IMPROVED DATA DRIVEN CONTROL CHARTS

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Abstract: Classical control charts for monitoring the mean are based on the assumption of normality. When normality fails, these control charts are no longer valid and serious errors often arise. Data driven control charts, which choose between the normal chart, a parametric one and a nonparametric chart, have recently been proposed to solve the problem. They also correct for estimation errors due to estimation of the parameters involved or, in the nonparametric chart, for estimation of the appropriate quantiles of the distribution. In many cases these data driven control charts are performing very well. However, when the data point towards the nonparametric chart no satisfactory solution is obtained unless the number of Phase I observations is very large. The problem is that accurate estimation of an extreme quantile in a nonparametric way needs a huge number of observations. Replacing the nonparametric individual chart by a nonparametric chart for grouped observations does the job. These improved data driven control charts are presented here. Ready-made formulas are given, which make implementation of the charts quite straightforward. An application on real data clearly shows the improvement: estimation of extreme quantiles is replaced by estimation of ordinary quantiles, which can be done in an accurate way for common sample sizes.

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1. Introduction

As long as normality holds, the classical control chart for monitoring the mean of a production process can be applied very well. When normality fails – as it unfortunately often does in the tails of the underlying distribution – quantities like the false alarm rate (FAR) or the average run length (ARL) may be far away from their prescribed values. Model errors of more than 400% are not uncommon, see e.g. Chan et al [13], Pappanastos and Adams [15], Albers et al [5, 6]. To avoid these errors as a first step the family of normal distributions can be embedded in a larger parametric model. The so called normal power family is a very useful and natural family for this purpose. It has an additional shape parameter, which is intimately related to the construction of control charts. This can be seen as follows.

Control limits are determined by quantiles of the underlying distribution. An upper limit (UL) and a lower limit (LL) are selected and an out-of-control (OoC) signal is produced as soon as a newly arriving measurement falls outside these limits. In order to get a prescribed FAR like 0.002, UL and LL are chosen as the upper and lower 0.001-quantiles of the underlying distribution.

Essentially the quantiles of the standardized normal power family are powers of the quantiles of the standard normal distribution. By taking a power larger than one heavier tails are modelled, while powers smaller than one result in lighter tails than the normal distribution. By taking different shape parameters at the lower and upper control limit, skewness of the distribution can be dealt with as well. Another parametric family with a similar flavor is the Burr distribution, see e.g. Yourstone and Zimmer [20].

Typically, the parameters of the proposed distribution are not known. Therefore, Phase I observations $X_1, ..., X_n$ need to be used to estimate the parameters. The price we have to pay for the extension to a larger parametric family is a larger stochastic error due to estimation of two extra parameters, one for the lower and one for the upper control limit.

When the parametric family still is too restricted to model the underlying distribution accurately enough, a nonparametric control chart comes into the picture. This topic by now has also received considerable attention in the literature, see e.g. Bakir and Reynolds [9], Bakir [7, 8], Chakraborti et al [11, 12], Chakraborti and Eryilmaz [10], Qiu and Hawkins [17, 18] and Albers and Kallenberg [1]. However, applying a nonparametric control chart may

result in an unacceptably large stochastic error. For instance, to construct the upper limit for a one-sided control chart (with group size $m = 1$) such that $FAR = 0.001$, we need to estimate the 0.999-quantile. Obviously, with e.g. 100 observations this cannot be done in a truly nonparametric way.

In Albers et al [5] the three control charts (normal, parametric, nonparametric) were combined by a data driven choice between them. This combined control chart works very well in many situations. The just mentioned problem what to do when the data tell us to apply the nonparametric chart and the number of observations is not very large, was partly solved there by introducing a modified nonparametric control chart. The nonparametric chart itself may lead to a randomized upper control limit, which equals ∞ with substantial probability. Although being theoretically correct, such a chart which may result in never producing an out-of-control signal is obviously totally unacceptable from a practical point of view. Therefore, in the modified version, the control limit ∞ was substituted by a very large quantity, in fact $X_{(n)} + S$, where $X_{(n)}$ denotes the largest observation from the Phase I observations $X_1, ..., X_n$ and S its sample standard deviation.

First of all, this modified version is not really a nonparametric control chart. Moreover, for sample sizes $n \leq 250$, the expected FAR occasionally turned out to be twice as large as it should be. Therefore, improvement of this part of the procedure should be very welcome.

Such an improvement can be obtained by considering at this stage a somewhat larger group size m. It has been shown in Albers and Kallenberg [4] that, surprisingly, rather than using the group averages, it is definitely preferable to compare the minimum for each group to a suitably chosen upper control limit and, similarly, the maximum to an analogous lower control limit. We call such a chart a minimum control chart or shortly a MIN chart. The reason for the preference of a MIN chart is that in the nonparametric case the sample mean or average turns out to be neither optimal nor easy to work with. Outside the normal model examples are easily found where MIN outperforms the average chart, while under normality the MIN chart behaves rather well and is substantially better than the individual chart for interesting shifts under out-of-control. More importantly for the present purpose, the estimation step for the average chart still requires uncomfortably large values of n. This sharply contrasts the situation with the MIN chart, since for that chart a much less extreme quantile has to be estimated. For example, for $m = 3$ and $p = 0.001$ the $(mp)^{1/m} = 0.144$-quantile comes in. For more details we refer to Albers and Kallenberg [4].

