}^{j(\ell)} = \pm N(j(\ell), i)\, a_0^{\mu_{j(\ell)}-\mu_i} a_i + g_{j(\ell),i} , \qquad g_{j(\ell),i} \in F_{i-1} .$$

Again the assertion follows from this and (i), and the lemma is proved.

For example (3.4) we compute $\mu_0 = \mu_1 = \mu_2 = 1$, $\mu_3 = 2$, and $\mu_4 = 3$. Corresponding to (3.5) the coefficients of x^2 resp. x^6 in $z_2(x)$ resp. $z_4(x)$ are $a_0 = a_0^{\mu_2}$ and $\sigma_4 a_0^3 = \pm a_0^{\mu_4}$. Further we see that the coefficient of the term of highest order in $z_4(x)$ which depends on a_3, is the coefficient of x^2, thus $f_{g(3)}^4 = f_4^4 = \sigma_4[3a_1^2 a_2 + 3a_0 a_2^2 + \sigma_4 a_0 a_3]$ in accordance with (ii).

Next, we define $M_n^* = \{z_n \in M_n \mid z_n$ is of degree exactly $g(n)\}$. Let $z_n(x) = \sigma \sum_{\nu=0}^{g(n)} d_\nu x^{g(n)-\nu} \in M_n^*$ be a given function, with $\sigma \in \{-1, 1\}$ such that $d_0 > 0$. Then, by definition there exist $a_0, ..., a_n$ such that

$$\sigma d_\nu = f_\nu^n(a_0, ..., a_n), \qquad \nu = 0, 1, ..., g(n). \tag{3.8}$$

The lemma shows that we can express the a_i uniquely as functions of the coefficients d_ν: by (3.5) we have $\sigma d_0 = \sigma_n a_0^{\mu_n}$. Setting $\sigma_n = \sigma$ we obtain

$$a_0 = d_0^{1/\mu_n}. \tag{3.9}$$

Because of $d_0 > 0$, (3.9) is always defined. From (3.7) we compute

$$a_i = \pm(d_{g(i)} - g_{n,i})/(N(n, i)\, a_0^{\mu_n - \mu_i}), \qquad i = 1,\ldots, n, \tag{3.10}$$

with polynomials $g_{n,i} \in F_{i-1}$.

Now we substitute $a_i = a_i(d_0,\ldots, d_{g(n)})$, $i = 0,\ldots, n$, into the $g(n) - n$ equations $\sigma d_i - f_i^n(a_0,\ldots, a_n) = 0$, $i \in I$, with

$$I = \{0, 1, 2,\ldots, g(n)\} \sim \{0, g(1), g(2),\ldots, g(n)\}. \tag{3.11}$$

This leads us to a system of equations

$$h_i(d_0,\ldots, d_{g(n)}) = 0, \qquad i \in I, \tag{3.12}$$

which must be satisfied by the coefficients d_i of $z_n(x) \in M_n^*$. On the other hand, if $d_0 > 0$ and the d_i solve (3.12) then from (3.9), (3.10) we can find $a_0,\ldots, a_n$ such that (3.8) holds. Thus, $z_n(x) \in M_n^*$. Thus, we have proved the following theorem.

THEOREM 1. $M_n^* = \{z_n(x) = \sum_{\nu=0}^{g(n)} d_\nu x^{g(n)-\nu} \mid h_i(d_0,\ldots, d_{g(n)}) = 0,\ i \in I\}$, *where I is defined by* (3.11) *and* $h_i(d_0,\ldots, d_{g(n)}) = \sigma d_i - f_i^n(a_0,\ldots, a_n)$, *with a_i given by* (3.9), (3.10).

Concerning the functions h_i we have the following theorem.

THEOREM 2. *Let $i \in I$ and* $s_i = \max\{k \mid g(k) < i\}$. *Then we have*

$$h_i = \sum_{m=1}^{m_i} K_m^i S_m^i, \tag{3.13}$$

with natural numbers m_i, rational numbers K_m^i, and

$$S_m^i = \prod_{k=1}^{i} d_{\ell(k)},$$

where $\ell(k) \in \{g(0), g(1),\ldots, g(s_i), i\}$, $\sum_{k=1}^{i} \ell(k) = i$.

We omit the extensive but elementary proof.

