29 July 2002

PHYSICS LETTERS A

Physics Letters A 300 (2002) 250–258

www.elsevier.com/locate/pla

# Skinner–Rusk approach to time-dependent mechanics

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Received 15 March 2002; accepted 28 May 2002

Communicated by A.P. Fordy

**Abstract**

The geometric approach to autonomous classical mechanical systems in terms of a canonical first-order system on the Whitney sum of the tangent and cotangent bundle, developed by Skinner and Rusk, is extended to the time-dependent framework. © 2002 Elsevier Science B.V. All rights reserved.

*Keywords:* Time-dependent mechanics; Skinner–Rusk formalism; Singular Lagrangians; Constraint algorithm

## 1. Introduction

In 1983, it was shown by Skinner and Rusk that the dynamics of an autonomous classical mechanical system, with configuration space $Q$, can be properly represented by a first-order system on the Whitney sum $T^*Q \oplus TQ$ [1–3]. If the system under consideration admits a Lagrangian description, with Lagrangian $L \in C^\infty(TQ)$, the corresponding first-order system on $T^*Q \oplus TQ$ is a Hamiltonian system with respect to a canonical presymplectic structure. The Skinner–Rusk formulation can be briefly summarized as follows. Denoting the projections of $T^*Q \oplus TQ$ onto $T^*Q$, respectively $TQ$, by $\mathrm{pr}_1$, respectively $\mathrm{pr}_2$, and putting $\omega = \mathrm{pr}_1^*\omega_Q$, with $\omega_Q$ the canonical symplectic form on $T^*Q$, one can consider the following equation:

$$i_Z\omega = d\mathcal{H}, \tag{1}$$

where $\mathcal{H} := \langle \mathrm{pr}_1, \mathrm{pr}_2\rangle - L \circ \mathrm{pr}_2$, and $\langle\ ,\ \rangle$ denotes the natural pairing between the dual bundles $T^*Q$ and $TQ$. If the given Lagrangian $L$ is regular, analysis of (1) shows that there exists a unique solution $Z$ which is tangent to the graph of the Legendre map $\mathrm{Leg}_L : TQ \to T^*Q$, $(q^A, v^A) \mapsto (q^A, \partial L/\partial v^A)$, where the $q^A$ are local coordinates on $Q$ and $(q^A, v^A)$ denote the corresponding bundle coordinates on $TQ$.

**Remark.** Here and in the sequel we will adopt the following definition for the *graph of a bundle mapping*. Let $E_1$ and $E_2$ denote two fibre bundles over the same base space $M$, and let $f : E_1 \to E_2$, $(m, e) \mapsto (m, \tilde{f}_m(e))$ be a fibre bundle mapping over the identity. Then we define the graph of $f$ as the image of the

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PII: S0375-9601(02)00777-6

mapping $f \times_M \mathrm{id}_{E_1} : E_1 \to E_2 \times_M E_1$, $(m, e) \in E_1 \mapsto (m, \tilde{f}_m(e), e)$.

The equations of motion induced by the vector field $Z$ are equivalent to the Euler–Lagrange equations for the system under consideration (see [2]). The important point now is that this equivalence between a Lagrangian system and the corresponding first-order system (1) on $T^*Q \oplus TQ$ also holds if the given Lagrangian $L$ is singular. In that case, in order to extract a consistent system of differential equations from (1), one will have to invoke a constraint algorithm. In fact, one of the main motivations for the work of Skinner and Rusk was precisely to show that the Dirac–Bergmann approach to singular Lagrangian systems can be properly described on the Whitney sum of the tangent and cotangent bundle of the configuration space, thereby avoiding some ambiguities occurring in the literature on the subject.

The Skinner–Rusk formalism has been applied by several authors in various contexts [4–7]. As one of the benefits of the formalism it turns out that it provides an appropriate setting for a geometric approach to constrained variational optimization problems. The latter are frequently encountered, for instance, in mathematical economics and in engineering. This has been demonstrated, in particular, for some optimal control problems in [6], and for the so-called vakonomic dynamics in [7] (see also [8]). The fact that it would be interesting to extend those results to the time-dependent framework, allowing for systems with an explicit time-dependence of the "forces" and/or the constraints, is the main motivation underlying the present work. More precisely, we will develop here a time-dependent version of the Skinner–Rusk formulation of dynamics, using the language of jet bundle theory [9] and cosymplectic geometry [10]. Among the virtues of this new formulation of time-dependent mechanics, we would like to stress the possibility it offers to model a large class of systems, also in areas such as economics and control theory. Applications for which this framework seems to be particularly well-suited include stabilization and trajectory tracking of mechanical systems by means of time-dependent transformations (see, e.g., [11]).