This new nonparametric control chart opens the possibility to fill the gap in the data driven control chart when n is not huge and the selection rule points towards the nonparametric chart. In this paper the data driven control chart is improved by adding at this stage the MIN chart.

The paper is organized as follows. When estimation comes in, quantities like FAR and ARL become random variables, as they are functions of the Phase I observations $X_1, ..., X_n$. In Section 2 criteria are given to express how far away the stochastic FAR and ARL are from their prescribed values. In particular, the bias and the so called exceedance probability are considered. Moreover, the concept of average time to signal (ATS) is introduced, which relates the individual chart with a chart based on groups. The improved data driven control chart is presented in Section 3. The three building blocks, the normal, the parametric and the MIN chart, are described, as well as the selection rule to choose between them. The ready-made formulas make implementation, for instance in *Excel*, quite straightforward. Therefore, the proposed methods are very accessible for practitioners.

In an application to real data in Section 4 it is shown how the improved data driven control chart works in practice. It is seen that for the upper limit the classical normal chart can be used, while the lower limit requires a non-parametric chart, due to the heavy lower tail. The individual nonparametric chart makes no sense here, while the MIN chart performs very well. A brief summary including theoretical results is given in Section 5.

2. Criteria

A one-sided control chart with an upper control limit such that $FAR = p$ is easily obtained when the continuous distribution F of the observations under in-control (IC) is known. One simply takes $UL = \overline{F}^{-1}(p)$, where $\overline{F}(x) = 1 - F(x)$ and hence $\overline{F}^{-1}(p)$ is the upper p-quantile of X, having distribution function F. For instance, when X is normally distributed with known mean μ and variance σ^2 we get $UL = \mu + u_p\sigma$ with $u_p = \overline{\Phi}^{-1}(p)$, the upper p-quantile of the standard normal distribution. Often one chooses $u_p = 3$ corresponding to $FAR = p = 0.00135$ and $ARL = 1/p = 741$. (Note that we assume independence of the observations and hence do not include autocorrelated time series.)

When the parameters of F (in the normal or parametric control chart) are unknown or F itself (in the nonparametric control chart) is unknown, we have to estimate the upper p-quantile using Phase I observations $X_1, ..., X_n$, which are

assumed to be IC. As a consequence, FAR and ARL are no longer deterministic quantities, but random variables, because they depend on $X_1, ..., X_n$. The idea is that the stochastic FAR, written as P_n or the stochastic $ARL = 1/P_n$ should be close to the prescribed p or $1/p$, respectively. Several criteria can be used for it. We may focus on the average behavior of the chart during a long series of separate applications by concentrating on the bias $EP_n - p$ (we do not consider the bias of the stochastic ARL, since $E\left(1/P_n\right)$ does not adequately summarize the run length properties of the chart, cf. e.g. Roes [19, p. 34], Quesenberry [16, p. 242] and Jones et al [14, p. 100]).

On the other hand we want to see how bad things can get in one given application of the chart. Then we look at the so called exceedance probabilities like $P(P_n > p(1 + \varepsilon))$ with ε small. They should be smaller than α, say, a second small quantity. The idea is that the probability that P_n is substantially larger than p (that is larger than $p(1+\varepsilon)$) is small: less than α. One may think of values for α and ε like 0.1 or 0.2.

To get a small bias or small exceedance probabilities, huge samples are needed when estimators of the parameters are simply plugged in into the control limits. Fortunately, corrections of the control limits are available reducing the bias and exceedance probabilities sufficiently well for common sample sizes, except for the nonparametric control charts. Indeed, as explained in the introduction, in the nonparametric case the stochastic error is so large that a truly nonparametric chart would imply that with substantial probability no OoC-signal is given at all. The remedy for it is the application of the MIN chart based on groups of observations with group size m. Since we are dealing with groups of m observations we consider in that case the so called average time to signal (ATS) which equals m/FAR and compare this for the IC-situation to the prescribed $1/p$. Here FAR refers to the failure rate for the group of m observations. So, when we have Phase II observations $X_{n+1}, ..., X_{n+m}$ and UL denotes the upper control limit of the MIN chart, we get (in the one sided case) $FAR = P\left(\min\left(X_{n+1}, ..., X_{n+m}\right) > UL\right)$. In practice UL should be estimated by $\widehat{UL}$, say, and (the corrected) $\widehat{UL}$ is chosen in such a way that $(1/m)$ times the observed FAR has a small bias w.r.t. p or that its exceedance probability is small.

For the two-sided control charts we simply combine the upper and lower control charts, replacing p by $p/2$. The observed FAR's for the two-sided control charts are defined as follows. For the normal and parametric chart we have

$$P_{nL} = P\left(X_{n+1} < \widehat{LL} | X_1, \ldots, X_n\right)$$

with $\widehat{LL} = \widehat{LL}_N$, say, for the normal chart and $\widehat{LL} = \widehat{LL}_P$ for the parametric chart and

$$P_{nU} = P\left(X_{n+1} > \widehat{UL}|X_1,\ldots,X_n\right)$$

with $\widehat{UL} = \widehat{UL}_N$ for the normal chart and $\widehat{UL} = \widehat{UL}_P$ for the parametric chart. For the nonparametric MIN chart we define

$$P_{nL} = P\left(\max\left(X_{n+1},...,X_{n+m}\right) < \widehat{LL}_{MIN}|X_1,...,X_n\right)$$

and

$$P_{nU} = P\left(\min\left(X_{n+1},...,X_{n+m}\right) > \widehat{UL}_{MIN}|X_1,...,X_n\right).$$

