Approximation Schemes for Wireless Networks

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Abstract. Wireless networks are created by the communication links between a collection of radio transceivers. The nature of wireless transmissions does not lead to arbitrary undirected graphs but to structured graphs which we characterize by the polynomially bounded growth property. In contrast to many existing graph models for wireless networks, the property of polynomially bounded growth is defined independently of geometric data such as positional information.

On such wireless networks, we present an approach that can be used to create polynomial-time approximation schemes for several optimization problems called the local neighborhood-based scheme. We apply this approach to the problems of seeking maximum (weight) independent sets and minimum dominating sets. These are two important problems in the area of wireless communication networks and are also used in many applications ranging from clustering to routing strategies. However, the approach is presented in a general fashion since it can be applied to other problems as well.

The approach for the approximation schemes is robust in the sense that it accepts any undirected graph as input and either outputs a solution of desired quality or correctly asserts that the graph presented as input does not satisfy the structural assumption of a wireless network (an NP-hard problem).

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1. *Introduction*

This article looks at the construction of some basic structures needed for efficient communication and organization strategies in large-scale wireless networks, namely, independent and dominating sets.

Wireless ad-hoc networks are advancing rapidly into our everyday life. The devices themselves get smaller and more embedded to the point where they are no longer visible to the eye; they receive into the background of our lives and perform tasks unattended and without much interaction from users. Numerous small battery-operated devices sense and interact with the environment and form a collaborative network by means of wireless multihop communication. Such a network realizes the vision of ubiquitous computing by creating a smart environment.

These ad-hoc and sensor networks, as well as other wireless networks, are modeled by communication graphs which give the communication links between the devices that are equipped with wireless transceivers. The characteristics of packet transmissions over a wireless medium create a network with certain structural properties. In this article, we look at graph models proposed for wireless communication networks in various degrees of granularity with respect to reality. While most graph models in the literature follow a geometric intuition, the unifying structure given in all these models is the polynomially bounded growth property, which we define and then use to characterize wireless networks. Generally speaking, bounded growth is defined and used independently of any metric.

In the context of efficient wireless networking, certain subgraphs play a prominent role. In this article, we consider independent and dominating sets. A subset of nodes is called *independent* if no two nodes from this set are connected, that is, no two nodes can communicate with one another directly. A subset is called *dominating* if all nodes of the network are in reach of at least one node from this subset. In other words, the notion of domination in a graph represents the fact that a broadcast from a communication device is received by all its neighbors. At the lowest level that is concerned by the actual internode communication, nodes in an independent set do not interfere with each other during simultaneous transmissions, and nodes in a dominating set of small cardinality can, for example, be used to efficiently reach the entire network by broadcasts from only these nodes.

We are interested in the problem of finding independent sets of maximum cardinality (and weight) and dominating sets of minimum size. A general solution approach for these problems is introduced which we call a *local neighborhood-based scheme*. The bounded growth property used to characterize wireless networks allows for strong results on the performance of the presented algorithms. Particularly, we show that the local neighborhood-based scheme applied to the maximum (weight) independent set and minimum dominating set problems results in polynomial-time approximation schemes (PTAS) on this graph class. The approximation schemes presented here have a runtime of $n^{O(1/\varepsilon \log 1/\varepsilon)}$.

Existing approaches for these and other problems in wireless networks usually require additional information like the exact positions of the devices in the Euclidean plane (see, e.g. Li [2003]). However, in many cases, such geometric information cannot be computed easily. Furthermore, assuming that the radio characteristics are equal diameter disks is quite idealistic. In reality, radio transceivers do not have omnidirectional antennas, and even small obstacles change the communication characteristics. Therefore, approaches that do not explicitly exploit geometric information (e.g., positions of the nodes) and that do not idealize the network structure (e.g. unit disk transmission ranges) are preferred. Our approach does not require geometric information nor does it idealize the wireless network structure.

Additionally, our approach is robust in the sense that the resulting algorithms accept any undirected graph as valid input, and then return a solution that meets the required bound on the approximation ratio or correctly assert that the input graph does not reflect a wireless communication structure. Especially when looking at applications in networks that are driven by unreliable communication links, robustness is an important property since it ensures meaningful output in any case.

The remainder of the article is organized as follows. Next we give some definitions needed later on and present some related work. In Section 2, we discuss graph classes used to model wireless communication networks and establish their common structure called polynomially bounded growth. Section 3 introduces the technique used to obtain approximation schemes in these graphs, and, in Section 4, we show how to modify the approach towards robust schemes. The article ends with a short conclusion.

1.1. PRELIMINARIES. Generally speaking, communication networks are modeled as undirected graphs $G = (V, E)$. The vertices V represent the communication devices or nodes, and two nodes are connected by an edge in E if they can communicate directly with one another.

Additionally, the vertices of the communication network may be weighted, that is, every vertex $v \in V$ has an assigned weight w_v, and we assume these weights to be positive values. In the context of wireless ad-hoc networks; these weights usually reflect residual energy or capabilities of a node for a specific task.

Two vertices of a graph are called independent if they are not adjacent to one another. A subset $I \subseteq V$ is called independent if all vertices are not connected. In other words, I is an independent set if the subgraph Gi induced by I contains no edges. When seeking such independent sets of maximum cardinality (or overall weight), we obtain the *maximum (weight) independent set* (MIS) problem.

A subset $D \subseteq V$ is called dominating if every vertex from V is contained in this subset or adjacent to a vertex from D. The resulting *minimum dominating set* (MDS) problem then asks for such a dominating set of minimum cardinality.

Note that a subset can be both independent and dominating. Such a set is then called a maximal independent set. Formally, an independent set is called *maximal* if it cannot be extended by the addition of any other vertex from the graph without violating the independence property. It is easy to verify that a maximal independent set is also dominating.

