On the magnetic field and the electrical potential generated by bioelectric sources in an anisotropic volume conductor

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Abstract—*The electrical conductivity in biological tissue is often dependent on the direction of the fibres. In the paper the influence of this anisotropic nature on the electrical potential and magnetic field generated by a current dipole is studied analytically. Three different methods are discussed. The volume conductor is described by piecewise homogeneous compartments and the interfaces between compartments are either parallel or perpendicular to one of the principal axes. To illustrate the methods, the influence of the anisotropic nature is computed for a two-layered medium. It turns out that the influence on both the potential and the magnetic field cannot be ignored. However, for some commonly used models of the head and torso, a certain component of the magnetic field is not influenced by the anisotropy.*

Keywords—*Anisotropy, Electrical Potential, Magnetic field, Volume conductor*

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1 Introduction

ELECTRIC ACTIVITY in the human body can be registered noninvasively by measuring the electrical potential distribution at the body surface or the magnetic field distribution outside the body. These measurements can be used to estimate strength and position of the source. To do so a model of the source and the surrounding body is required. It is usual to model the source by one or more current dipoles. Normally the volume conductor is modelled by compartments, which are piecewise homogeneous and isotropic. However, many biological tissues have some degree of directional organisation and would therefore be expected to behave anisotropically. Skeletal muscle is strongly anisotropic and conductivity ratios as high as 14:1 can exist in these tissues (GEDDES and BAKER, 1967). STANLEY *et al.* (1986) showed in a study on dogs that the anisotropic nature of the muscle layer around the rib-cage has a strong effect on the relationship between torso and epicardial potentials, and if this is taken into account it greatly improves the agreement between calculated and measured torso potentials.

Their calculations are based on a model of the muscle layer introduced by RUSH (1967). Because the muscles are essentially directed parallel to the body surface but otherwise almost uniformly distributed over all angles, Rush assumed that the muscle layer as a whole is essentially isotropic over the two directions parallel to the body surface but with a different higher resistivity in the perpendicular directions. In models used to explain the potential distribution due to electric activity in muscle fibres or human nerves, the anisotropic nature of the passive surrounding muscle tissue is usually taken into account, e.g. ALBERS *et al.* (1986). However, the influence on the magnetic field has, to our knowledge, not yet been reported in the literature. Recently MIZUTANI and KURIKI (1988) reported the measurement of magnetic responses in the vicinity of the neck evoked in the spinal cord. ERNÉ *et al.* (1988) reported the measurement of the magnetic activity of the peripheral nerve in the vicinity of the arm in the area between the elbow and shoulder. Both types of measurements are used for localisation of the source. Because skeletal muscles in the cases mentioned are a part of the volume conductor, it is important to estimate the extent of the influence of the anisotropy on the various components of the magnetic field.

The heart is mildly anisotropic, the electrical conductivity in the direction of the fibres being roughly a factor three larger than that across the fibres. However, the fibres are wound in such a complicated fashion that no overall preferred direction can be readily discerned. Although the structure of the heart muscle is far too complicated to permit a realistic theoretical estimate of the anisotropy, its influence on the wavefronts has been studied, e.g. COLLI-FRANZONE *et al.* (1982), GONELLI and AGNELLO, (1988).

The head tissues are also expected to be anisotropic (NICHOLSON, 1973). In the cerebellar cortex, for example, the Purkinje cells are roughly orientated normal to the pial

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surface, their dendrites forming a system of parallel planes. Measurements of the conductivity in cat cerebellar cortex showed that the conductivity in the direction normal to the cortical surface has a value which is approximately twice as large as the conductivity parallel to the surface (YEDLIN *et al.*, 1974) and in the cerebral white matter this factor is nine (NICHOLSON, 1965). HOELTZELL and DIJKER, (1979) also demonstrated the existence of cortical anisotropy in the somatosensory cortex of the cat. They found that the anisotropy varied with the depth. NICHOLSON and FREEMAN (1975) measured the conductivity in frog and toad cerebella and found that the anuran cerebellum is anisotropic. According to RUSH and DRISCOLL (1968) the skull also has a greater resistivity transversely than in directions parallel to its surface. Up to now, studies on the influence of anisotropy on the magnetic field have been limited to the search for aspects of the sources which cannot be seen in the potential (ROTH and WIKSWO, 1986) and to the magnetic field generated by a current dipole situated in an infinite homogeneous anisotropic medium (PETERS *et al.* 1988). In the latter case the conductivity was defined by two values only, namely one in the direction of the fibres and the other in the direction across the fibres.