In all cases we have

$$P_n = P_{nL} + P_{nU}.$$

3. Improved Data Driven Control Chart

To present the improved data driven control chart we start with describing its building blocks: the normal chart, the parametric chart and the minimum chart, the latter being a nonparametric chart. They are obtained by starting with $UL = \overline{F}^{-1}(p)$ and $LL = F^{-1}(p)$ for the normal and parametric chart, and $UL = \overline{F}^{-1}(p^{1/m})$ and $LL = F^{-1}(p^{1/m})$ for the minimum chart, thus giving $FAR = p$ for all three charts. Next appropriate estimators are plugged in, based on Phase I observations $X_1,...,X_n$. We write $\overline{X} = n^{-1}\sum X_i$ and $S = \sqrt{S^2}$ with $S^2 = (n-1)^{-1}\sum\left(X_i - \overline{X}\right)^2$. This gives the uncorrected control limits $\overline{X} \pm u_{p/2}S$ in the normal case, $\overline{X} \pm c(\widehat{\gamma})u_{p/2}^{1+\widehat{\gamma}}S$ in the parametric case (see (3.1) and (3.2) for definition of $c(\gamma)$) and $\overline{F}_n^{-1}(p^{1/m}), F_n^{-1}(p^{1/m})$ for the minimum chart, where F_n denotes the empirical distribution function based on $X_1,...,X_n$. Finally correction terms are added according to the prescribed aim, to reduce the bias or the exceedance probability. Here we simply present the formulas. For more information and derivation of these charts we refer to Albers and Kallenberg [2, 3, 4] and references therein. Although the derivations of the charts are rather complicated, as to the computational aspects of the control limits it should be remarked that the actual control limits are explicitly given here and they can be obtained quite straightforwardly, for instance using *Excel*, because all required quantities (like the inverse of the standard normal distribution function and the Gamma function) are available in *Excel*.

Aim	$\widehat{LL}_N, \widehat{UL}_N$
$EP_n = p$	$\overline{X} \pm u_{p/2} S \left\{ 1 + \dfrac{u_{p/2}^2 + 3}{4n} \right\}$
$\Pr\left(P_{nL} > \frac{p}{2}(1+\varepsilon)\right) \leq \alpha,$ $\Pr\left(P_{nU} > \frac{p}{2}(1+\varepsilon)\right) \leq \alpha$	$\overline{X} \pm u_{p/2} S \left\{ 1 + \dfrac{u_\alpha \left(\frac{1}{2} + u_{p/2}^{-2}\right)^{1/2}}{\sqrt{n}} - \dfrac{\varepsilon}{u_{p/2}^2} \right\}$
$\Pr\left(\frac{1}{P_{nL}} < \frac{1}{p/2}(1-\varepsilon)\right) \leq \alpha,$ $\Pr\left(\frac{1}{P_{nU}} < \frac{1}{p/2}(1-\varepsilon)\right) \leq \alpha$	$\overline{X} \pm u_{p/2} S \left\{ 1 + \dfrac{u_\alpha \left(\frac{1}{2} + u_{p/2}^{-2}\right)^{1/2}}{\sqrt{n}} - \dfrac{\varepsilon}{u_{p/2}^2 (1-\varepsilon)} \right\}$

Table 1: Two-sided (corrected) normal control limits

3.1. Normal Control Charts

The two-sided control limits for the (corrected) normal control chart are given in Table 1. The control limits are denoted by $\widehat{LL}_N$ and $\widehat{UL}_N$, where the subscript N refers to "normality". In case of the lower limit $\widehat{LL}_N$ the $-$ sign should be read and for $\widehat{UL}_N$ the $+$ sign is used. An OoC-signal is given when a new incoming Phase II observation X_{n+1}, say, is outside the interval $\left(\widehat{LL}_N, \widehat{UL}_N\right)$. The one-sided versions are simply obtained by replacing $p/2$ in the control limits by p and taking the corresponding one-sided intervals.

3.2. Parametric Control Charts

To embed the normal distributions in a larger family with heavier or lighter tails we consider essentially powers of the standard normal quantiles as the new quantiles. The family of distributions obtained in this way is called the normal power family. More precisely, replace u_p (for $0 < p < 1/2$) by

$$c(\gamma) u_p^{1+\gamma}, \tag{3.1}$$

where $\gamma > -1$ and where $c(\gamma)$ is a normalizing constant (to make the variance equal to one) given by

$$c(\gamma) = \pi^{1/4} 2^{-(1+\gamma)/2} \Gamma(\gamma + \tfrac{3}{2})^{-1/2} \tag{3.2}$$

with Γ the Gamma-function. It is immediately seen that $\gamma = 0$ leads to the standard normal quantile u_p.