A *polynomial-time approximation scheme* (PTAS) is an algorithm which, in addition to an input instance, requires a parameter $\varepsilon > 0$, which then returns a solution with a relative error of at most $1 + \varepsilon$ with respect to an optimal solution. The running time of such algorithms is allowed to depend on ε but should be polynomial in $n := |V|$ for fixed $\varepsilon > 0$. For example, a PTAS for the MDS problem returns a dominating set of cardinality at most $(1+\varepsilon)$ times the cardinality of a minimum cardinality dominating set.

Let $V' \subseteq V$ be a subset of vertices in G. In the following, we use $G[V']$ to denote the the subgraph induced by V'. In case of a weighted graph, we define the weight of a subset V' by $W(V') := \sum_{v \in V'} w_v$.

Furthermore, we denote by $\Gamma(v)$ the closed neighborhood of a vertex $v \in V$, that is, $\Gamma(v) := \{u \in V \mid (u, v) \in E\} \cup \{v\}$. Analogously, for $V' \subseteq V$, let $\Gamma(V') := \bigcup_{v \in V'} \Gamma(v)$. For $r \in \mathbb{N}$, we call $\Gamma_r(v) := \Gamma(\Gamma_{r-1}(v))$ the recursively defined r-th neighborhood of $v \in V$, with $\Gamma_0(v) := \{v\}$.

1.2. RELATED WORK ON GEOMETRIC INTERSECTION GRAPHS. Most work on optimization algorithms for wireless networks has been done using geometric intersection graphs as underlying models for the communication network [Nieberg and Hurink 2004]. In this context, unit disk graphs are probably the most prominent class of graphs used [Li 2003]. These graphs are defined by looking at the intersections of equal diameter disks in the plane. Several variants and modifications to this basic model exist (see Section 2). The MIS and MDS problems considered here remain *NP*-hard on unit disk graphs [Clark et al. 1990].

An important detail when using disk graphs is the encoding of the input instance. Basically, there are two ways of describing a geometric intersection graph, by its adjacency and by its geometric information. While the first presents the graph as an undirected graph, the latter conveys more information that can be exploited explicitly by respective algorithms. Note that this is a significant distinction because determining for a given graph whether it is a disk graph is an *NP*-complete problem [Breu and Kirkpatrick 1998; Kratochvil 1997], and therefore computing a representation that gives feasible positions to each node in a disk graph (of which we have adjacency information only) is an intractable problem.

In case geometric information is available, we can use geometric separation and a shifting strategy to obtain a PTAS for many problems on (unit) disk graphs. This strategy gives, for example, a PTAS for the MIS, MDS, and vertex cover problems on UDGs [Chan 2003; III et al. 1998] and the minimum connected dominating set problem on UDGs [Cheng et al. 2003]. Combined with a dynamic programming approach, the shifting strategy also gives a PTAS for the MIS problem on disk graphs with arbitrary radii [Erlebach et al. 2005]. Also using separation alongside the geometric positions of the vertices, a constant-factor approximation algorithm for the minimum weight dominating set problem is presented in Ambuehl et al. [2006].

Without geometric information, a robust PTAS for the MIS and MDS problems on unit disk graphs are presented in Nieberg et al. [2004] and Nieberg and Hurink [2005]. These schemes are the basis for the approach presented here.

2. *Wireless Graph Models*

A wireless network is created by the communication links between a collection of radio transceivers. The nature of wireless transmissions does not lead to an arbitrary undirected graph but to a structured graph. In this section, we introduce the class of graphs with polynomially bounded growth and show that many other classes of graphs used to model wireless communication networks satisfy the bounded growth property. While most existing wireless graph models rely on geometric information of the nodes and their transmission ranges, bounded growth is defined independently of any geometric information as follows.

Definition 2.1. Let $G = (V, E)$ be a graph. If there exists a function $f(.)$ such that every r-neighborhood in G contains at most $f(r)$ independent vertices, then

G is *f-growth-bounded*. Furthermore, we say that G has *polynomially* bounded growth if for some constant $k \geq 1$, $f(r) = O(r^k)$ holds.

Graphs of bounded growth are sometimes also called graphs of bounded (local) independence. Note that the growth function $f(.)$ only depends on the radius of the neighborhood and not on the number of vertices in G. Thus, for constant r, the number of independent vertices in $\Gamma_r(v)$ is bounded by a constant for any $v \in V$. It is straightforward to verify that the bounded growth property of a graph is closed under taking vertex-induced subgraphs.

2.1. GEOMETRIC INTERSECTION GRAPH MODELS. A wireless network is created by nodes that are equipped with a radio transceiver and that are placed in the real world. The environment, especially the positions of the nodes, has to be accounted for, and it is no surprise that most wireless graph models are therefore defined for the Euclidean space. We present several geometrically inspired graph models in various degrees of granularity with respect to reality and establish the polynomially bounded growth property for these. In the following, we use $\|.\|$ to denote the Euclidean distance in $\mathbb{R}^2$.

Usually, the models result from geometric intersection or containment graphs which give the general idea behind these models. Next, we introduce these graph models in general, and then specify additional characteristics in order to justify them for the purpose of modeling wireless communication networks.

We assume that the vertices of the graph, that is, in our case, the wireless nodes, are placed in the 2-dimensional Euclidean plane. In other words, there exists a mapping $p : V \to \mathbb{R}^2$ which gives each vertex $v \in V$ its location $p_v \in \mathbb{R}^2$. Furthermore, each wireless node has a certain area which is covered by its radio. For every $v \in V$, let this area be represented by $A_v \subset \mathbb{R}^2$. As a consequence, another vertex $u \in V$ can receive a transmission, and thus a message from v, if and only if $p_u \in A_v$ holds.

To present the possible communication or interference between wireless nodes, two different graph models are considered. The first one is the *containment model*, where the set of edges is characterized by

$$(u, v) \in E \iff p_u \in A_v.$$

This model gives the possible direct communication between nodes and results in a directed graph model. If we only look at the coverage areas of the models, we can also define the *intersection model* as follows:

$$(u, v) \in E \iff A_v \cap A_u \neq \varnothing.$$

With this symmetric model, interference during simultaneous transmissions can be explored. If two nodes transmit at the same time, a node in the nonempty intersection receives both transmissions simultaneously and may thus not be able to reconstruct the messages.