In the present study biological tissue is treated as continuous and therefore we are only concerned with gross distributions of currents, potentials and magnetic fields. We shall study three different analytical techniques for solving electrical potential and magnetic field problems. We shall not pay attention to purely numerical methods such as the boundary-element and the finite-element methods, usually applied to solve the basic equations for more complex configurations of the volume conductor. It will be shown that the boundary-element method cannot be used for anisotropic media.

Although the class of problems that can be solved analytically is restricted, the advantages of an analytical approach are:

(*a*) each parameter may be varied continuously
(*b*) general conclusions can be drawn, for instance with regard to the component of the magnetic field which will be least influenced by anisotropy
(*c*) results can be used for verification of results obtained by purely numerical methods.

First we shall present a comprehensive survey of the mathematical equations which are obeyed by the magnetic field $\boldsymbol{H}$ and the electrical potential V. A complete set of equations for both $\boldsymbol{H}$ and V will be given, where these are coupled, even in the case that the volume conductor is chosen to be homogeneous and of infinite extent. In the present study some examples of the solution of the set of equations for $\boldsymbol{H}$ and V are given and from these we can find some rules of thumb that may be used when dealing with layered structures where anisotropy is involved.

2 Assumptions and fundamental equations

Before describing the different techniques used for solving electrical potential and magnetic field problems the fundamental equations and assumptions are given. The volume conductor is described by compartments which are piecewise homogeneous and which are either isotropic or anisotropic. Furthermore, the magnetic permeability of each compartment is μ_0. As shown by PLONSEY and HEPPNER (1967), in solving the magnetic and electric fields caused by physiological current sources in a physiological conductor, the quasistatic approach may be used.

We choose our co-ordinate system to be Cartesian. The conductivity of each tissue is described by a tensor conductivity $\bar{\boldsymbol{\sigma}}$, which is defined by the diagonal elements σ_x, σ_y and σ_z only. In general σ_x, σ_y and σ_z are compartment-dependent constants. The total current density $\boldsymbol{J}$ within an anisotropic volume conductor will be the sum of the current density in the active cellular regions, the so-called primary current density $\vec{J}_p$ and the ohmic current density $\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{E}$, implying

$$\boldsymbol{J} = \boldsymbol{J}_p + \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{E} \tag{1}$$

In the quasistatic approach the equation of continuity reads

$$\text{div } \boldsymbol{J} = 0 \tag{2a}$$

yielding

$$\text{div } (\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{E}) = -\text{div } \boldsymbol{J}_p \tag{2b}$$

The electric field $\boldsymbol{E}$ is related to the electrical potential V by

$$\boldsymbol{E} = -\text{grad } V \tag{3}$$

Introducing V in eqn. (2*b*) and rewriting the left-hand term in full yields

$$\left(\sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2}\right) V = \text{div } \boldsymbol{J}_p \tag{4}$$

Eqn. 4 is the fundamental equation for the electrical potential V. The fundamental equation for the magnetic field $\boldsymbol{H}$ can be deduced from the quasistatic Maxwell equations

$$\text{rot } \boldsymbol{H} = \boldsymbol{J} \tag{5a}$$

and

$$\text{div } \boldsymbol{H} = 0 \tag{5b}$$

Combining eqn. 1 and the rule of vector analysis:

$$\text{rot} \quad \text{rot } \boldsymbol{H} = \text{grad div } \boldsymbol{H} - \Delta \boldsymbol{H}$$

results in

$$\Delta \boldsymbol{H} = -\text{rot } \boldsymbol{J}_p - \text{rot}(\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{E}) \tag{5c}$$

Introducing the electrical potential V in eqn. 5*c* and writing the left-hand term in full yields the fundamental equation for the magnetic field

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \boldsymbol{H} = -\text{rot } \boldsymbol{J}_p + \text{rot } (\bar{\boldsymbol{\sigma}} \cdot \text{grad } V) \tag{6}$$