Let $X_{(1)} \leq \ldots \leq X_{(n)}$ be the order statistics of $X_1, \ldots, X_n$. The estimator

of γ at the upper tail, $\widehat{\gamma}_U$, is given by ($ent(x)$ is the integer part of x)

$$\widehat{\gamma}_U = 1.1218 \log \left(\frac{X_{(ent(0.95n+1))} - \overline{X}}{X_{(ent(0.75n+1))} - \overline{X}} \right) - 1.$$

Note that $\left(X_{(ent(0.95n+1))} - \overline{X} \right) / \left(X_{(ent(0.75n+1))} - \overline{X} \right)$ estimates

$$\left(c(\gamma) u_{0.05}^{1+\gamma} \right) / \left(c(\gamma) u_{0.25}^{1+\gamma} \right) = (u_{0.05}/u_{0.25})^{1+\gamma}$$

and that $1/\log (u_{0.05}/u_{0.25}) = 1.1218$. Similarly, the estimator of γ at the lower tail, $\widehat{\gamma}_L$, is given by

$$\widehat{\gamma}_L = 1.1218 \log \left(\frac{\overline{X} - X_{(n-ent(0.95n))}}{\overline{X} - X_{(n-ent(0.75n))}} \right) - 1.$$

By taking estimators of γ according to the tail under consideration allowance is made for possible skewness of the distribution. The uncorrected control limits are given by $\overline{X} \pm c(\widehat{\gamma}) u_{p/2}^{1+\widehat{\gamma}} S$. Before presenting the corrected control limits we introduce some notation, which is needed for the correction terms (see Table 2) :

$$C_1(\gamma, u_p) = -1.23 - 0.63\gamma + 0.73\gamma^2 + 0.74u_p - 0.08\gamma u_p - 0.14\gamma^2 u_p,$$

$$C_2(\gamma) = \left(\frac{u_{a_n}}{u_{b_n}} \right)^{1+\gamma} - 2.4387^{1+\gamma}$$

$$\text{with } a_n = 1 - \frac{ent(0.95n+1)}{n+1}, b_n = 1 - \frac{ent(0.75n+1)}{n+1},$$

$$C_3(\gamma, u_p) = -76.37 - 120.12\gamma - 81.93\gamma^2 + 35.53u_p + 53.71\gamma u_p + 37.18\gamma^2 u_p,$$

$$A(\gamma, u_p) = -4.00 - 12.54\gamma - 10.02\gamma^2 + 2.91u_p + 6.47\gamma u_p + 4.42\gamma^2 u_p.$$

For a detailed explanation of these formulas we refer to Albers et al [5, 6].

The two-sided control limits for the (corrected) parametric control chart are given in Table 2. They are denoted by $\widehat{LL}_P$ and $\widehat{UL}_P$, where the subscript P refers to "parametric". In case of the lower limit $\widehat{LL}_P$ the $-$ sign should be read and $\widehat{\gamma} = \widehat{\gamma}_L$ should be inserted, while for $\widehat{UL}_P$ the $+$ sign is used and $\widehat{\gamma} = \widehat{\gamma}_U$. An OoC-signal is given when a new incoming Phase II observation X_{n+1} is outside the interval $\left(\widehat{LL}_P, \widehat{UL}_P \right)$. The one-sided versions are simply obtained by replacing $p/2$ in the control limits by p and taking the corresponding one-sided intervals.

Aim	$\widehat{LL}_P, \widehat{UL}_P$
$EP_n = p$	$\overline{X} \pm S\left\{c(\widehat{\gamma})u_{p/2}^{1+\widehat{\gamma}} - C_1(\widehat{\gamma}, u_{p/2})C_2(\widehat{\gamma}) + \dfrac{C_3(\widehat{\gamma}, u_{p/2})}{n}\right\}$
$\Pr\left(P_{nL} > \frac{p}{2}(1+\varepsilon)\right) \leq \alpha,$ $\Pr\left(P_{nU} > \frac{p}{2}(1+\varepsilon)\right) \leq \alpha$	$\overline{X} \pm S\left\{c(\widehat{\gamma})u_{p(1+\varepsilon)/2}^{1+\widehat{\gamma}} + \dfrac{A(\widehat{\gamma}, u_{p/2})u_\alpha}{\sqrt{n}}\right\}$
$\Pr\left(\frac{1}{P_{nL}} < \frac{1}{p/2}(1-\varepsilon)\right) \leq \alpha,$ $\Pr\left(\frac{1}{P_{nU}} < \frac{1}{p/2}(1-\varepsilon)\right) \leq \alpha$	$\overline{X} \pm S\left\{c(\widehat{\gamma})u_{p/\{2(1-\varepsilon)\}}^{1+\widehat{\gamma}} + \dfrac{A(\widehat{\gamma}, u_{p/2})u_\alpha}{\sqrt{n}}\right\}$

Table 2: Two-sided (corrected) parametric control limits based on the normal power family

3.3. Minimum Control Charts

The two-sided control limits for the (corrected) minimum control chart are given in Table 3. They are denoted by $\widehat{LL}_{MIN}$ and $\widehat{UL}_{MIN}$, where the subscript MIN refers to "minimum". An OoC-signal is given when for a new incoming group of Phase II observation $X_{n+1}, ..., X_{n+m}$ either $\min(X_{n+1}, ..., X_{n+m}) > \widehat{UL}_{MIN}$ or $\max(X_{n+1}, ..., X_{n+m}) < \widehat{LL}_{MIN}$. Before presenting the minimum control limits we introduce the following notation:

$$r = ent\left(n\{m(p/2)\}^{1/m}\right),$$

k is determined by $\begin{pmatrix} r-k-1+m \\ m \end{pmatrix} \leq m(p/2)\begin{pmatrix} n+m \\ m \end{pmatrix}$
$$< \begin{pmatrix} r-k+m \\ m \end{pmatrix},$$

$$\lambda = \frac{m(p/2)\begin{pmatrix} n+m \\ m \end{pmatrix} - \begin{pmatrix} r-k-1+m \\ m \end{pmatrix}}{\begin{pmatrix} r-k+m \\ m \end{pmatrix} - \begin{pmatrix} r-k-1+m \\ m \end{pmatrix}},$$