The resulting graphs are called intersection and containment *coverage area graphs*. If we consider each undirected edge to be a two-way edge between the respective vertices, it is easy to see that the containment graph is completely contained in the intersection graph for the same set of vertices and coverage areas. In the following, we refer to a set of positions and corresponding coverage areas as *geometric representation* of an intersection or containment graph.

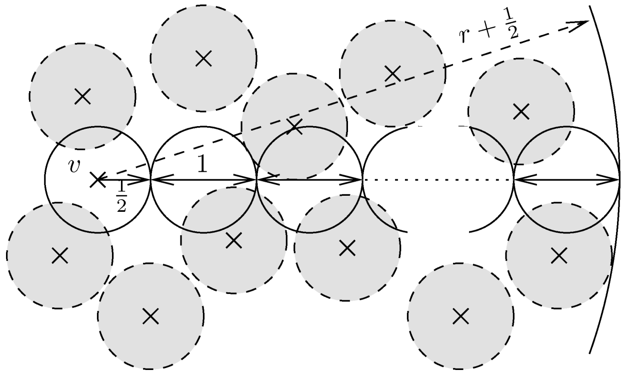

FIG. 1. Proof of Lemma 2.3.

Throughout the following, we do not differentiate strictly between the containment and intersection graph models and consider all edges to be undirected. We now consider specially structured coverage area graphs.

(Unit) Disk Graphs. In practice and real-world settings, the coverage areas follow the laws of physics, especially the laws of radio wave propagation with respect to the environment in which the network operates. The most basic model used for wireless communication is a unit disk graph. Suppose that all wireless nodes are equal and are placed in an ideal environment, that is, all nodes send with the same transmission radius and have the same circular coverage area. By proper scaling, we may assume the diameter of the disks to be of unit length, and then define a unit disk graph as follows.

Definition 2.2. A graph $G = (V, E)$ is a *unit disk graph* if there exists a map $p : V \to \mathbb{R}^2$ satisfying

$$(u, v) \in E \iff \|p_u - p_v\| \leq 1.$$

In other words, a UDG is the intersection graph of unit diameter disks in the Euclidean plane. Note that Definition 2.2 actually characterizes both intersection and containment graphs, and all edges are bidirectional. Unit disk graphs have polynomially bounded growth, which follows from a simple geometric packing argument given in Figure 1.

LEMMA 2.3. *Let $G = (V, E)$ be a unit disk graph. Then, G is of $(2r + 1)^2$ bounded growth.*

PROOF. From the Definition 2.2 of a UDG, we conclude that any $w \in \Gamma_r(v)$, $v \in V$ satisfies

$$\|p_w - p_v\| \leq r.$$

Let $I \subset \Gamma_r(v)$ denote an independent set in the r-neighborhood of v. The unit disks corresponding to vertices in I are pairwise disjoint and are all contained in a disk of larger radius R with $R = (r + 1/2)$ around $f(v)$. This implies

$$|I| \leq \frac{\pi R^2}{\pi (1/2)^2} = (2r + 1)^2$$

as claimed. $\square$

While UDGs as models for wireless networks are quite idealistic, they are the basis for more realistic models. If the wireless nodes are able to adjust their transmission power, still in ideal settings, then different circular coverage areas emerge, and we obtain a disk graph.

If the ratio of the diameters of the largest and smallest disks in such a disk graph are bounded by a constant, we again obtain a graph of polynomially bounded growth. This can easily be seen by adapting the geometric packing argument used in the proof of Lemma 2.3.

Quasi-Disk Graphs. While unit disk graphs are widely used to obtain strong theoretic results for graph algorithms, one might argue that they are not very realistic since ideal assumptions are made for the radio propagation.

Refining the idea behind disk graphs by no longer limiting the reasons for different radii to the transmission power, but also to environmental reasons like objects, we can define a Quasi-Disk Graph (QDG, [Kuhn et al. 2003]) as follows.

Definition 2.4. A graph $G = (V, E)$ is a *quasi-disk graph* if there exist two values $0 < c^- \leq c^+$ and a map $p : V \to \mathbb{R}^2$, satisfying $(u, v) \in E$, if $\|p_u - p_v\| \leq c^-$ and $(u, v) \notin E$ if $\|p_u - p_v\| > c^+$.

In such a quasi-disk graph, there are two disks D^+ and D^- of radius c^+ and c^-, respectively, that can be placed around each vertex position such that $D^- \subseteq A_v \subseteq D^+$ holds for the coverage area A_v of that vertex. A more intuitive characterization of a quasi-disk graph based on transmissions is as follows.

—Two vertices $u, v \in V$, which are sufficiently close to each other, that is, $\|p_u - p_v\| \leq c^-$ holds, always receive each other's messages.
—Two vertices $\bar{u}, \bar{v} \in V$ that are too far apart, that is, $\|p_{\bar{u}} - p_{\bar{v}}\| > c^+$, cannot communicate directly.

If $c^- < \|p_u - p_v\| \leq c^+$ holds for two vertices $u, v \in V$, the existence of an edge is not explicitly defined but depends on the concrete shapes of A_v and A_u. Furthermore, effects like fading and the resulting unreliable transmission characteristics can be incorporated into this model.

By slightly adjusting the geometric packing argument in the proof of Lemma 2.3, it is clear that a QDG also has polynomially bounded growth when the ratio of the diameters is bounded.

In practice, it may not be possible to give a radius on the transmission range where coverage can be guaranteed for all wireless nodes, for example, when these are mounted on concrete walls. Leaving the idea of circles that reflect the coverage area, we only consider the area itself. We assume that the size (or volume) of the coverage area of each vertex is bounded and that it does not stretch too far from each vertex position. Again, the geometric argument can be adapted to show polynomially bounded growth as it does not depend on disk shapes.