Eqn. 6 shows that in general $\boldsymbol{H}$ is coupled to the electrical potential V. Applying the operator

$$\left(\sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2}\right)$$

to eqn. 6 and combining eqns. 6 and 4 yields

$$\left(\sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \boldsymbol{H}$$
$$= -\left(\sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2}\right) \text{rot } \boldsymbol{J}_p$$
$$+ \text{rot } (\boldsymbol{\sigma} \cdot \text{grad div } \boldsymbol{J}_p) \tag{7}$$

If different compartments are involved it is necessary to formulate the conditions for $\boldsymbol{H}$ and V at the boundaries between different compartments:

(*a*) V is continuous
(*b*) the normal component of $\bar{\boldsymbol{\sigma}} \cdot \nabla V$ is continuous
(*c*) because all compartments have the same permeability, the normal component of $\boldsymbol{H}$ is continuous
(*d*) the tangential components of $\boldsymbol{H}$ are continuous.

The field equations and conditions stated above are valid for every configuration of compartments and every sort of primary current density. In this study the source is a small plane current dipole layer. At distances larger than the dimensions of the layer, the source acts as a dipole

$$p = \int \tau e_n \, dS = \int J_p a \, dS \qquad (8)$$

where τ is the dipole density, e_n is the normal vector on the layer and a is the thickness of the layer.

3 Techniques to solve the fundamental equations

3.1 *The scale transformation*

Several authors described a method in which the solution of eqn. 4 is found with the help of a scale transformation of the co-ordinate system, e.g. KUNZ and MORAN (1958) and RUSH (1967). The co-ordinate system is transformed to a primed system such that x' is defined by

$$x' = x \frac{\sqrt{\sigma_y \sigma_z}}{\sigma} \qquad (9)$$

y' and z' being defined correspondingly. To keep the dimensions of x and x' the same σ is introduced, but the value is a free choice. If both the currents and the potential are chosen to be invariant, i.e. they are the same at corresponding points of the primed and unprimed system, then the x-component of the current density, being a current divided by a surface, transforms as

$$(J'_p)_x = (J_p)_x \frac{\sigma^2}{\sigma_x \sqrt{\sigma_y \sigma_z}}$$

The y- and z-components transform analogously. As a consequence in the primed system eqn. 4 will read

$$\Delta' V' = \frac{\text{div}\, 'J'_p}{\sigma} \text{ and } J'_p = \sigma \nabla' V' \qquad (10)$$

In other words, the method of scale transformation leads to a constructed space, where the relationships between J'_p and V' are similar to those in a homogeneous isotropic medium. The solution of eqn. 10 followed by the inverse transformation to the original co-ordinate system results in the expression for the potential in the anisotropic medium. For instance, the potential generated by a dipole in the origin of a homogeneous isotropic medium of infinite extent reads

$$V' = \frac{1}{4\pi\sigma} \frac{x'p'_x + y'p'_y + z'p'_z}{[(x')^2 + (y')^2 + (z')^2]^{3/2}} \qquad (11)$$

The inverse transformation, where p_x, being a current times a distance, transforms as

$$p'_x = p_x \frac{\sqrt{\sigma_y \sigma_z}}{\sigma}$$

leads to the potential generated by a dipole in the origin of a homogeneous anisotropic medium of infinite extent

$$V = \frac{1}{4\pi\sqrt{\sigma_x \sigma_y \sigma_z}} \frac{\dfrac{xp_x}{\sigma_x} + \dfrac{yp_y}{\sigma_y} + \dfrac{zp_z}{\sigma_z}}{\left(\dfrac{x^2}{\sigma_x} + \dfrac{y^2}{\sigma_y} + \dfrac{z^2}{\sigma_z}\right)^{3/2}} \qquad (12)$$

The method of scale transformation is not an adequate method for solving H, as is clearly shown by eqn. 7. Although the first operation in eqn. 7

$$\left(\sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2}\right)$$

is transformed properly, the second

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

is not.

In the next section we shall comment on another approach to find a solution for V and H in an anisotropic volume conductor.