$B(n, p^*, j) = P(Z \leq j), b(n, p^*, j) = P(Z = j)$ with Z binomial(n, p^*),

$q(p^*) = (mp^*)^{1/m}$,

$k(p^*, \alpha)$ is determined by $B(n, q(p^*), r - k(p^*, \alpha) - 1) \leq \alpha$
$$< B(n, q(p^*), r - k(p^*, \alpha)),$$

$$\lambda(p^*, \alpha) = \frac{\alpha - B(n, q(p^*), r - k(p^*, \alpha) - 1)}{b(n, q(p^*), r - k(p^*, \alpha))},$$

$\widetilde{k} = k((1+\varepsilon)p/2, \alpha), \widetilde{\lambda} = \lambda((1+\varepsilon)p/2, \alpha),$

Aim	$\widehat{LL}_{MIN}, \widehat{UL}_{MIN}$
$E\frac{P_n}{m} = p$	$\widehat{LL}_{MIN} : (1-\lambda)X_{(r-k)} + \lambda X_{(r+1-k)}$ $\widehat{UL}_{MIN} : (1-\lambda)X_{(n+k+1-r)} + \lambda X_{(n+k-r)}$
$\Pr\left(\frac{P_{nL}}{m} > \frac{p}{2}(1+\varepsilon)\right) \le \alpha,$ $\Pr\left(\frac{P_{nU}}{m} > \frac{p}{2}(1+\varepsilon)\right) \le \alpha$	$\widehat{LL}_{MIN} : (1-\widetilde{\lambda})X_{(r-\widetilde{k})} + \widetilde{\lambda} X_{(r+1-\widetilde{k})}$ $\widehat{UL}_{MIN} : (1-\widetilde{\lambda})X_{(n+\widetilde{k}+1-r)} + \widetilde{\lambda} X_{(n+\widetilde{k}-r)}$
$\Pr\left(\frac{m}{P_{nL}} < \frac{1}{p/2}(1-\varepsilon)\right) \le \alpha,$ $\Pr\left(\frac{m}{P_{nU}} < \frac{1}{p/2}(1-\varepsilon)\right) \le \alpha$	$\widehat{LL}_{MIN} : (1-\overline{\lambda})X_{(r-\overline{k})} + \overline{\lambda} X_{(r+1-\overline{k})}$ $\widehat{UL}_{MIN} : (1-\overline{\lambda})X_{(n+\overline{k}+1-r)} + \overline{\lambda} X_{(n+\overline{k}-r)}$

Table 3: Two-sided (corrected) minimum control limits

$$\overline{k} = k\left(\frac{p}{2(1-\varepsilon)}, \alpha\right), \overline{\lambda} = \lambda\left(\frac{p}{2(1-\varepsilon)}, \alpha\right)$$

3.4. Selection Rule

The idea behind the selection rule is to stay as long as possible in the classical normal chart, to move to the parametric chart if the tails are too heavy or too light and to take the nonparametric MIN chart when the parametric family presumably fails too. The data are telling us which chart to use. Since control limits are determined by the behavior in the far tail (p is small), the natural yardstick to decide between the charts is the standardized maximum of the Phase I observations $(X_{(n)} - \overline{X})/S$ for the upper limit and similarly the standardized minimum $\left(\overline{X} - X_{(1)}\right)/S$ for the lower limit.

In view of a possible modest outlier it may seem risky to use the (standardized) largest and/or smallest observation in the selection rule. Quite the contrary is the case. When an outlier is present, we will take the more safe nonparametric minimum control chart, where the most extreme observations are not involved in determining the control limits. Note that extreme outliers may have a "masking effect", but such kind of outliers are not expected to be present, because the observations $X_1, ..., X_n$ are authorized as being IC and hence nonconforming extreme outliers should have been detected before.

The next point is the determination of the cut-off points. Distributions with heavier tails may give serious problems with the IC behavior leading to invalid control charts. Distributions with thinner tails are conservative in the IC case with as consequence a loss in the OoC. Because IC-errors are more serious than those in the OoC case and moreover, since a positive model error as large as p or larger can easily occur, whereas the negative model error is at most $-p$, we take the selection rules *unbalanced*. In particular, for the upper control limit

we will prefer the normal control chart when

$$u_{(-0.7+0.5\log n)/n} \le \frac{X_{(n)} - \overline{X}}{S} \le u_{5/(n\sqrt{n})}.$$