To demonstrate the above technique, we derive the h_i for example (3.4). System (3.8) becomes:

$$\sigma d_0 = \sigma_4 a_0^3, \qquad \sigma d_1 = \sigma_4 3 a_0^2 a_1,$$
$$\sigma d_2 = \sigma_4(3a_0^2 a_2 + 3a_1^2 a_0), \qquad \sigma d_3 = \sigma_4(6a_0 a_1 a_2 + a_1^3),$$
$$\sigma d_4 = \sigma_4(3a_1^2 a_2 + 3a_0 a_2^2) + a_0 a_3, \qquad \sigma d_5 = \sigma_4 3 a_1 a_2^2 + a_1 a_3,$$
$$\sigma d_6 = \sigma_4 a_2^3 + a_2 a_3 + a_4.$$

By (3.9), (3.10) we get ($\sigma = \sigma_4$):

$$a_0 = d_0^{1/\mu_n} = d_0^{1/3}, \qquad a_1 = d_1/(3a_0^2), \qquad a_2 = (d_2 - 3a_1^2 a_0)/(3a_0^2),$$

$$a_3 = \sigma(d_4 - 3a_1^2 a_2 - 3a_0 a_2^2)/a_0, \qquad a_4 = \sigma(d_6 - a_2^3) - a_2 a_3.$$

Substituting these expressions into the equations $\sigma d_3 = \sigma_4(6a_0 a_1 a_2 + a_1^3)$ and $\sigma d_5 = \sigma_4 3a_1 a_2^2 + a_1 a_3$ we did not use yet (we have $I = \{3, 5\}$) results in the following equations which have the form prescribed by Theorem 2:

$$h_3(d_0, d_1, d_2, d_3) = (27d_0^2 d_3 - 18d_0 d_1 d_2 + 5d_1^3)/(27d_0^2) = 0$$
$$h_5(d_0, d_1, d_2, d_4, d_5) = (81d_0^4 d_5 + 3d_0 d_1^3 d_2 - 27d_0^3 d_1 d_4 - d_1^5)/(81d_0^4) = 0.$$

We close this section by a representation theorem for the closure $\overline{M}_n{}^*$ of $M_n{}^*$ (in the topology of pointwise convergence). Clearly $M_n{}^* \subsetneq M_n \subsetneq \overline{M}_n{}^* = \overline{M}_n$. While in M_n a best Chebyshev approximation does not exist for every function in $C[a, b]$, in $\overline{M}_n$ there is always a best approximation (cf. [9]). Let

$$A = \{d = (d_0, ..., d_{g(n)})^T \in \mathbb{R}^{g(n)+1} \mid h_i(d_0, ..., d_{g(n)}) = 0, i \in I, d_0 \neq 0\}$$

and $\bar{A}$ the closure of A. The following theorem holds:

THEOREM 3. *The closure $\overline{M}_n$ of M_n is given by*

$$\overline{M}_n = \overline{M}_n{}^* = \left\{ \sum_{\nu=0}^{g(n)} d_\nu x^{g(n)-\nu} \;\middle|\; d \in \bar{A} \right\}.$$

Proof. The assertion follows immediately from the fact that a sequence of polynomials converges pointwise if and only if all the sequences of the coefficients converge.

4. A Numerical Method

The following method for computing locally best approximations is also applicable to more general linear approximation problems with nonlinear constraints.

Let $[a, b]$ be a compact real interval, f a function, continuous on $[a, b]$. If we use the representation of $M_n{}^*$ given by Theorem 1, the determination of a best approximation from $M_n{}^*$ to f is equivalent to the following optimization problem:

$$\text{Maximize } \chi(d_0, ..., d_{g(n)}, \epsilon) = -\epsilon \qquad (4.1)$$

subject to the constraints

$$\begin{aligned} &\sum_{\nu=0}^{g(n)} d_\nu x^{g(n)-\nu} - f(x) - \epsilon \leqslant 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad x \in [a, b] \\ -&\sum_{\nu=0}^{g(n)} d_\nu x^{g(n)-\nu} + f(x) - \epsilon \leqslant 0, \end{aligned} \tag{4.2}$$

$$h_i(d_0, \ldots, d_{g(n)}) = 0, \qquad i \in I. \tag{4.3}$$

First, we consider the discrete problem where $[a, b]$ in (4.2) is replaced by a finite subset $B \subset [a, b]$. Thus the number of constraints (4.2) is finite. The method we have used is a combination of two other optimization methods (cf. [5, 13]):

(a) The sequential unconstrained minimization technique (SUMT) ([1–4]).