Our starting point is a fibre bundle $\pi : E \to \mathbb{R}$, with $E$ representing the evolution space of a mechanical system. Although it is quite common in treatments of time-dependent mechanics to fix a trivialization of $\pi$, i.e., to work on a direct product space $\mathbb{R} \times Q$ $(= E)$, we will not select such a trivialization here. The natural space to consider then for the treatment of time-dependent Lagrangian mechanics is the first jet space $J^1\pi$, with the Lagrangian of the system being given as a function $L \in C^\infty(J^1\pi)$. The immediate candidate for replacing the direct sum $T^*Q \oplus TQ$ in the Skinner–Rusk model for autonomous systems, seems to be the fibred product $J^1\pi^* \times_E J^1\pi$, where $J^1\pi^*$ is the "dual" of the affine bundle $J^1\pi$ (for this notion of dual, see, e.g., [12]). It turns out, however, that this is not an appropriate choice for the following reasons. First, there does not exist a natural pairing between $J^1\pi$ and $J^1\pi^*$, needed for the construction of the time-dependent analogue of the "Hamiltonian" $\mathcal{H}$ appearing in (1) and, secondly, there is no canonical 2-form on $J^1\pi^*$ to take over the role of the symplectic form $\omega_Q$ in the autonomous picture. To overcome these difficulties we will show that the appropriate space to consider is the fibred product $T^*E \times_E J^1\pi$.

Two final remarks are in order here. First of all, although we will restrict ourselves to Lagrangian systems, it is clear that, in analogy with the autonomous case (cf. [1]), the treatment can be easily extended to more general time-dependent mechanical systems, with forces not necessarily derivable from a potential. Secondly, it is interesting to note that the ideas developed here also admit a further extension to classical field theory, as has been demonstrated in a recent paper by de León et al. [13].

The present Letter is organized as follows. In the next section we briefly recall the jet bundle approach to time-dependent Lagrangian and Hamiltonian mechanics, including the description of the constraint algorithm in case of a singular Lagrangian. In Section 3 we then develop the Skinner–Rusk formalism for time-dependent systems. We end with some conclusions and an outlook on future work along these lines.

## 2. Non-autonomous Lagrangian systems

### 2.1. The regular case

Let $\pi : E \to \mathbb{R}$ be a fibre bundle (the evolution space), with $\dim E = n + 1$ and local bundle coordinates $(t, q^A)$, $A = 1, \dots, n$. Consider the corresponding 1-jet space $J^1\pi$, with coordinates $(t, q^A, \dot q^A)$ and associated projections $\pi_1 : J^1\pi \to \mathbb{R}$ and $\pi_{1,0} : J^1\pi \to E$. Given a time-dependent Lagrangian $L : J^1\pi \to \mathbb{R}$, the Euler–Lagrange equations read in local coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^A}\right) - \frac{\partial L}{\partial q^A} = 0. \tag{2}$$

These equations can be rewritten in geometrical terms as follows. Define the Poincaré–Cartan 1-form and 2-form

$$\Theta_L = L\tilde\eta + \tilde S^*(dL), \qquad \omega_L = -d\Theta_L,$$

where $\tilde\eta = \pi_1^*(dt)$ and $\tilde S = (dq^A - \dot q^A\,dt) \otimes (\partial/\partial \dot q^A)$ is the canonical vertical endomorphism on $J^1\pi$ (see [9]). The action of $\tilde S$ on 1-forms is denoted by $\tilde S^*$. Eqs. (2) can then be expressed as

$$i_X\omega_L = 0, \qquad i_X\tilde\eta = 1. \tag{3}$$

If the given Lagrangian is regular, then $\omega_L$ has maximal rank and the pair $(\omega_L, \tilde\eta)$ determines a *cosymplectic structure* on $J^1\pi$, i.e., both forms are closed and satisfy the conditions $\omega_L^n \wedge \tilde\eta \neq 0$, $\omega_L^{n+1} = 0$ (cf. [10]). It then follows that (3) admits a unique solution, called the *Euler–Lagrange vector field* for $L$, and which we will denote by $X_L$. It is a second-order vector field, i.e., $\tilde S(X_L) = 0$ and $i_{X_L}\tilde\eta = 1$, and a direct computation shows that integral curves of $X_L$ determine solutions of the Euler–Lagrange equations (2) and vice versa.