To see what this implies, let $X_1, \ldots, X_n$ be i.i.d. random variables with a standard normal distribution. Then

$$P\left(X_{(n)} < u_{(-0.7+0.5\log n)/n}\right) = \left(1 - \frac{-0.7 + 0.5\log(n)}{n}\right)^n$$

$$\approx \exp\left(0.7 - 0.5\log n\right) \approx \frac{2}{\sqrt{n}},$$

and

$$P\left(X_{(n)} > u_{5/(n\sqrt{n})}\right) = 1 - \left(1 - \frac{5}{n\sqrt{n}}\right)^n \approx 1 - \exp\left(-\frac{5}{\sqrt{n}}\right) \approx \frac{5}{\sqrt{n}}.$$
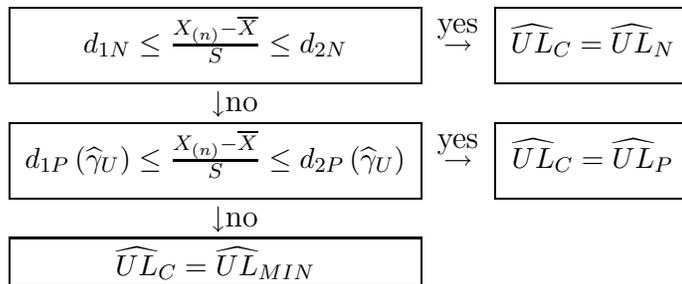
Hence, when normality holds we stay with high probability at the normal chart. When the standardized maximum $(X_{(n)} - \overline{X})/S$ is very large, this indicates that the tail may be heavier than the normal one and we change to the parametric or nonparametric chart. Similarly we change for relatively small values of $(X_{(n)} - \overline{X})/S$, indicating a thinner tail behavior. That the latter is less severe, is expressed in the probability $2/\sqrt{n}$, compared to $5/\sqrt{n}$.

The total two-sided improved data driven control chart is given by the following scheme. Let

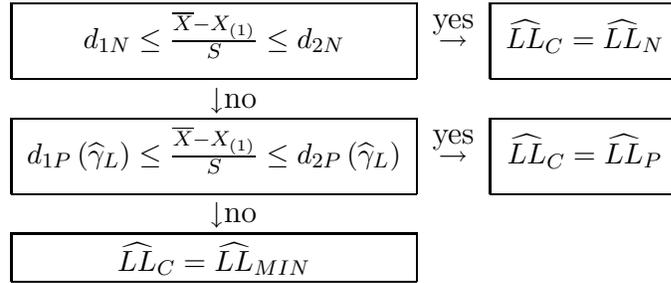$$d_{1N} = u_{(-0.7+0.5\log n)/n}, d_{2N} = u_{5/(n\sqrt{n})},$$

$$d_{1P}\left(\widehat{\gamma}\right) = c\left(\widehat{\gamma}\right) u_{(-0.2+0.5\log n)/n}^{1+\widehat{\gamma}}, d_{2P}\left(\widehat{\gamma}\right) = c\left(\widehat{\gamma}\right) u_{3/(n\sqrt{n})}^{1+\widehat{\gamma}}$$

and denote the lower and upper limit by $\widehat{LL}_C$ and $\widehat{UL}_C$, respectively. The upper limit is obtained as follows

$$\boxed{d_{1N} \le \frac{X_{(n)} - \overline{X}}{S} \le d_{2N}} \quad \overset{\text{yes}}{\rightarrow} \quad \boxed{\widehat{UL}_C = \widehat{UL}_N}$$

$$\downarrow \text{no}$$

$$\boxed{d_{1P}\left(\widehat{\gamma}_U\right) \le \frac{X_{(n)} - \overline{X}}{S} \le d_{2P}\left(\widehat{\gamma}_U\right)} \quad \overset{\text{yes}}{\rightarrow} \quad \boxed{\widehat{UL}_C = \widehat{UL}_P}$$

$$\downarrow \text{no}$$

$$\boxed{\widehat{UL}_C = \widehat{UL}_{MIN}}$$

Here $\widehat{UL}_N, \widehat{UL}_P$ and $\widehat{UL}_{MIN}$ are taken from Tables $1 - 3$ according to the required aim: bias, exceedance with FAR or exceedance with ARL or ATS.

For the lower limit $\widehat{LL}_C$ we get

$$\boxed{d_{1N} \le \dfrac{\overline{X} - X_{(1)}}{S} \le d_{2N}} \quad \overset{\text{yes}}{\to} \quad \boxed{\widehat{LL}_C = \widehat{LL}_N}$$

$$\downarrow \text{no}$$

$$\boxed{d_{1P}\left(\widehat{\gamma}_L\right) \le \dfrac{\overline{X} - X_{(1)}}{S} \le d_{2P}\left(\widehat{\gamma}_L\right)} \quad \overset{\text{yes}}{\to} \quad \boxed{\widehat{LL}_C = \widehat{LL}_P}$$

$$\downarrow \text{no}$$

$$\boxed{\widehat{LL}_C = \widehat{LL}_{MIN}}$$

Here $\widehat{LL}_N, \widehat{LL}_P$ and $\widehat{LL}_{MIN}$ are taken from Tables $1-3$ according to the required aim: bias, exceedance with FAR or exceedance with ARL or ATS.