2.2. Graphs Based on Metric Spaces. While wireless networks operate in the Euclidean space, we can extend the previous graph models to intersection graphs induced by other metrics. Analogously to unit disk graphs, we immediately obtain the following characterization of unit ball graphs.

Definition 2.5. Let $M = (X, d)$ be a metric space with a distance function $d : X^2 \to \mathbb{R}$. A graph $G = (V, E)$ is a *unit ball graph* (UBG) if there exists a mapping $p : V \to X$ such that

$$(u, v) \in E \iff d(p_u, p_v) \leq 1$$

holds. The pair (M, p) is called a representation of G.

However, in this context, note that any undirected graph is such a unit ball graph by taking the shortest-path distance as the metric on $V = X$. So a UBG is not necessarily growth bounded.

Nevertheless, further restricting the metric space, we can identify a large class of metric unit ball graphs with polynomially bounded growth. Such a restriction uses a bound on the growth of the metric space M, which is defined as follows [Assouad 1983; Gupta et al. 2003].

Definition 2.6. Let $M = (X, d)$ be a metric space. The *doubling dimension* ρ of M is the smallest ρ such that every ball of radius r can be completely covered by at most 2^ρ balls of radius $r/2$. If ρ is bounded by a constant, we say that M is *doubling*.

Analogously, we refer to a unit ball graph as doubling if there exists a representation where the underlying metric space is doubling. The following lemma now shows that doubling UBGs are growth-bounded.

LEMMA 2.7. *Let $G = (V, E)$ be a unit ball graph with a representation (M, p) such that the metric space $M = (X, d)$ has doubling dimension ρ. Then, G has f-bounded growth with $f(r) = O(r^\rho)$.*

PROOF. For a vertex $v \in V$ and a radius $r \geq 0$, consider the neighborhood $\Gamma_r(v)$, and let $I \subset \Gamma_r(v)$ denote an independent set therein.

Both $\Gamma_r(v)$ and I are contained in a ball $\{v' \in V \mid d(p_v, p_{v'}) \leq r\}$ of radius r around p_v. Also, for every $u \in I$, the ball with radius $1/2$ around p_u, given by $\{v' \in V \mid d(p_u, p_{v'}) \leq 1/2\}$, does not contain another vertex from I. In other words, for all $u \in I$, these balls are mutually disjoint.

The number of balls of radius $1/2$ needed to cover the ball of radius r around p_v and thus $\Gamma_r(v)$ is at most $2^{\rho \log(2r)} = O(r^\rho)$, and the claim follows. □

Also, the geometric graph models can be generalized to a doubling metric space. This more general characterization allows us to use a distance function not only based on geometric distance, but also on characteristics of the wave propagation of the wireless medium. We can thus relate signal strength, distance, and transmission characteristics to obtain a suitable metric intersection graph model for the wireless communication network.

In the remainder of this article, we assume the graph to be of polynomially bounded growth. With respect to the problems of this article, in Clark et al. [1990], it is shown that both the MAX-IS and MIN-DS problems remain *NP*-hard even on unit disk graphs, and thus on graphs of bounded growth, as well.

3. *Local Neighborhood-Based Approximation Schemes*

In this section, we present an approach for approximation schemes that does not rely on positional information of the vertices but assumes only knowledge of the adjacency of each vertex in the graph. We assume the graph $G = (V, E)$ to be of polynomially bounded growth, thus completely abandoning any underlying geometric structure. Let p denote the growth polynomial of G.

Denote by $\mathcal{P}(V)$ the set of all subsets of vertices. We then define two functions $I : \mathcal{P}(V) \to \mathcal{P}(V)$ and $D : \mathcal{P}(V) \to \mathcal{P}(V)$ which return an independent set of maximum cardinality and a dominating set of minimum cardinality, respectively, for the subset given as argument. For a subset $V' \subseteq V$, the set $I(V')$ is independent in V', and $D(V')$ dominates V'. For $D(V')$, the inclusion $D(V') \subseteq V'$ needs not to hold. Therefore, in the following, the function $D(.)$ is always computed with respect to the entire underlying graph G. However, it is easy to see that $V' \subseteq \Gamma(D(V'))$ and that $D(V') \subseteq \Gamma(V')$ hold.

Using the previous definitions, we are interested in a polynomial time approximation of $I(V)$ and $D(V)$ within a factor of $1 + \varepsilon$ for any given $\varepsilon > 0$.

In order to simplify the notation, for some vertex $v \in V$ and its r-th neighborhood $\Gamma_r(v)$, we use $I_r(v)$ and $D_r(v)$ to denote the independent set $I(\Gamma_r(v))$ and the dominating set $D(\Gamma_r(v))$ for $\Gamma_r(v)$. Further on, in case the vertex v is unambiguous, we omit indexing the respective neighborhoods and subsets with this central vertex.

Suppose the radius r of a neighborhood Γ_r is bounded. Then by the definition of bounded growth graphs and the fact that any maximal independent set is also dominating,

$$|D_r| \leq |I_r| \leq p(r)$$

holds. With this bound on the cardinality of the locally optimal solutions, it becomes clear that we can obtain both optimal solutions I_r and D_r in time $n^{O(p(r))} = n^{O(r)}$ for this neighborhood Γ_r.

The following algorithms work by creating optimal partial solutions inside neighborhoods of bounded radius, and then combining these partial solutions without violating feasibility. To describe the approaches taken in a more general way, we introduce the basic definition of collections of d-separated subsets in G as follows.

Definition 3.1. For a graph $G = (V, E)$, let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of subsets of vertices $S_i \subseteq V, i = 1, \ldots, k$ with the following property: for any two vertices $s \in S_i$ and $\bar{s} \in S_j, i \neq j$,

$$d_G(s, \bar{s}) > d$$

holds. We refer to $\mathcal{S}$ as a *d-separated* collection of subsets.