There also exists an alternative Hamiltonian description of the problem. Consider the Legendre map $\mathrm{Leg}_L : J^1\pi \to T^*E$, defined by $\mathrm{Leg}_L(j_t^1\phi)(v) = (\Theta_L)_{j_t^1\phi}(\tilde v)$ for $j_t^1\phi \in J^1\pi$, $v \in T_{\phi(t)}E$ and for any $\tilde v \in T_{j_t^1\phi}J^1\pi$ such that $\pi_{1,0*}(\tilde v) = v$. Let $V\pi$ denote the subbundle of $TE$ consisting of $\pi$-vertical tangent vectors, and denote its annihilator in $T^*E$ by $(V\pi)^0$. Consider the quotient bundle $J^1\pi^* = T^*E/(V\pi)^0$, which is called the dual of $J^1\pi$, with associated projections $\nu : T^*E \to J^1\pi^*$, $\tilde\pi_{1,0} : J^1\pi^* \to E$ and $\tilde\pi_1 : J^1\pi^* \to \mathbb{R}$. Finally, denote by $\mathrm{leg}_L : J^1\pi \to J^1\pi^*$ the composition $\mathrm{leg}_L = \nu \circ \mathrm{Leg}_L$. If $L$ is regular, then $\mathrm{Leg}_L$ is an immersion and $\mathrm{leg}_L$ is a local diffeomorphism.

Assume now that the Lagrangian $L$ is hyperregular, that is, $\mathrm{leg}_L$ is a global diffeomorphism. Consider then the map $h = \mathrm{Leg}_L \circ \mathrm{leg}_L^{-1}$. The mapping $h : J^1\pi^* \to T^*E$ is a section of the projection $\nu$, i.e., $\nu \circ h = \mathrm{id}_{J^1\pi^*}$ and is called a Hamiltonian of the system. Next, denote by $\omega_E$ the canonical symplectic two-form on $T^*E$ and let $\omega_h = h^*\omega_E$ be its pull-back to $J^1\pi^*$ under $h$. If $\eta_1 := (\tilde\pi_1)^*dt$, then $(\omega_h, \eta_1)$ defines a cosymplectic structure on $J^1\pi^*$. In addition, one has that $\mathrm{leg}_L^*(\omega_h) = \omega_L$ and $\mathrm{leg}_L^*\eta_1 = \tilde\eta$. It then easily follows that the solution $X$ of (3) is $\mathrm{leg}_L$-related to the (unique) solution $Y$ of the equations

$$i_Y\omega_h = 0, \qquad i_Y\eta_1 = 1. \tag{4}$$

Note that, always under the assumption of (hyper)regularity of $L$, the vector fields $X_L$, respectively $Y$, are precisely the Reeb vector fields corresponding to the cosymplectic structures $(\omega_L, \tilde\eta)$, respectively $(\omega_h, \eta_1)$.

### 2.2. The singular case: the constraint algorithm

Suppose now that the given Lagrangian $L$ is degenerate, in the sense that the Hessian matrix $(\partial^2 L/\partial\dot q^A\partial\dot q^B)$ is singular. We confine ourselves to the case where this matrix has constant rank everywhere, say $r$. The pair $(\omega_L, \tilde\eta)$ then satisfies the following relations (cf. [14,15]):

$$\omega_L^r \wedge \tilde\eta \neq 0, \qquad \omega_L^{r+1} \wedge \tilde\eta = 0, \qquad \omega_L^{r+2} = 0.$$

It follows that $2r \leqslant \operatorname{rank}\omega_L \leqslant 2r + 2$. In general, Eqs. (3) will not admit a global solution $X$. Moreover, if a solution exists it will not be unique. Therefore, in order to determine a consistent dynamics for such a system (if it exists) one has to apply a constraint algorithm which, at least in the case of Lagrangian mechanics, should be supplemented with the “second-order equation condition”. For completeness, we will now briefly sketch the constraint algorithm described in [14,15], which is an adaptation to the time-dependent setting of the well-known geometric constraint algorithm for presymplectic systems developed by Gotay and Nester [16,17].

With a view on its application later on, we will describe the constraint algorithm here in the general framework of a structure $(M, \Omega, \eta)$ consisting of a smooth manifold $M$, a closed 2-form $\Omega$ and a closed 1-form $\eta$, satisfying

$$\Omega^r \wedge \eta \neq 0, \qquad \Omega^{r+1} \wedge \eta = 0, \qquad \Omega^{r+2} = 0,$$

for some $r < \dim M$. On $M$ we then consider the equations

$$i_X\Omega = 0, \qquad i_X\eta = 1. \tag{5}$$

One can prove that there exists a vector $X_x \in T_xM$ satisfying these equations at the point $x$ iff $\operatorname{rank}\Omega_x = 2r$ (see [14]). In particular, it follows that (5) admits a global (but not necessarily unique) solution $X$ iff $\Omega$ has constant rank $2r$, in which case the given pair $(\Omega, \eta)$ defines a so-called *precosymplectic structure* on $M$. If this is not the case, the constraint algorithm proceeds as follows. Put $P_1 := M$ and consider the set

$$P_2 := \bigl\{x \in M \mid \exists X_x \in T_xM \text{ such that } i_{X_x}\Omega_x = 0,\ i_{X_x}\eta_x = 1\bigr\}.$$