(b) A method for solving problems of the type: maximize $g(x)$ subject to the linear constraints $\ell_i(x) \leqslant 0$, $i = 1, \ldots, p$. Examples of suitable methods are the gradient–projection method [10] or the conjugate–gradient–projection method [5].

SUMT is suitable for problems of the following type: maximize $g(x)$ subject to $v_i(x) \leqslant 0$, $i = 1, \ldots, p$, $w_j(x) = 0$, $j = 1, \ldots, q$. To solve this problem a sequence of unconstrained problems is solved: let $\{\rho_\nu\}$ be a sequence, $\rho_\nu > 0$, $\lim_{\nu\to\infty} \rho_\nu = 0$. For each ν a maximum point x^ν of

$$g_\nu(x) = g(x) + \rho_\nu \sum_{i=1}^{p} (v_i(x))^{-1} - \rho_\nu^{-1/2} \sum_{i=1}^{q} w_i^2(x) \tag{4.4}$$

is determined and the sequence $\{x^\nu\}$ is expected to converge to the solution of the given problem. Convergence can only be proved under rather strong conditions (cf. [2]) which do not hold in our case.

A disadvantage of SUMT is that linear constraints give rise to nonlinear terms in g_ν. This suggests to handle only the nonlinear constraints (4.3) as does SUMT and to solve linearly constrained problems instead of unconstrained:

Assume $\{\rho_\nu\}$ as above. For each ν the following problem is solved by a method of type (b):

Maximize $\chi_\nu(d_0, \ldots, d_{g(n)}, \epsilon) = -\epsilon - \rho_\nu^{-1} \sum_{i\in I} h_i^2(d_0, \ldots, d_{g(n)})$ subject to the constraints (4.2) ($[a, b]$ was replaced by B!).

There are two ways to treat the continuous problem:

Starting with a finite subset $B \subset [a, b]$ further constraints are added until (4.2) holds within a given precision.

By solving a discrete problem once an estimate is obtained for a locally best approximation as well as the positions of the extrema of the error-function. To improve this estimate, the method of Newton (cf. [13]) or another iterative method may be applied.

The solution of the unconstrained problem (corresponding to $\rho_0 = \infty$) is expected to be a suitable initial point for the iteration. By means of the conditions given in [7] it can be tested whether a locally best approximation is found or not. In general it will be impossible to decide if the approximation is also globally best. Taking other initial points one can attempt to find further locally best approximations.

5. Examples and Numerical Results

Now, for $n = 3$ and $n = 4$ we will give a list of all H-polynomials and the respective constraints. Let $\sigma = \pm 1$ and $z_2(x) = a_0 x^2 + a_1 x + a_2$. An upper index denotes the degree of the H-polynomial, a second upper index numbers different polynomials of same degree.

$n = 3$. Besides the class $z_3^{(3)}(x) = p_3(x)$ of all polynomials of third degree, we have just one H-polynomial of fourth degree

$$z_3^{(4)} = \sigma(z_2(x))^2 + a_3 = \sum_{\nu=0}^{4} d_\nu x^{4-\nu}$$

where the d_i satisfy

$$h_3(d_0, d_1, d_2, d_3) = (8d_0^2 d_3 - 4d_0 d_1 d_2 + d_1^3)/(8d_0^2) = 0.$$

$n = 4$. Besides $z_4^{(4)}(x) = p_4(x)$ we have five H-polynomials:

(a) $z_4^{(5.1)}(x) = z_3^{(4)}(x)x + a_4 = \sum_{\nu=0}^{5} d_\nu x^{5-\nu}$ with the same constraint as $z_3^{(4)}(x)$.