3.2 *Secondary sources at the boundaries*

The solutions of eqns. 4 and 6 read

$$V(r) = \frac{p \cdot r}{4\pi\sigma r^3} + \frac{1}{4\pi} \int_{\substack{\text{all}\\ \text{space}}} \frac{1}{\bar{\sigma}(r')} \frac{\text{div}\,(\bar{\sigma}(r')\,\text{grad}\, V(r'))}{|r - r'|} d^3r' \qquad (13)$$

$$H(r) = \frac{p \times r}{4\pi r^3} + \frac{1}{4\pi} \int_{\substack{\text{all}\\ \text{space}}} \frac{\text{rot}\,(\bar{\sigma}(r')\,\text{grad}\, V(r'))}{|r - r'|} d^3r' \qquad (14)$$

If all compartments are homogeneous and isotropic, these volume integrals can be transformed into the following surface integrals as derived, respectively, by BARNARD *et al.* (1967) and GESELOWITZ (1970)

$$V(\bar{r}) = \frac{p \cdot r}{4\pi\sigma r^3} - \frac{1}{4\pi} \sum_i \int_{S_i} \Delta\sigma_i V_i \,\text{grad}' \left(\frac{1}{|r - r'|}\right) \cdot dS_i \qquad (15)$$

$$H(r) = \frac{p \times r}{4\pi r^3} - \frac{1}{4\pi} \sum_{S_i} \int_{S_i} \Delta\sigma_i V_i \,\text{grad}' \left(\frac{1}{|r - r'|}\right) \times dS_i \qquad (16)$$

where V_i is the electrical potential at the boundary S_i and $\sum_{S_i}$ means summation over all interfaces S_i, the outer boundary included
σ is the conductivity at the point r
$\Delta\sigma_i$ is the difference in conductivity of the regions at both sides of the interface numbered i.

These equations show that the contribution of the volume currents can be considered equivalent to the influence of secondary sources which lie at the interfaces between regions of different conductivities. The orientation of these secondary sources is normal to the interfaces and their strengths are proportional to the local potential and the conductivity difference of the successive compartments. The numerical computations of the integral equations can be carried out by means of the boundary-element method. It is in general not possible to transform the volume integrals into surface integrals if one or more of the compartments behaves anisotropically, because div ($\bar{\sigma} \cdot$ grad V) and rot ($\bar{\sigma} \cdot$ grad V) cannot be expressed as the divergence and the rotation, respectively, of the gradient of a scalar. Therefore the idea of secondary sources is no longer applicable. Numerical solutions of V and H will require a three-dimensional integral approximation instead of the two-dimensional boundary-element method used in the case of isotropy.

However, there are some special configurations in which one of the components of H can be computed with the procedure mentioned above. These configurations appear in electrophysiology in cases where the conductivity in the anisotropic compartment is defined by two values only, one in the direction of the fibres and one at right angles to them. Suppose, for example, that $\sigma_z = \sigma_a$ and $\sigma_x = \sigma_y = \sigma_t$ in a compartment which is bounded by an isotropic compartment with conductivity σ_0. Because in this case (rot $(\bar{\sigma} \cdot \text{grad } V))_z = (\sigma_t \text{ rot grad } V)_z$ we can handle the volume integral in the expression for H_z in the same way as in the isotropic case, and

$$H_z(\mathbf{r}) = \frac{(\mathbf{p} \times \mathbf{r})_z}{4\pi r^3} - \frac{1}{4\pi} \times \int_S (\sigma_t - \sigma_0) V \left(\text{grad} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times d\mathbf{S} \right)_z \quad (17)$$

This means that the influence of the volume conductor on the component of the magnetic field in the direction of the fibres can be interpreted as caused by a secondary source density $(\sigma_t - \sigma_0)V$. If the boundary is a surface normal to the fibres the last term on the right hand side in eqn. 17 is zero, which means that the component of $\mathbf{H}$ in the direction of the fibres is neither influenced by the anisotropy, nor by the inhomogeneity. This last situation is found if the head is modelled by concentric spherical shells. Here the proper co-ordinates are spherical polar co-ordinates. If the cortex in this model is described as an anisotropic sheet with $\sigma_r = \sigma_a$ and $\sigma_\phi = \sigma_\theta = \sigma_t$ then the radial component of the magnetic field H_r is equal to the source term only. A similar situation is found if the muscles around the rib cage are modelled as a cylindrical shell. The proper co-ordinates are cylindrical and in the muscle layer $\sigma_r = \sigma_a$ and $\sigma_z = \sigma_\phi = \sigma_t$. Again it follows that the contribution of the volume currents to the radial component of $\mathbf{H}$ is zero.