According to the previous observation, this set can be equivalently characterized by $P_2 = \{x \in P_1 \mid \operatorname{rank}\Omega_x = 2r\}$. We then assume that $P_2$ is an embedded submanifold of $P_1$ $(= M)$ and we denote the natural inclusion by $j_2 : P_2 \hookrightarrow P_1$. We are then assured that Eqs. (5) admit a solution $X$ defined at all points of $P_2$, but $X$ need not be tangent to $P_2$ and, hence, does not necessarily induce a dynamics on $P_2$. We therefore have to continue the constraint algorithm by considering the subset

$$\begin{aligned} P_3 &:= \bigl\{x \in P_2 \mid \exists X_x \in T_xP_2 \text{ such that } i_{X_x}\Omega(x) = 0, i_{X_x}\eta(x) = 1\bigr\} \\ &= \bigl\{x \in P_2 \mid \eta_x \in \flat(T_xP_2)\bigr\}, \end{aligned}$$

where $\flat$ is the bundle morphism defined by

$$\flat : TM \to T^*M, \quad v \in T_xM \mapsto i_v\Omega_x + (i_v\eta_x)\eta_x.$$

Assuming $P_3$ is a submanifold of $P_2$, with inclusion map $j_3 : P_3 \hookrightarrow P_2$, it follows that there exists a vector field $X$ on $P_2$, which satisfies (5) at points of $P_3$. Again, however, such an $X$ need not be tangent to $P_3$, and one may have to repeat the above procedure. In this way, a descending sequence of submanifolds

$$\cdots \overset{j_{\ell+1}}{\hookrightarrow} P_\ell \overset{j_\ell}{\hookrightarrow} \cdots \overset{j_4}{\hookrightarrow} P_3 \overset{j_3}{\hookrightarrow} P_2 \overset{j_2}{\hookrightarrow} P_1\ (= M)$$

is generated, with

$$P_\ell := \bigl\{x \in P_{\ell-1} \mid \eta_x \in \flat(T_xP_{\ell-1})\bigr\} \quad (\ell \geqslant 2), \tag{6}$$

and where $P_\ell$ is called the *$\ell$-ary constraint submanifold*. If this sequence terminates at a nonempty set, in the sense that for some finite $k \geqslant 1$ we have $P_{k+1} = P_k$, but $P_k \neq P_{k-1}$, then $P_k$ is called the *final constraint submanifold*, which we denote by $P_f$. Now, it may still happen that $\dim P_f = 0$ (i.e., $P_f$ is a discrete set), in which case the given problem admits no proper dynamics. However, if $\dim P_f > 0$, then we know by construction that there exists a vector field $X$ on $M$, defined along $P_f$, which is tangent to $P_f$ and satisfies the equations

$$i_X\Omega_{|P_f} = 0, \qquad i_X\eta_{|P_f} = 1,$$

i.e., the given dynamical problem admits a consistent solution on $P_f$. In general, however, this solution will not be unique: given a solution $X$, for any smooth section $Y$ of the bundle $(\ker\Omega \cap \ker\eta) \cap TP_f$ over $P_f$, $X + Y$ is also a solution.

If we are dealing with a time-dependent Lagrangian system, i.e., with $M = J^1\pi$, $\Omega = \omega_L$, $\eta = \tilde\eta$ $(= dt)$, this is not the full story. First of all, we then also have to impose the so-called "second-order differential equation" condition, i.e., we are only interested in a solution $X$ which determines a system of second-order ordinary differential equations. Secondly, as in the autonomous case, one can develop a similar constraint algorithm on the Hamiltonian side (i.e., on $J^1\pi^*$) and, under a suitable assumption regarding the given Lagrangian, one can show that both descriptions are equivalent. For more details, we again refer to [14,15].

In the next section we will show how the above can be translated into a Skinner–Rusk type formulation for time-dependent Lagrangian systems.

## 3. Skinner–Rusk formulation

We start again from a fibre bundle $\pi : E \to \mathbb{R}$, with $n$-dimensional fibre, and its first jet space $J^1\pi$. Bundle coordinates on $E$ and $J^1\pi$ are denoted by $(t, q^A)$ and $(t, q^A, \dot q^A)$, respectively. Canonical coordinates on $T^*E$ will be written as $(t, q^A, \tau, p_A)$ and the canonical symplectic form on $T^*E$ then reads $\omega_E = dq^A \wedge dp_A + dt \wedge d\tau$. We now consider the fibred product of the bundles $T^*E$ and $J^1\pi$ over $E$, i.e., $T^*E \times_E J^1\pi$, with projections $\mathrm{pr}_1 : T^*E \times_E J^1\pi \to T^*E$, $\mathrm{pr}_2 : T^*E \times_E J^1\pi \to J^1\pi$ and $\mathrm{pr} : T^*E \times_E J^1\pi \to E$. The natural bundle coordinates on $T^*E \times_E J^1\pi$ are $(t, q^A, \tau, p_A, \dot q^A)$.