4. Application

We apply the improved data driven control chart on a real life example concerning the production of electric shavers by Philips. In an electrochemical process razor heads are formed. The measurements concern the thickness of these razor heads on a particular spot on the head. Available are two samples of 835 measurements each. In Albers et al [5] this data set has been utilized to illustrate the data driven approach of that paper with the whole first sample of 835 observations as Phase I observations. Instead of using the whole first sample we now take only 100 observations of the first sample, that is we have $n = 100$ here. The sample mean of this Phase I sample equals 42.79 and the sample standard deviation is 3.07, while the smallest and largest observations are 31.03 and 49.95, respectively. Hence , we obtain $(x_{(n)} - \overline{x})/s = 2.33, (\overline{x} - x_{(1)})/s = 3.83$. Direct calculation gives $d_{1N} = 2.14$ and $d_{2N} = 2.58$, implying that the upper limit is given by the normal chart. Application of Table 1 with $p/2 = 0.001, \alpha = 0.2$ and $\varepsilon = 0.2$ gives as control limits

$$\text{bias} \quad \widehat{UL}_C = 52.580 \,,$$

$$\text{exceedance } FAR \quad \widehat{UL}_C = 52.705 \,,$$

$$\text{exceedance } ATS \quad \widehat{UL}_C = 52.655.$$

We apply the control charts on the remaining 1570 observations. Figure 1 shows the result, where the horizontal lines give the various upper control limits.

Next we consider the lower part of the improved data driven control chart. Since $\left(\overline{x} - x_{(1)}\right)/s = 3.83 > d_{2N} = 2.58$, the data tell us that the lower tail is too heavy to use the normal chart. Direct calculation gives $\widehat{\gamma}_L = 0.558$ and $d_{2P}\left(\widehat{\gamma}_L\right) = 3.699$. Because $\left(\overline{x} - x_{(1)}\right)/s = 3.83 > d_{2P}\left(\widehat{\gamma}_L\right) = 3.699$, the
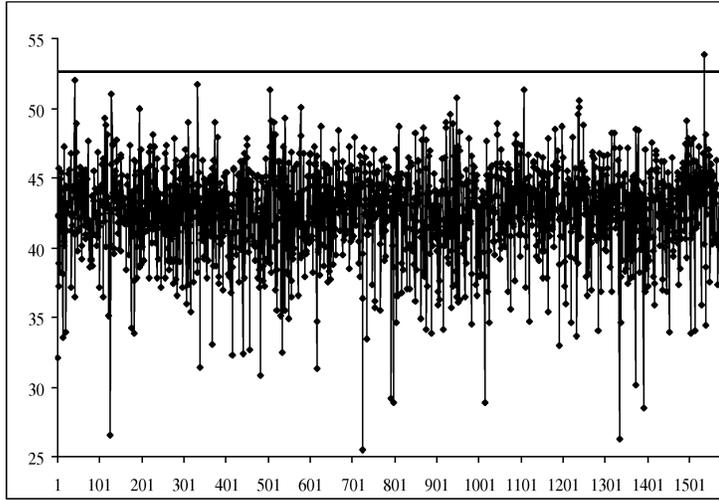
Figure 1: Improved data driven control chart (upper part) of the remaining 1570 measurements of the thickness of razor heads with $\widehat{UL}_C = 52.580$ (bias corrected), $\widehat{UL}_C = 52.705$ (exceedance FAR corrected) and $\widehat{UL}_C = 52.655$ (exceedance ATS corrected)

parametric chart cannot be used either and hence the nonparametric MIN chart should be used at the lower part. We take $m = 3$. As we have $n = 100$ and $p/2 = 0.001$, we therefore arrive at $r = 14$. This already shows that even for a value like $m = 3$ no longer very extreme order statistics come in as with the individual chart, but ordinary ones. Indeed, the $p/2 = 0.001$ upper quantile is replaced by the $q = (mp/2)^{1/m} = 0.144$ upper quantile. While the 0.001 upper quantile cannot be estimated nonparametricly in a sensible way with 100 observations, the 0.144 upper quantile can be estimated very well. This is the improvement we were looking for.

Since $\alpha = 0.2$ and $\varepsilon = 0.2$, we get $k = 1, \lambda = 0.72$ for the bias correction, $\widetilde{k} = 2, \widetilde{\lambda} = 0.74$ for the exceedance case when correcting FAR and $\overline{k} = 2, \overline{\lambda} = 0.95$ for the exceedance correction w.r.t. ATS. In passing it is seen that indeed protection against exceedance effects requires a larger correction than against bias. Noting that $x_{(12)} = 39.09, x_{(13)} = 39.76$ and $x_{(14)} = 39.82$, it easily follows that the lower limits are given by

$$\text{bias} \quad \widehat{LL}_C = 39.803\,,$$
$$\text{exceedance } FAR \quad \widehat{LL}_C = 39.586\,,$$
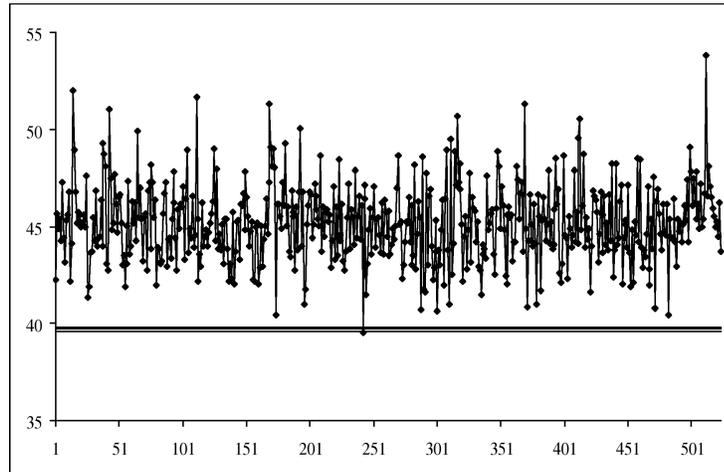$$\text{exceedance } ATS \quad \widehat{LL}_C = 39.727.$$

Figure 2: Improved data driven control chart (lower part) of the 523 triples contained in the remaining 1570 measurements of the thickness of razor heads with $\widehat{LL}_C = 39.803$ (bias corrected), $\widehat{LL}_C = 39.587$ (exceedance FAR corrected) and $\widehat{LL}_C = 39.727$ (exceedance ATS corrected)

We apply the control charts on the 523 triples contained in the remaining 1570 observations. Figure 2 shows the result, where the horizontal lines give the several upper control limits.