3.3 *Solving the homogeneous equations*

In this section a method is given to compute the magnetic field distribution due to a dipolar source embedded in a volume conductor consisting of anisotropic homogeneous compartments which have a simple geometry. This method implies that the homogeneous equations are solved taking the source into account by means of conditions for V and $\mathbf{H}$ at the singularity (see also AMELSFORT and SCHARTEN (1986)). The fundamental equation for V according to eqn. 4 is

$$\left(\sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2} \right) V = \text{div } \mathbf{J}_p \quad (18)$$

In a source-free region the right-hand side is zero and the equation is called the homogeneous equation. The homogeneous equation for $\mathbf{H}$ is, according to eqn. 6,

$$\left(\sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} + \sigma_z \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{H} = \mathbf{O} \quad (19)$$

To find the conditions at the source we will first consider a small dipole layer, which is infinitely thin, in the plane $z = 0$ centred at the origin. On crossing a dipole layer there is a discontinuity in V. This is clear if we take into account that the potential generated by a homogeneous dipole layer in a homogeneous isotropic medium is linearly proportional to the solid angle subtended by the dipole layer and the solid angle makes a jump of 4π when passing the layer. The normal component of the electric field $E_z = -\partial V/\partial z$ is continuous. The discontinuity in V can be found by the following procedure. Integrating eqn. 18 from $z = \Delta z$ to $z = \infty$, where $\Delta z > 0$, yields

$$\int_{\Delta z}^{\infty} \left(\sigma_x \frac{\partial^2 V}{\partial x^2} + \sigma_y \frac{\partial^2 V}{\partial y^2} \right) dz + \sigma_z \frac{\partial V(\Delta z)}{\partial z} = \int_{\Delta z}^{\infty} \text{div } \mathbf{J}_p \, dz = -J_p(\Delta z)$$

The first term on the left-hand side is finite. A similar expression is found in the region $z < 0$. Subsequent integration of $\partial V/\partial z$ from $z = -\Delta z$ to $z = \Delta z$ and taking the limit for $\Delta z \to 0$ gives

$$\lim_{z \downarrow 0} V(z) - \lim_{z \uparrow 0} V(z) = \frac{\tau}{\sigma_z}$$

In the limit that the area of the dipole layer goes to zero, we can describe the source by a current dipole p_z and the jump in the potential is

$$p_z \frac{\delta(x)\delta(y)}{\sigma_z}$$

From div $\mathbf{H} = 0$ it follows that $\mathbf{H}$ is a continuous function and consequently H_z is continuous at $z = 0$. Combining the continuity of $\mathbf{H}$ with the fact that the x- and y-components of rot $\mathbf{H}$ are zero yields that $\partial H_z/\partial z$ is continuous at $z = 0$.

If the source is a dipole at the origin, orientated in the x-direction, then the electric field component in the z-direction suffers a discontinuity at the source. To compute this discontinuity, eqn. 18 is integrated from $z = -\Delta z$ to $z = \Delta z$ and the limit for $\Delta z \to 0$ is taken yielding

$$\sigma_z \left(\lim_{z \downarrow 0} \frac{\partial V}{\partial z} - \lim_{z \uparrow 0} \frac{\partial V}{\partial z} \right) = \frac{\partial}{\partial x} p_x \delta(x)\delta(y)$$

From the combination of the facts that $\mathbf{H}$ is continuous and $(\text{rot } \mathbf{H})_x = p_x$ it follows that $\partial H_z/\partial z$ makes a jump of $\partial p_x/\partial y$ at the source.