On $T^*E \times_E J^1\pi$ we define the 2-form $\omega$ as the pullback of the canonical symplectic form on

$T^*E$, i.e., $\omega = \mathrm{pr}_1^* \omega_E$, and the 1-form $\eta = (\pi \circ \mathrm{pr})^*(dt) = \mathrm{pr}_2^* \tilde{\eta}$. For simplicity we will sometimes write $\eta = dt$. Recall that the affine bundle $J^1\pi$ can be identified with an affine subbundle of $TE$ whose underlying set is given by $\{v \in TE \mid \langle dt, v\rangle = 1\}$. In coordinates, the natural embedding $j : J^1\pi \hookrightarrow TE$ reads $j(t, q^A, \dot{q}^A) = (t, q^A, 1, \dot{q}^A)$.

Given a Lagrangian $L \in C^\infty(J^1\pi)$, we can define the following function on $T^*E \times_E J^1\pi$:

$$\mathcal{H} = \langle \mathrm{pr}_1, j \circ \mathrm{pr}_2\rangle - \mathrm{pr}_2^* L,$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between vectors and covectors on $E$. In coordinates this becomes $\mathcal{H} = p_A\dot{q}^A + \tau - L(t, q^A, \dot{q}^A)$. Putting

$$\omega_{\mathcal{H}} = \omega + d\mathcal{H} \wedge \eta,$$

we can then consider the following equations:

$$i_Z\omega_{\mathcal{H}} = 0, \qquad i_Z\eta = 1. \tag{7}$$

Let us try to find out, in coordinates, what kind of dynamics is encoded by (7). For that purpose, we write $Z$ as

$$Z = Z_t \frac{\partial}{\partial t} + Z_{q^A} \frac{\partial}{\partial q^A} + Z_\tau \frac{\partial}{\partial \tau} + Z_{p_A} \frac{\partial}{\partial p_A} + Z_{\dot{q}^A} \frac{\partial}{\partial \dot{q}^A}.$$

From the second equation in (7) we deduce that $Z_t = 1$, and the first equation then becomes

$$\begin{aligned} i_Z\omega_{\mathcal{H}} &= i_Z\omega + Z(\mathcal{H})\,dt - d\mathcal{H} \\ &= \left(Z(\mathcal{H}) + \frac{\partial L}{\partial t} - Z_\tau\right) dt \\ &\quad + \left(\frac{\partial L}{\partial q^A} - Z_{p_A}\right) dq^A \\ &\quad - \left(p_A - \frac{\partial L}{\partial \dot{q}^A}\right) d\dot{q}^A + \left(Z_{q^A} - \dot{q}^A\right) dp_A \\ &= 0. \end{aligned}$$

This immediately gives $Z_{q^A} = \dot{q}^A$ and $Z_{p_A} = \partial L/\partial q^A$, together with the following constraint equations: $p_A = \partial L/\partial \dot{q}^A$. These constraints determine a submanifold of $T^*E \times_E J^1\pi$ which, for convenience, we will denote by $M_L$. With the above expressions for $Z_{q^A}, Z_{p_A}$ and $Z_t$, we see that the remaining condition $Z(\mathcal{H}) + \partial L/\partial t - Z_\tau = 0$ is identically satisfied at all points of $M_L$, irrespective of the value of the components $Z_\tau$ and $Z_{\dot{q}^A}$. Note that the relation $Z_{q^A} = \dot{q}^A$ reflects the second-order differential equation property.

The previous analysis already shows that (7) locally admits a solution $Z$, defined in points of the submanifold $M_L$ of $T^*E \times_E J^1\pi$. In fact, we have a whole family of solutions since the components $Z_\tau$ and $Z_{\dot{q}^A}$ can still be chosen arbitrarily.

In order to obtain consistent equations of motion, however, we have to impose the condition that $Z$ be tangent to the submanifold $M_L$, that is, the functions $Z(p_A - \partial L/\partial \dot{q}^A)$ should vanish at points of $M_L$ for all $A = 1, \ldots, n$. We now have that

$$\begin{aligned} Z\left(p_A - \frac{\partial L}{\partial \dot{q}^A}\right) &= \frac{\partial L}{\partial q^A} - \frac{\partial^2 L}{\partial t \partial \dot{q}^A} - \dot{q}^B \frac{\partial^2 L}{\partial q^B \partial \dot{q}^A} \\ &\quad - Z_{\dot{q}^B} \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}. \end{aligned} \tag{8}$$

Clearly, if $L$ is regular, the vanishing of (8) fixes all the components $Z_{\dot{q}^A}$ as functions of $(t, q^A, \dot{q}^A)$ on $M_L$. If not, one will have to apply a constraint algorithm.