(b) $z_4^{(5.2)}(x) = \sigma z_3^{(3)}(x)\, z_2(x) + a_4 = \sum_{\nu=0}^{5} d_\nu x^{5-\nu}$ with

$$h_4(d_0, d_1, d_2, d_3, d_4) = (64d_0^3 d_4 - 32d_0^2 d_1 d_3 + 24 d_0 d_1^2 d_2 - 16 d_0^2 d_2^2 - 5d_1^4)/(64 d_0^3) = 0.$$

(c) $z_4^{(6.1)}(x) = \sigma(z_3^{(3)})^2 + a_4 = \sum_{\nu=0}^{6} d_\nu x^{6-\nu}$ with

$$h_4(d_0, d_1, d_2, d_3, d_4) = 0, \qquad h_4 \text{ as for } z_4^{(5.2)},$$

and

$$h_5(d_0, d_1, d_2, d_3, d_5)$$
$$= (64d_0^4d_5 - 32d_0^3d_2d_3 + 16d_0^2d_1d_2^2$$
$$- 8d_0d_1^3d_2 + 8d_0^2d_1^2d_3 + d_1^5)/(64d_0^4) = 0.$$

(d) $z_4^{(6.2)}(x) = \sigma z_3^{(4)}(x)\, z_2(x) + a_4 = \sum_{\nu=0}^{6} d_\nu x^{6-\nu}$ with

$$h_3(d_0, d_1, d_2, d_3)$$
$$= (27d_0^2d_3 - 18d_0d_1d_2 + 5d_1^3)/(27d_0^2) = 0$$

and

$$h_5(d_0, d_1, d_2, d_4, d_5)$$
$$= (81d_0^4d_5 + 3d_0d_1^3d_2 - 27d_0^3d_1d_4 - d_1^5)/(81d_0^4) = 0.$$

(e) $z_4^{(8)}(x) = \sigma(z_3^{(4)}(x))^2 + a_4 = \sum_{\nu=0}^{8} d_\nu x^{8-\nu}$ with

$$h_3(d_0, d_1, d_2, d_3)$$
$$= (32d_0^2d_3 - 24d_0d_1d_2 + 7d_1^3)/(32d_0^2) = 0,$$

$$h_5(d_0, d_1, d_2, d_4, d_5)$$
$$= (256d_0^4d_5 + 20d_0d_1^3d_2 - 128d_0^3d_1d_4 - 7d_1^5)/(256d_0^4) = 0,$$

$$h_6(d_0, d_1, d_2, d_4, d_6)$$
$$= (4096d_0^5d_6 - 2048d_0^4d_2d_4 + 512d_0^3d_2^3 + 512d_0^3d_1^2d_4$$
$$- 192d_0^2d_1^2d_2^2 - 16d_0d_1^4d_2 + 7d_1^6)/(4096d_0^5) = 0,$$

and

$$h_7(d_0, d_1, d_2, d_4, d_7)$$
$$= (2048d_0^6d_7 - 256d_0^4d_1d_2d_4 + 64d_0^3d_1d_2^3 + 96d_0^3d_1^3d_4$$
$$- 24d_0^2d_1^3d_2^2 - 8d_0d_1^5d_2 + 3d_1^7)/(2048d_0^6) = 0.$$

It is essential to retain the powers of d_0 in the denominator: if $d_0 = d_1 = 0$, all numerators become zero for each choice of the other d_i. This may result in numerical instability of the method if the denominators in the equations are omitted.

To test the method a FORTRAN IV program has been written and a number of examples have been treated. Though no correction of rounding errors took place, except for $z_4^{(8)}$ the results were satisfactory in all cases. Some of the results are given in Table I below. The results indicate that systematic computation of approximations to standard functions by *H*-polynomials (as is done in [6] for ordinary polynomials and rationals) could be advantageous. Note that for the Gamma function $\Gamma(x)$ the deviation for

TABLE I

Function	Interval	Error	$z_4^{(4)}$	$z_4^{(5.1)}$	$z_4^{(5.2)}$	$z_4^{(6.1)}$	$z_4^{(6.2)}$	$z_5^{(5)}$
10^x	[0, 1]	rel.	2.99	2.60	2.73	3.52	3.57	4.01
$\frac{\tan(\frac{1}{4}\Pi x^{1/2})}{x^{1/2}}$	[0, 1]	rel.	5.50	4.67	5.66	5.58	5.78	6.64
$\Gamma(x)$	[2, 3]	abs.	4.24	3.33	4.93	4.60	5.42	5.27

$z_4^{(6.2)}$ is even smaller than for $z_5^{(5)}$, the full class of polynomials of degree five or less. The numbers in the table are the values $-\log_{10}(\text{err})$ with err the corresponding (relative or absolute) maximum errors.

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