The technique that can be used to actually solve the homogeneous equations using the boundary conditions at both the source and the interfaces between various compartments is to apply integral transformations to both the equations and the conditions. The type of integral transformation which will be suitable depends on the symmetry of the problem. For the planar geometry the Fourier transformation is an appropriate choice. The Fourier transform is defined as the function

$$\tilde{f}(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-ixk_x} e^{-iyk_y} dx \, dy \quad (20)$$

The inverse transformation is then defined as

$$f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_x, k_y) e^{ixk_x} e^{iyk_y} dk_x \, dk_y \quad (21)$$

The transform of $p\delta(x)\,\delta(y) = \tilde{p}$. The transforms of the elementary functions $\partial f/\partial x$ and $\partial f/\partial y$ are respectively $ik_x\tilde{f}$ and $ik_y\tilde{f}$. If we call

$$\kappa^2 = k_x^2 + k_y^2 \text{ and } S^2 = \frac{\sigma_x}{\sigma_z} k_x^2 + \frac{\sigma_y}{\sigma_z} k_y^2$$

the Fourier transforms of eqns. 4 and 7 in the source-free regions read

$$\left(S^2 - \frac{\partial^2}{\partial z^2} \right) \tilde{V} = 0 \quad (22)$$

$$\left(S^2 - \frac{\partial^2}{\partial z^2} \right) \left(\kappa^2 - \frac{\partial^2}{\partial z^2} \right) \tilde{H}_z = 0 \quad (23)$$

with solutions of the form

$$\tilde{V} = K_1 e^{Sz} + K_2 e^{-Sz} \quad (24)$$

$$\tilde{H}_z = K_3 e^{\kappa z} + K_4 e^{-\kappa z} + K_5 e^{Sz} + K_6 e^{-Sz} \quad (25)$$

The constants K_1–K_6 are typical for the various compartments, where K_1, K_2, K_5 and K_6 are related through the ohmic currents. These relationships can be deduced from the Fourier transform of eqn. 6:

$$(\kappa^2 - S^2)\tilde{H}_z = -(\sigma_x - \sigma_y)k_x k_y \tilde{V}$$

which gives the relationships

$$(\kappa^2 - S^2)K_5 = -(\sigma_x - \sigma_y)k_x k_y K_1$$
$$(\kappa^2 - S^2)K_6 = -(\sigma_x - \sigma_y)k_x k_y K_2 \quad (26)$$

The solution of $\tilde{H}_z$ and $\tilde{V}$ is found if the following boundary conditions are met

(*a*) $\tilde{V}$ and $\tilde{H}_z$ vanish at infinity

(*b*) The conditions at the interfaces between compartments, mentioned in section 2, in the transformed domain, read: $\tilde{V}$; $\sigma_z\, \partial\tilde{V}/\partial z$; $\tilde{H}_z$ and $\partial\tilde{H}_z/\partial z$ are continuous

(c) (i) If the source is orientated in the z-direction then the conditions at the source in the transformed domain read: $\tilde{V}$ makes a jump of p_z/σ_z while $\partial V/\partial z$, $\tilde{H}_z$ and $\partial\tilde{H}_z/\partial z$ are continuous

(ii) If the source is orientated in the x-direction then the conditions are: $\partial\tilde{V}/\partial z$ makes a jump of $ik_x p_x/\sigma_z$; $\tilde{H}_z$ is continuous and $\partial\tilde{H}_z/\partial z$ makes a jump of $ik_y p_x$.

Using the stated equations and boundary conditions, $\tilde{V}$ and H_z can be found. By rewriting eqn. 5*a* and 5*b* expressions are obtained from which $\tilde{H}_x$ and $\tilde{H}_y$ can be deduced:

$$\tilde{H}_x = -\frac{1}{\kappa^2}\left(ik_y \sigma_z \frac{\partial\tilde{V}}{\partial z} - ik_x \frac{\partial\tilde{H}_z}{\partial z}\right) \quad (27)$$

$$\tilde{H}_y = \frac{1}{\kappa^2}\left(ik_x \sigma_z \frac{\partial\tilde{V}}{\partial z} + ik_y \frac{\partial\tilde{H}_z}{\partial z}\right) \quad (28)$$

The final expressions for V and $\boldsymbol{H}$ are obtained by applying the inverse transformation (eqn. 21). In most cases the inverse transformation has to be carried out numerically. An exception is the two-layered medium pictured in Fig. 1, which can be handled analytically.