### 3.1. The regular case

Let us assume that the given Lagrangian $L$ is regular. The previous analysis tells us that the system (7) admits a solution $Z$ on $M_L$ and it follows from the expression for the components $Z_t, Z_{q^A}, Z_{p_A}, Z_{\dot{q}^A}$ that any integral curve of $Z$ on $M_L$ determines a solution $(q^A(t))$ of the corresponding Euler–Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = 0 \quad (A = 1, \ldots, n).$$

However, the solution $Z$ is not unique since the component $Z_\tau$ is still undetermined. This, of course, is not surprising since $\partial/\partial\tau$ belongs to $\ker \omega_{\mathcal{H}} \cap \ker \eta$, i.e., $i_{\partial/\partial\tau}\omega_{\mathcal{H}} = 0$ and $i_{\partial/\partial\tau}\eta = 0$. In order to obtain a unique dynamics, we now impose an additional constraint

$$\tau = L - \dot{q}^A \frac{\partial L}{\partial \dot{q}^A}.$$

Together with the constraints $p_A = \partial L/\partial \dot{q}^A$, these are (locally) the defining equations of a submanifold of $T^*E \times_E J^1\pi$, namely the graph of the (extended)

Legendre map

$$\mathrm{Leg}_L : J^1\pi \to T^*E,$$
$$\left(t, q^A, \dot q^A\right) \mapsto \left(t, q^A, L - \dot q^A \frac{\partial L}{\partial \dot q^A}, \frac{\partial L}{\partial \dot q^A}\right)$$

(for the intrinsic definition of $\mathrm{Leg}_L$, see, e.g., [12], and for the notion of graph considered here, see the remark in the Introduction). We denote the graph of $\mathrm{Leg}_L$ by $\mathrm{graph}_L$. Clearly, $\mathrm{graph}_L \subset M_L$ and if we now require that $Z$ should be tangent to $\mathrm{graph}_L$, it readily follows that

$$Z_\tau = Z\left(L - \dot q^A \frac{\partial L}{\partial \dot q^A}\right), \tag{9}$$

which uniquely fixes $Z_\tau$. Note that the differential equation corresponding to the $\tau$-component of $Z$ represents the so-called "energy-balance" equation from time-dependent mechanics.

The above construction was carried out on an arbitrary natural bundle chart of $T^*E \times_E J^1\pi$. Using a standard argument it then follows that $Z$ is in fact well-defined on the whole of $\mathrm{graph}_L$. Defining the mapping

$$\mathrm{Leg}_L \times_E \mathrm{id}_{J^1\pi} : J^1\pi \to T^*E \times_E J^1\pi,$$
$$\left(t, q^A, \dot q^A\right) \mapsto \left(t, q^A, L - \dot q^A \frac{\partial L}{\partial \dot q^A}, \frac{\partial L}{\partial \dot q^A}, \dot q^A\right),$$

we see that $\mathrm{Im}(\mathrm{Leg}_L \times_E \mathrm{id}_{J^1\pi}) = \mathrm{graph}_L$ and it is not difficult to verify that the unique solution $Z$ of (7), defined on $\mathrm{graph}_L$, and the Euler–Lagrange vector field $X_L$ on $J^1\pi$ are related by

$$(\mathrm{Leg}_L \times_E \mathrm{id}_{J^1\pi})_*(X_L)_x = Z_{\bar x},$$

where $\bar x = (\mathrm{Leg}_L \times_E \mathrm{id}_{J^1\pi})(x)$, for all $x \in J^1\pi$.

The previous discussion can now be summarized by the following proposition.

**Proposition 3.1.** *For a regular Lagrangian $L$, system* (7) *admits a unique solution $Z$ defined on, and tangent to,* $\mathrm{graph}_L$, *and the induced vector field on* $\mathrm{graph}_L$ *is* $(\mathrm{Leg}_L \times_E \mathrm{id}_{J^1\pi})$*-related to the Euler–Lagrange vector field on $J^1\pi$.*

Recall that there exists a canonical projection $\nu : T^*E \to J^1\pi^*$ $(= T^*E/(V\pi)^0)$ which, in coordinates, reads $\nu(t, q^A, \tau, p_A) = (t, q^A, p_A)$ (cf. Section 2.1 and [12]). Let us assume that $L$ is hyperregular such that the mapping $\mathrm{leg}_L := \nu \circ \mathrm{Leg}_L$ is a global diffeomorphism. Consider the fibred product $J^1\pi^* \times_E J^1\pi$, with associated projections $\lambda_1$ and $\lambda_2$ onto $J^1\pi^*$ and $J^1\pi$, respectively. We can then define the following projection:

$$\nu \times_E \mathrm{id}_{J^1\pi} : T^*E \times_E J^1\pi \to J^1\pi^* \times_E J^1\pi,$$
$$\left(t, q^A, \tau, p_A, \dot q^A\right) \mapsto \left(t, q^A, p_A, \dot q^A\right).$$