It is easy to see that the subsets of any d-separated collection, $d \geq 0$ are mutually disjoint. An example of a 2-separated collection is given in Figure 2. The grey areas mark the different subsets that make up the collection; vertices which are not part of it, and thus separate the subsets, are white.

3.1. Maximum Independent Set. We now present an approach that gives the PTAS for the maximum independent set problem on a graph $G = (V, E)$ of polynomially bounded growth. The basic idea of is simple. We start with an arbitrary vertex $v \in V$ and consider for $r = 0, 1, 2, \ldots,$ the r-th neighborhoods Γ_r and optimal independent sets $I_r \subseteq \Gamma_r$ therein. We then keep expanding the neighborhoods as long as

$$|I_{r+1}| > (1 + \varepsilon) \cdot |I_r|$$

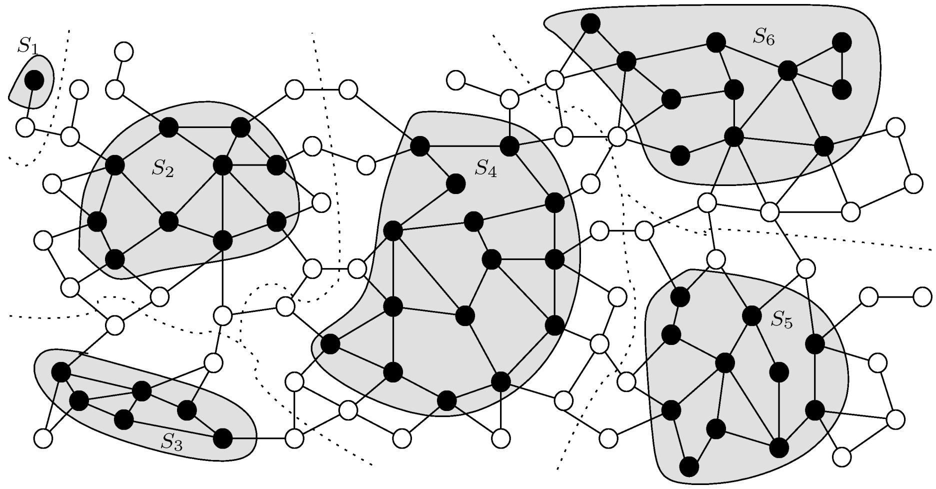

FIG. 2. Example of a 2-separated collection $\mathcal{S} = \{S_1, \ldots, S_6\}$.

holds. Let $\bar{r}$ denote the smallest $r \geq 0$ for which this condition is violated. Such an $\bar{r}$ indeed exists, and it is bounded by a constant that only depends on ε.

LEMMA 3.2. *Let $G = (V, E)$ be a graph of polynomially p-bounded growth. There exists a constant $c = c(\varepsilon)$ such that $\bar{r} \leq c$.*

PROOF. Let $r < \bar{r}$. By definition of $\bar{r}$, we then have for r

$$|I_r| > (1+\varepsilon)|I_{r-1}| > \cdots > (1+\varepsilon)^r |I_0| = (1+\varepsilon)^r.$$

Since the graph G is of polynomial bounded growth, we also have $|I_r| \leq p(r)$. By comparison, that is,

$$p(r) \geq |I_r| > (1+\varepsilon)^r,$$

the claim follows. Using the inequality $\log(1+\varepsilon) > 1/2 \cdot \varepsilon$ for sufficiently small ε, we can bound the constant c by $O(1/\varepsilon \log 1/\varepsilon)$. $\square$

To achieve an independent set for the graph G, we iteratively apply this scheme. Each time the expansion is stopped, we remove the neighborhood $\Gamma_{\bar{r}+1}$ from G and combine $I_{\bar{r}}$ with the partial solution I obtained thus far. A detailed summary of the approach is given by Algorithm 1, and we now prove its correctness and polynomial complexity starting with the independence property of the returned solution.

Since $I_{\bar{r}} \subseteq \Gamma_r$, when we remove $\Gamma_{\bar{r}+1}$, the set $\Gamma_{\bar{r}}$ is 1-separated from all sets to be calculated in the remaining process. This implies that the created sets $\Gamma_{\bar{r}}$, at the completion of the algorithm, form a 1-separated collection. In other words, each $v \in V \setminus \Gamma_{\bar{r}+1}$ has no neighbor in $\Gamma_{\bar{r}}$, and thus not in $I_{\bar{r}} \subseteq \Gamma_{\bar{r}}$. A direct consequence is the following lemma.

LEMMA 3.3. *The solution I created by Algorithm 1 forms an independent set in the graph G.*

In order to motivate that all operations can be completed in polynomial runtime, note that we only need to consider a single iteration as the number of new central vertices picked in the algorithm is limited by n. Due to the definition of a

—**Input:** $G = (V, E)$ poly. growth-bounded, $\varepsilon > 0$
—**Output:** $(1 + \varepsilon)$-approx. Max. Independent Set I

```
I := ∅;
while V ≠ ∅ do
    Pick v ∈ V;
    r := 0;
    while |I_{r+1}(v)| > (1 + ε) · |I_r(v)| do
        r := r + 1;
    end while
    I := I ∪ I_r(v);
    V := V \ Γ_{r+1}(v);
end while
```

FIG. 3. Algorithm 1: PTAS maximum independent set.

polynomially bounded growth graph and the constant established by Lemma 3.2, every operation can be completed in polynomial time with the degree depending only on $\varepsilon > 0$ and not on the size of the graph. It remains to show that the cardinality of the independent set I meets the desired approximation guarantee of $(1 + \varepsilon)$.