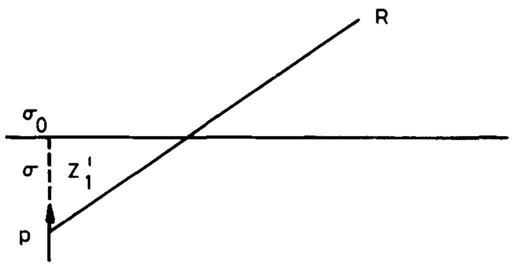

Fig. 1 *Current dipole* ***p*** *located below an interface separating two regions having different conductivities. The origin of the co-ordinate system is at the dipole. The lower region is anisotropic, the fibres are parallel to the interface $z = z_1$. The upper region is isotropic of conductivity σ_0. The point of observation is R*

The actual computation will be given as an example. In the region $z < z_1$ the medium is anisotropic wth the fibres perpendicular to the boundary $z = z_1$. The conductivity normal to the fibres is $\sigma_x = \sigma_y = \sigma_t$ and the conductivity in the direction of the fibres is $\sigma_z = \sigma_a$, which means that $S = \lambda\kappa$, where $\lambda = \sqrt{\sigma_t/\sigma_a}$.

In the region $z > z_1$ the medium is isotropic with conductivity σ_0, implying that $S = \kappa$. In both regions $\sigma_x = \sigma_y$ and therefore according to eqn. 34 $K_5 = K_6 = 0$. Consequently, taking into account the fact that $\tilde{V}$ and $\tilde{H}_z$ vanish at infinity, eqns. 24 and 25 read in the various regions

$z > z_1$	$\tilde{V} = C_2 e^{-\kappa z}$	$\tilde{H}_z = C_4 e^{-\kappa z}$
$0 < z < z_1$	$\tilde{V} = B_1 e^{\kappa\lambda z} + B_2 e^{-\kappa\lambda z}$	$\tilde{H}_z = B_3 e^{\kappa z} + B_4 e^{-\kappa z}$
$z < 0$	$\tilde{V} = A_1 e^{\kappa\lambda z}$	$\tilde{H}_z = A_3 e^{\kappa z}$

Using the boundary conditions at $z = 0$ and $z = z_1$ we find straightforwardly for $z > z_1$

$$\tilde{V} = \frac{\lambda p_z}{\lambda\sigma_a + \sigma_0} e^{-\kappa z - \kappa(\lambda - 1)z_1} \text{ and } \tilde{H}_z = 0$$

From these expressions $\tilde{H}_z$ and $\tilde{H}_y$ can be deduced.

In this special situation the inverse transformation can be performed analytically. Substituting $k_x = \kappa \cos\alpha$ and $k_y = \kappa \sin\alpha$ and using integrals from Gradshteyn and Ryzhik (1965) yields in the region where $z > z_1$

$$V = \frac{1}{2\pi}\frac{\lambda p_z}{\lambda\sigma_a + \sigma_0}\frac{z + z_1(\lambda - 1)}{\bar{R}^3},$$

$$\text{where } \bar{R}^2 = x^2 + y^2 + (z + z_1(\lambda - 1))^2 \quad (29)$$

$$H_x = \frac{1}{2\pi}\frac{-\sigma_0\lambda p_z}{\lambda\sigma_a + \sigma_0}\frac{y}{\bar{R}^3}$$

$$H_y = \frac{1}{2\pi}\frac{\sigma_0\lambda p_z}{\lambda\sigma_a + \sigma_0}\frac{x}{\bar{R}^3}$$

$$H_z = 0$$

The same procedure using conditions (*c*)(ii) at the source give $\boldsymbol{H}$ and V in the case where the dipole is parallel to the boundary, i.e. normal to the fibres, yielding in the region where $z > z_1$

$$V = \frac{1}{2\pi}\frac{xp_x}{\lambda\sigma_a + \sigma_0}\frac{1}{\bar{R}^3}$$

and

$$H_z = \frac{1}{4\pi}\frac{yp_x}{R^3}, \text{ with } R^2 = x^2 + y^2 + z^2 \quad (30)$$

4 Discussion

Three methods are discussed to solve volume conductor problems taking anisotropy into account. The number of problems which can be solved is restricted because all compartments are considered to be homogeneous and the interfaces are either parallel or perpendicular to the fibres. Although the example given in Section 3.3 where $p = p_z e_z$ was solved by means of Fourier transformation, due to the cylinder symmetry of the problem Hankel transformation is also appropriate.