From the discussion above we deduce that, along $\mathrm{graph}_L$, the vertical distribution determined by the projection $\nu \times_E \mathrm{id}_{J^1\pi}$, i.e., $\ker(\nu \times_E \mathrm{id}_{J^1\pi})_*$, is invariant under $Z$ in the sense that

$$\left[Z, \ker(\nu \times_E \mathrm{id}_{J^1\pi})_*\right] \subset \ker(\nu \times_E \mathrm{id}_{J^1\pi})_*.$$

Hence, in the hyperregular case, the solution vector field $Z$ is projectable onto $J^1\pi^* \times_E J^1\pi$. Its projection locally reads

$$\begin{aligned}(\nu \times_E \mathrm{id}_{J^1\pi})_*(Z) = {} & \frac{\partial}{\partial t} + \dot q^A \frac{\partial}{\partial q^A} \\ & + Z_{\dot q^A}\left(t, q^A, \dot q^A\right) \frac{\partial}{\partial \dot q^A} \\ & + \frac{\partial L}{\partial q^A}\left(t, q^A, \dot q^A\right) \frac{\partial}{\partial p_A}.\end{aligned}$$

This is the unique vector field on $J^1\pi^* \times_E J^1\pi$ determined by the equations

$$i_{\tilde Z}(\lambda_1^* \omega_h) = 0, \qquad i_{\tilde Z}(\lambda_1^* \eta_1) = 1,$$

where we recall that $h = \mathrm{Leg}_L \circ \mathrm{leg}_L^{-1}$ and $\omega_h = h^*\omega_E$ (cf. Section 2.1).

In the general case, however, it is not possible to represent the dynamics of the non-autonomous problem corresponding to $L$ by a first-order system on $J^1\pi^* \times_E J^1\pi$.

### 3.2. *The singular case*

Returning to the beginning of this section, let us now assume that $L$ is not regular. Observe that, with $\omega := \mathrm{pr}_1^* \omega_E$, we have

$$\omega_{\mathcal H}^2\ (= \omega_{\mathcal H} \wedge \omega_{\mathcal H}) = \omega \wedge \omega + 2\omega \wedge d\mathcal H \wedge \eta,$$

and, in general,

$$\omega_{\mathcal H}^k = \omega^k + k\omega^{k-1} \wedge d\mathcal H \wedge \eta, \quad \forall k.$$

Herewith, it is straightforward to check that the pair $(\omega_{\mathcal H}, \eta)$ satisfies the following relations:

$$\omega_{\mathcal H}^n \wedge \eta \neq 0, \qquad \omega_{\mathcal H}^{n+1} \wedge \eta = 0, \qquad \omega_{\mathcal H}^{n+2} = 0.$$

Indeed, we have

$$\begin{aligned}
\omega_{\mathcal{H}}^n \wedge \eta &= \omega^n \wedge \eta = (-1)^{n(n+1)/2} n!\, dq^1 \wedge \cdots \wedge dq^n \\
&\quad \wedge dp_1 \wedge \cdots \wedge dp_n \wedge \eta \neq 0, \\
\omega_{\mathcal{H}}^{n+1} \wedge \eta &= (-1)^{(n+1)(n+2)/2}(n+1)! \\
&\quad \times dt \wedge dq^1 \wedge \cdots \wedge dq^n \\
&\quad \wedge d\tau \wedge dp_1 \wedge \cdots \wedge dp_n \wedge \eta = 0, \\
\omega_{\mathcal{H}}^{n+2} &= \omega^{n+2} + (n+2)\omega^{n+1} \wedge d\mathcal{H} \wedge dt = 0.
\end{aligned}$$

This implies, in particular, that $2n \leqslant \operatorname{rank} \omega_{\mathcal{H}} \leqslant 2(n+1)$, where we recall that $\dim E = n+1$.

Putting $M_1 := T^*E \times_E J^1\pi$, we can now apply the constraint algorithm described in Section 2.2 to the triplet $(M_1, \omega_{\mathcal{H}}, \eta)$. First of all, we consider the set

$$\begin{aligned}
M_2 = \bigl\{ & x \in M_1 \mid \exists Z_x \in T_x M_1 \text{ such that} \\
& i_{Z_x}\omega_{\mathcal{H}}(x) = 0,\ i_{Z_x}\eta(x) = 1 \bigr\},
\end{aligned}$$

which can be equivalently characterized by $M_2 = \{x \in M_1 \mid \operatorname{rank} \omega_{\mathcal{H}}(x) = 2n\}$. In local coordinates we find