THEOREM 3.4. *Let $I^* := I(V)$ be a maximum independent set in a graph $G = (V, E)$. The independent set I computed by Algorithm 1 satisfies*

$$(1 + \varepsilon) \cdot |I| \geq |I^*|.$$

PROOF. Suppose inductively that Algorithm 1 computes a $(1+\varepsilon)$-approximate independent set $I' \subseteq V \setminus \Gamma_{\bar{r}+1}$ for $G' = G[V \setminus \Gamma_{\bar{r}+1}]$.

By definition of $\bar{r}$, we have

$$|I_{\bar{r}+1}| \leq (1 + \varepsilon) \cdot |I_{\bar{r}}|.$$

Since the cardinality of the part of the optimal set I^* which lies in $G[\Gamma_{\bar{r}+1}]$ is bounded by the cardinality of $I_{\bar{r}+1}$, we get

$$|\Gamma_{\bar{r}+1} \cap I^*| \leq |I_{\bar{r}+1}| \leq (1 + \varepsilon) \cdot |I_{\bar{r}}|.$$

Further, by inductive assumption, I' is $(1+\varepsilon)$-approximately optimal for G'. Therefore,

$$|V(G') \cap I^*| \leq |I(V(G'))| \leq (1 + \varepsilon) \cdot |I'|.$$

Adding the two inequalities, we obtain

$$|I^*| \leq (1 + \varepsilon) \cdot (|I_{\bar{r}}| + |I'|) = (1 + \varepsilon) \cdot |I|. \qquad \square$$

Note that Algorithm 1 actually returns a $(1 + \varepsilon)$-approximate independent set for any undirected graph given as input. We have only used the specific structure of polynomially bounded growth in Lemma 3.2 in order to bound the radius of the largest neighborhood we need to consider during execution. However, the criterion to stop expanding a neighborhood is met eventually in any undirected graph, possibly while considering an $O(n)$-neighborhood or eventually G itself.

Coming back to the runtime of Algorithm 1, we see that the time needed for completion of the algorithm is dominated by the constant $c = c(\varepsilon)$ of Lemma 3.2. To be more precise, the runtime is dominated by the time needed to construct an optimal solution for $\Gamma_{\bar{r}+1}(v)$, $\bar{r} \leq c$. The overall runtime of the approach is thus $n^{O(1/\varepsilon \log 1/\varepsilon)}$.

Maximum Weight Independent Set. The previous approximation scheme can easily be adapted for the case that each vertex $v \in V$ is given a weight w_v. Without loss of generality, we assume $w_v > 0$ for every $v \in V$. Recall that in this case, we are interested in an independent set $I \subseteq V$ of high total weight $W(I) := \sum_{i \in I} w_i$ in G. We now present the necessary modifications to the previous algorithm in order to obtain an independent set of weight at least $(1+\varepsilon)^{-1}$ the maximum total weight of an independent set in the graph.

The algorithm again follows the idea of expanding the local neighborhood of a central vertex v. This time, however, the central vertex v is chosen to be a vertex with maximum weight $w_v = \max\{w_i \mid i \in V\}$ in the remaining graph G. Then, we compute an independent set $I_r \subseteq \Gamma_r(v)$ of maximum weight for increasing radii r until the criterion

$$W(I_{r+1}) > (1+\varepsilon) \cdot W(I_r)$$

is violated. Let $\bar{r}$ denote the smallest $r \geq 0$ for which this is the case.

LEMMA 3.5. *Let $G = (V, E)$ be a polynomially p-bounded growth graph. There exists a constant $c = c(\varepsilon)$ such that $\bar{r} \leq c$.*

PROOF. Adapting the proof of Lemma 3.2, for $r < \bar{r}$, we get

$$W(I_r) = \sum_{i \in I_r} w_i \leq \sum_{i \in I_r} w_{\max} = |I_r| \cdot w_{\max},$$

and

$$W(I_r) > (1+\varepsilon) \cdot W(I_{r-1}) > \cdots > (1+\varepsilon)^r \cdot W(I_0) = (1+\varepsilon)^r \cdot w_{\max},$$

respectively. Since $|I_r| \leq p(r)$, combining the two inequalities again yields the claim. □

As a consequence, the running time of this algorithm remains polynomial in the weighted case with the same time complexity as for the unweighted case. Furthermore, the approximation ratio can be guaranteed by the following theorem.

THEOREM 3.6. *The adapted algorithm, that is, choosing a central vertex of maximum weight in G in each round, creates an independent set I of weight*

$$(1+\varepsilon) \cdot W(I) \geq W(I^*),$$

where I^* denotes an optimal solution to the maximum weight independent set problem on G.

PROOF. Let $V' := V \setminus \Gamma_{\bar{r}+1}(v)$ and assume inductively that $I' \subseteq V'$ is a $(1+\varepsilon)$-approximate weighted independent set in $G[V']$. Clearly, $I_{\bar{r}} \cup I'$ is an independent set in G.

For the weighted independent set in the neighborhood $\Gamma_{\bar{r}+1}(v)$, we have

$$W(I^* \cap \Gamma_{\bar{r}+1}(v)) \leq W(I_{\bar{r}+1}) \leq (1+\varepsilon) \cdot W(I_{\bar{r}})$$

from the criterion to stop expanding the neighborhoods.

For the weight $W(I) = W(I_{\bar{r}} \cup I')$ of the overall solution set I returned by the algorithm, we then get

$$\begin{aligned} W(I^*) &= W((I^* \cap \Gamma_{\bar{r}+1}(v) \cup (I^* \cap V')) \\ &= W(I^* \cap \Gamma_{\bar{r}+1}(v)) + W(I^* \cap V') \\ &\leq (1+\varepsilon) \cdot W(I_{\bar{r}}) + (1+\varepsilon) \cdot W(I') \\ &= (1+\varepsilon) \cdot W(I_{\bar{r}} \cup I') \\ &= (1+\varepsilon) \cdot W(I). \end{aligned}$$ □

3.2. MINIMUM DOMINATING SET. In this section, we present a polynomial time approximation scheme for the minimum dominating set problem on graphs of polynomially p-bounded growth. The approach also follows the implicit separation idea of the PTAS for the MAX-IS problem. Again, we use local neighborhoods of bounded radius and optimal partial solutions therein to obtain a PTAS. However, while in the previous section, the separation strategy and the overall approximation followed by rather simple arguments, for the MIN-DS problem, some attention has to be paid to the manner in which the local neighborhoods are created and put together.