The two-layered medium served as a simple example. However, it may be an adequate model to explain the potential and magnetic field caused by an electrical activity in a slice of the hippocampus (Tesche *et al.*, 1988) or the cerebellum (Okada and Nicholson, 1988) immersed in a saline solution. The magnitude will be affected by the anisotropic nature of the tissue as described by eqns. 29 and 30. The scale transformation can also be used to explain the potential distribution in the two-layered medium depicted in Fig. 1. The lower region can be described by primed co-ordinates to the effect that the potential in this region is given by $\sigma\Delta' V' = \text{div}'\, J'_p$, in the upper region it is given by $\Delta V = 0$.

If we construct our primed co-ordinate system such that the boundary conditions at the interface between the primed and the isotropic region read

V is continuous

and

$$\sigma \frac{\partial V'}{\partial z'} = \sigma_0 \frac{\partial V}{\partial z} \quad (31)$$

then we have a set of conditions and equations for the potential which is identical to the one which describes the potential distribution due to a dipole in a two-layered medium in which both layers are isotropic. This problem was already solved by the method of images (NUNEZ, 1981). The potential in the upper material of conductivity σ_0 due to a source in the lower material of conductivity σ is

$$\frac{2\sigma}{\sigma_0 + \sigma} V_H \quad (32)$$

where V_H is the potential which would have been generated by the same source in a homogeneous medium of conductivity σ. The potential in the lower material is

$$\left(1 + \frac{\sigma - \sigma_0}{\sigma + \sigma_0} \frac{R}{S}\right) V_H$$

where R is the distance from the source to the point of observation and S is the distance from the image of the source to the point of observation. V_H is given in eqn. 11.

Condition 14 is met at the interface, because at the interface $R = S$. Condition 15 is met if we transform condition 13 in primed co-ordinates

$$\sigma_z \frac{\partial V}{\partial z} = \sigma_z \frac{\sqrt{\sigma_x \sigma_y}}{\sigma} \frac{\partial V'}{\partial z'} = \sigma_0 \frac{\partial V}{\partial z} \quad (33)$$

From this latter expression it follows that a correct choice of σ is given by

$$\sigma^2 = \sigma_z \sqrt{\sigma_x \sigma_y} \quad (34)$$

Let the source be a dipole with a component in the fibre direction and a component perpendicular to the fibres, i.e.

$$\boldsymbol{p} = p_x \boldsymbol{e}_x + p_z \boldsymbol{e}_z$$

According to eqn. 9 the transformation from the primed to the unprimed co-ordinates reads

$$x' = x;\ y' = y \text{ and } z' = \lambda z, \text{ where } \lambda = (\sigma_t/\sigma_a)^{1/2} \quad (35)$$

On the interface the distribution of secondary sources is linearly proportional to the distribution found in the case that the distance between the source and the boundary is diminished with a factor λ. As a consequence the secondary sources give rise to a potential for which in the upper region the condition $\Delta V = 0$ still holds. This implies that the whole procedure is legitimate.

In the upper region we find after transforming the primed co-ordinates of the lower region to the original co-ordinates in expression (32)

$$V = \frac{1}{2\pi} \frac{x p_x + \lambda p_z (z + z_1(\lambda - 1))}{(\lambda\sigma_a + \sigma_0)(x^2 + y^2 + (z + z_1(\lambda - 1))^2)^{3/2}} \quad (36)$$

which is in accordance with expressions 29 and 30. If the fibres are parallel to the interface then an appropriate scale transformation is not found for which, after inverse transformation, the condition that the ohmic currents give rise to such a potential distribution that $\Delta V = 0$ still holds. Although the method given in Section 3.3 can be used in this case it leads to Fourier transforms which cannot be transformed analytically.

An important point in favour of magnetic measurements is that certain components of the magnetic field are not influenced at all by the anisotropy. For instance, if the head is modelled by anisotropic spherical shells, and the conductivity in the radial direction differs from that in the tangential direction, then the radial component of the magnetic field is neither influenced by the inhomogeneities nor by the anisotropies. On the other hand, the potential is influenced by both the inhomogeneities and the anisotropies. The latter especially is difficult to take into account because the tensor conductivity is impossible to measure in human beings.

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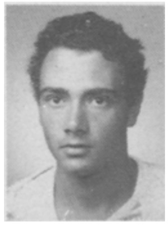

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