5. Conclusion

The "open end" in the data driven control chart, when the number of Phase I observations is not huge and a nonparametric approach is chosen by the data, is filled in with the improved data driven control chart. The application in Section 4 clearly shows the improvement. When we would have used the truly nonparametric control chart in the lower limit, this limit would be $-\infty$ with probability 0.9 and $x_{(1)} = 31.03$ with probability 0.1 (in the bias case). Obviously, such a rule can never be recommended in practice: with very high probability there will be never a signal at all! The modified nonparametric control chart gives some kind of remedy, but is not a truly nonparametric chart. In the application of Section 4 it gives as lower limit: $x_{(1)} - s = 27.96$ with probability 0.9 and $x_{(1)} = 31.03$ with probability 0.1. The limit 27.96 turns out to cause 3 times a OoC-signal, while 31.03 results in 9 signals.

In Albers et al [5] the theoretical behavior of the data driven control chart is studied in detail. It has been shown there that the behavior of the data driven control chart is asymptotically equivalent to the behavior of each specific control chart in its own domain. Since these results are a consequence of the selection rule and the selection rule in the improved data driven control chart is the same, these theoretical results apply for the improved data driven control chart as well. Hence, in contrast to for instance the classical normal chart, the improved data driven control chart is valid for all distributions. In each of the three situations (normality, normal power family, outside the normal power family) its out-of-control behavior is asymptotically as good as that of the specific corresponding control chart. For more details of the proof and precise theoretical statements we refer to Albers et al [5].

References

[1] W. Albers, W.C.M. Kallenberg, Empirical nonparametric control charts: estimation effects and corrections, *Journal of Applied Statistics*, **31** (2004), 345-360.

[2] W. Albers, W.C.M. Kallenberg, Self adapting control charts, *Statistica Neerlandica*, **60** (2006), 292-308.

[3] W. Albers, W.C.M. Kallenberg, Control charts in new perspective, *Sequential Analysis*, (2007), To Appear.

[4] W. Albers, W.C.M. Kallenberg, Minimum control charts, *Journal of Statistical Planning and Inference*, (2007), To Appear.

[5] W. Albers, W.C.M. Kallenberg, S. Nurdiati, Parametric control charts, *Journal of Statistical Planning and Inference*, **124** (2004), 159-184.

[6] W. Albers, W.C.M. Kallenberg, S. Nurdiati, Exceedance probabilities for parametric control charts, *Statistics*, **39** (2005), 429-443.

[7] S.T. Bakir, A distribution-free Shewhart quality control chart based on signed-ranks, *Quality Engineering*, **16** (2004), 611-621.

[8] S.T. Bakir, Distribution-free quality control charts based on signed-rank-like statistics, *Communications in Statistics: Theory and Methods*, **35** (2006), 743-757.

[9] S.T. Bakir, M. R. Reynolds, Jr., A nonparametric procedure for process control based on within-group ranking, *Technometrics*, **21** (1979), 175-183.

[10] S. Chakraborti, S. Eryilmaz, A nonparametric Shewhart-type signed-rank control chart based on runs, *Communications in Statistics: Simulation and Computation* (2007), To Appear.

[11] S. Chakraborti, P. van der Laan, S.T. Bakir, Nonparametric control charts: an overview and some results, *Journal of Quality Technology*, **33** (2001), 304-315.

[12] S. Chakraborti, P. van der Laan, M.A. van de Wiel, A class of distribution-free control charts, *Journal of the Royal Statistical Society. Series C.*, **53** (2004), 443-462.

[13] L.K. Chan, K.P. Hapuarachchi, B.D. Macpherson, Robustness of $\overline{X}$ and R charts, *IEEE Transactions on Reliability*, **37** (1988), 117-123.

[14] L.A. Jones, C.W. Champ, S.E. Rigdon, The run length distribution of the CUSUM with estimated parameters, *Journal of Quality Technology*, **36** (2004), 95-108.

[15] E.A. Pappanastos, B.M. Adams, Alternative designs of the Hodges-Lehmann control chart, *Journal of Quality Technology*, **28** (1996), 213-223.

[16] C.P. Quesenberry, The effect of sample size on estimated limits for $\overline{X}$ and X control charts, *Journal of Quality Technology*, **25** (1993), 237-247.

[17] P. Qiu, D. Hawkins, A rank based multivariate CUSUM procedure, *Technometrics*, **43** (2001), 120-132.

[18] P. Qiu, D. Hawkins, A nonparametric multivariate cumulative sum procedure for detecting shifts in all directions, *Journal of the Royal Statistical Society. Series D*, **52** (2003), 151-164.

[19] C.B. Roes, *Shewhart-type Charts in Statistical Process Control*, Ph.D. Thesis, University of Amsterdam (1995).

[20] S.A. Yourstone, W.J. Zimmer, Non-normality and the design of control charts for averages, *Decision Sciences*, **23** (1992), 1099-1113.