The main part of the algorithm now consists of iteratively constructing dominating sets for the local neighborhoods Γ_r and to stop if the cardinality of these dominating sets does not grow much anymore. To be more precise, we stop expanding the radius r of the neighborhoods if

$$|D_{r+2}(v)| > (1+\varepsilon) \cdot |D_r(v)|$$

is violated.

Note the possibility that vertices outside a subset are able to dominate vertices inside this subset. The local dominating sets are always created with respect to G, and for neighborhoods Γ_r, we have $D_r \subseteq \Gamma_{r+1}$. Therefore, in order to have sufficient separation, we need to consider a 2-separated structure given by the neighborhoods.

The approach constructs a 2-separated collection of neighborhoods given by the $\Gamma_{\bar{r}}(v)$, where $\bar{r}$ denotes the radius upon stopping to expand the neighborhoods. At this point, we remove $\Gamma_{\bar{r}+2}$ from G and keep $D_{\bar{r}+2}$ as part of the solution. The overall separation can then easily be seen by inductive argumentation: each neighborhood removed from the remaining set V of vertices satisfies the separation property with respect to previously removed neighborhoods and the resulting reduced set V for the following iteration. The algorithm from the this approach is given by Algorithm 2 (Figure 4).

The following lemma then gives a lower bound for partial dominating sets of 2-separated collections with respect to an optimal dominating set for a graph G.

LEMMA 3.7. *Let $D^* := D(V)$ be a minimum dominating set in a graph $G = (V, E)$. For a 2-separated collection $\mathcal{S} = \{S_1, \ldots, S_k\}$ in G, it is*

$$|D^*| \geq \sum_{i=1}^{k} |D(S_i)|.$$

PROOF. For each subset $S_i \in \mathcal{S}$, consider the neighborhood $\Gamma(S_i)$. As a direct consequence of the definition of 2-separated subsets, these neighborhoods are pairwise disjoint. Furthermore, any vertex outside $\Gamma(S_i)$ has distance more than one to all vertices in S_i. Therefore, $D^* \cap \Gamma(S_i)$ dominates S_i. $D(S_i) \subseteq \Gamma(S_i)$ also

—**Input:** $G = (V, E)$ poly. growth-bounded, $\varepsilon > 0$
—**Output:** $(1+\varepsilon)$-approx. Min. Dominating Set D

$D := \varnothing$;
while $V \neq \varnothing$ **do**
  Pick $v \in V$;
  $r := 0$;
  while $|D_{r+2}(v)| > (1+\varepsilon) \cdot |D_r(v)|$ **do**
    $r := r + 1$;
  end while
  $D := D \cup D_{r+2}(v)$;
  $V := V \setminus \Gamma_{r+2}(v)$;
end while

FIG. 4. Algorithm 2: PTAS minimum dominating set.

dominates S_i using a minimum number of vertices. Therefore, we obtain

$$|D^* \cap \Gamma(S_i)| \geq |D(S_i)|.$$

Combining this observation for all subsets of $\mathcal{S}$, we get

$$|D^*| \geq \sum_{i=1}^{k} |D^* \cap \Gamma(S_i)| \geq \sum_{i=1}^{k} |D(S_i)|,$$

as claimed. $\square$

Looking at such a 2-separated collection together with related subsets and bounded cardinality dominating sets for these, we extend the Lemma 3.7 to receive an upper bound.

COROLLARY 3.8. *Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a 2-separated collection in $G = (V, E)$, and let $T_1, \ldots, T_k$ be subsets of V with $S_i \subseteq T_i$ for all $i = 1, \ldots, k$. If there exists a bound $\rho \geq 1$ such that*

$$|D(T_i)| \leq \rho \cdot |D(S_i)|$$

holds for all $i = 1, \ldots, k$, then the set $T := \bigcup_{i=1}^{k} D(T_i)$ satisfies

$$|T| \leq \rho \cdot |D^*|,$$

where D^* denotes a minimum dominating set in G.

PROOF. $|\bigcup_{i=1}^{k} D(T_i)| \leq \sum_{i=1}^{k} |D(T_i)| \leq \rho \cdot \sum_{i=1}^{k} |D(S_i)| \leq \rho \cdot |D^*|$. $\square$

The partial solutions taken from the respective $(\bar{r}+2)$-neighborhoods $\Gamma_{\bar{r}+2}$ in each iteration thus satisfy the desired approximation guarantee. Furthermore, the following lemma establishes the overall domination property for the partial solutions $D(\Gamma_{\bar{r}+2})$.

LEMMA 3.9. *The set D returned by Algorithm 2 dominates the graph G.*

PROOF. Let N_i denote the neighborhoods $\Gamma_{\bar{r}+2}(v)$ removed from V in each iteration $i = 1, \ldots, k$ of the algorithm. Then, since we stop the algorithm when $V = \varnothing$ is reached, it is $\bigcup_{i=1}^{k} N_i = V$, and each of these neighborhoods N_i is dominated by $D(N_i)$. Therefore, $\bigcup_{i=1}^{k} D(N_i)$ dominates G. $\square$

So far, we see that the solution set D returned by Algorithm 2 is a global $(1+\varepsilon)$-approximate dominating set for the input graph G. At this point, we would like to recall that this approximation guarantee is valid for any undirected graph G even if it does not satisfy the bounded growth condition.

It remains to show that the approximation algorithm has polynomial runtime. Clearly, the number of times we have to pick a new central vertex, and construct local neighborhoods and dominating sets for these is bounded by $n = |V|$. We may thus limit the further discussion to one iteration only. We focus without loss of generating on the polynomially growth-bounded graph $G = (V, E)$ in the first iteration and again show that the radius of the largest neighborhood we need to consider is bounded by a constant.

LEMMA 3.10. *Let $G = (V, E)$ be a polynomially p-bounded growth graph. There exists a constant $c = c(\varepsilon)$ such that the radius r of any neighborhood $\Gamma_r(v)$ considered in the algorithm is bounded by c, i.e. $r \leq c$.*

PROOF. It is $|D(\Gamma_0(v))| = |D(\Gamma_1(v))| = 1$ as the central vertex v dominates itself and all its neighbors. Consider the criterion for stopping to expand the neighborhoods in Algorithm 2, and consider the inequality $|D(\Gamma_{r+2})| > (1+\varepsilon) \cdot |D(\Gamma_r)|$ for all $r \leq \bar{r}$. Suppose that r is an even number, we have

$$\begin{aligned} p(r+2) \geq |D(\Gamma_{r+2})| \; &> \; (1+\varepsilon) \cdot |D(\Gamma_r)| \\ &> \; \cdots > (1+\varepsilon)^{r/2} \cdot |D(\Gamma_0)| = (\sqrt{1+\varepsilon})^r. \end{aligned}$$

If r is odd, the chain ends at $D(\Gamma_1)$. Since $\varepsilon > 0$, and thus $\sqrt{1+\varepsilon} > 1$, the inequalities have to be violated eventually in both cases. Thus, the bound c on the largest radius of the neighborhoods depends only on the approximation guarantee ε. $\square$

If the input graph of Algorithm 2 is of polynomially bounded growth, each iteration has polynomial running time. The complexity of the computation of $D(\Gamma_r(v))$ dictates the overall runtime of the algorithm. Since any maximal independent set is also dominating, this can be achieved in $n^{O(1/\varepsilon \log 1/\varepsilon)}$.

4. *Robustness*

We now present a simple way to make the previous approximation schemes *robust*. In this case, the robust algorithm accepts any undirected graph as valid input and either returns a desired approximate solution or outputs a polynomial certificate showing that the input graph does not satisfy the structural assumption of p-bounded growth. A robust algorithm must produce correct output regardless of the input [Raghavan and Spinrad 2003]. More precisely, robust algorithms are defined as follows.

Definition 4.1. Let $\mathcal{A}$ be an algorithm defined on $\mathcal{G}$, f be a function on $\mathcal{G}$, and $\mathcal{U} \subset \mathcal{G}$. Then $\mathcal{A}$ computes f robustly (*on* $\mathcal{G}$) if, for all instances $x \in \mathcal{U}$, the algorithm $\mathcal{A}$ returns $f(x)$, and, for all instances $x \in \mathcal{G} \setminus \mathcal{U}$, the algorithm $\mathcal{A}$ returns either $f(x)$ or a certificate showing that $x \notin \mathcal{U}$.

Of course, the notion of a robust algorithm is especially interesting when $\mathcal{A}$ has polynomial running time with respect to the size of the input, and the decision as to whether an instance belongs to the subclass $\mathcal{U} \subset \mathcal{G}$ is not so easy to decide. In our situation, $\mathcal{G}$ is the set of all undirected graphs and $\mathcal{U}$ are all those graphs that have polynomially bounded growth. The function f then computes a $(1+\varepsilon)$-approximate subset of the vertices, depending on the problem.

In the previous section, we have seen that the approximation algorithms introduced actually yield a PTAS when the instance reflects a graph of polynomially p-bounded growth. We thus continue the discussion only for the case where the undirected graph $G = (V, E)$ presented to the algorithm does not satisfy the characterization of a polynomially bounded growth graph.

Observe that both approximation algorithms return an independent or dominating subset, I or D, respectively, for G also in the case where G is a general graph. We only use specific properties of bounded growth when deriving the polynomial runtime of the algorithms. In both problems, the polynomial runtime of the approximation algorithms results from the bound $p(r)$ on the size of any independent set and an optimal dominating set in the r-th neighborhood, that is, $|D_r| \leq |I_r| \leq p(r)$. If, during execution of the algorithm, a neighborhood Γ_r contains an independent set of size larger than $p(r)$, we can use this neighborhood as a polynomial certificate showing non-membership in the class of p-bounded growth graphs.

By definition of polynomial bounded growth graphs, any independent set in Γ_r has to satisfy the given bound on the cardinality. For dominating sets, this bound obviously does not hold. However, before considering dominating sets for a neighborhood Γ_r, we can use a simple greedy approach to quickly compute a maximal independent set as a starting point. This independent set is then used to guarantee the polynomial runtime of the algorithm, making the approach for the MIN-DS approximation scheme robust in the prior sense as well.

We can thus apply the approximation schemes to any undirected graph which is believed to be of polynomially bounded growth without risk of failure, that is, exponential running time, if this assumption is wrong. In wireless ad-hoc networks, this gives us the advantage to indirectly account for all the uncertainties and the dynamic behavior which govern the resulting applications: there may exist wireless communication graphs in very harsh environments for which the polynomially bounded growth assumption does not hold. In this case, we at least receive feedback.

Furthermore, robust algorithms can be combined in sequence where the computed solution is used as input for further algorithms, that is, as building blocks of more general algorithms. In other words, robustness is preserved by composition. This is to be contrasted to the non-robust case where an algorithm need not produce an output or even terminate if the input does not satisfy the additional structural properties and some produced output may not even be a feasible solution to the problem at hand.

5. *Conclusions*

In this article, we presented an approach that yields polynomial-time approximation schemes for the maximum independent set and minimum dominating set problems on graphs with polynomially bounded growth. Such graphs can be used to model wireless communication networks independent of any geometric information. The approaches presented are robust, making them well-suited for the dynamic and sometimes unpredictable nature of wireless transmissions.

We believe that the local neighborhood-based approach presented in this article can be applied to a variety of different, related problems on graphs of polynomially bounded growth.

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