$$\begin{aligned}
\omega_{\mathcal{H}}^{n+1}(x) &= \omega^{n+1}(x) + (n+1)\omega^n \wedge d\mathcal{H} \wedge \eta(x) \\
&= \omega^{n+1}(x) - (n+1)\omega^n \wedge \frac{\partial \mathcal{H}}{\partial \tau}\, d\tau \wedge \eta(x) \\
&\quad + (n+1)\frac{\partial \mathcal{H}}{\partial \dot{q}^A}\omega_E^n \wedge d\dot{q}^A \wedge \eta(x) \\
&= (-1)^{n(n+1)/2}(n+1)!\frac{\partial \mathcal{H}}{\partial \dot{q}^A}\, dq^1 \wedge \cdots \wedge dq^n \\
&\quad \wedge dp_1 \wedge \cdots \wedge dp_n \wedge d\dot{q}^A \wedge \eta(x),
\end{aligned}$$

such that $x \in M_2$ if and only if

$$\frac{\partial \mathcal{H}}{\partial \dot{q}^A}_{|x} \equiv \left( p_A - \frac{\partial L}{\partial \dot{q}^A} \right)_{|x} = 0, \quad A = 1, \ldots n.$$

Observe that $M_2$ coincides with the submanifold $M_L$ introduced at the beginning of this section. By construction, there exists a vector field $Z$ on $M_1$, defined along $M_2$, which verifies Eqs. (7) at points of $M_2$. But in general $Z$ will not be tangent to $M_2$ and so we then have to proceed with the constraint algorithm by considering the set

$$\begin{aligned}
M_3 = \bigl\{ & x \in M_2 \mid \exists Z_x \in T_x M_2 \text{ such that} \\
& i_{Z_x}\omega_{\mathcal{H}}(x) = 0,\ i_{Z_x}\eta(x) = 1 \bigr\}.
\end{aligned}$$

Assuming that $M_3$ is a smooth submanifold, there will be a vector field $Z$ defined along $M_3$ and tangent to $M_2$, satisfying (7) at each point of $M_3$. Continuing this way, we obtain a descending sequence of submanifolds of $M_1$ that, in the favorable case, will stop at a final constraint submanifold $M_f$ on which there exists a consistent solution of the given dynamical problem (cf. Section 2.2). The constraint submanifolds $M_\ell$ can still be characterized in an algebraic way similar to (6), with the map $\flat : TM_1 \to T^*M_1$ being induced here by the pair $(\omega_{\mathcal{H}}, \eta)$.

As in the autonomous case, we thus see that the constraint algorithm for time-dependent singular Lagrangian systems can be properly developed in terms of the structure $(T^*E \times_E J^1\pi, \omega_{\mathcal{H}}, \eta)$. To complete the picture, we have the following result which shows that this description is equivalent to the standard one based on the structure $(J^1\pi, \omega_L, \tilde{\eta})$.

**Proposition 3.2.** *Let $\{P_\ell\}_{\ell \geqslant 1}$, respectively $\{M_\ell\}_{\ell \geqslant 1}$, denote the sequence of constraint submanifolds generated by applying the constraint algorithm to $(J^1\pi, \omega_L, \tilde{\eta})$, respectively $(T^*E \times_E J^1\pi, \omega_{\mathcal{H}}, \eta)$. Then, for each $i = 1, 2, \ldots,$ we have that $\varphi_{i+1} \equiv \mathrm{pr}_{2|M_{i+1}} : M_{i+1} \to P_i$ is a surjective submersion such that the following diagram commutes*:

$$\begin{tikzcd}
M_1 := T^*E \times_E J^1\pi \arrow[drr, "\mathrm{pr}_2"] & & \\
M_2 \arrow[u, "j'_2"] \arrow[rr, "\varphi_2"] & & P_1 := J^1\pi \\
M_3 \arrow[u, "j'_3"] \arrow[rr, "\varphi_3"] & & P_2 \arrow[u, "j_2"'] \\
\vdots & & \vdots
\end{tikzcd}$$

(*where $j_\ell : P_\ell \to P_{\ell-1}$ and $j'_\ell : M_\ell \to M_{\ell-1}$ are the natural embeddings of the respective constraint submanifolds*). *Moreover, if there is a final constraint submanifold $M_f := M_k \subset M_1$ (for some $k \geqslant 2$) on which there exists a consistent solution $Z$ of* (7), *then $Z$ projects under $\varphi_k$ onto a solution of* (3) *on the final constraint submanifold $P_f := P_{k-1}$ in $J^1\pi$ and, conversely, any solution of* (3), *defined on $P_f$, is the projection of a vector field on $M_f$ which is a solution of* (7).

The proof of this proposition essentially relies on the following two facts. First of all, the projection $\mathrm{pr}_2 : T^*E \times_E J^1\pi \to J^1\pi$ has the appropriate "almost regularity" properties, that is: (i) $\mathrm{pr}_2$ is a surjective submersion, and (ii) the fibres of this submersion are connected submanifolds of $T^*E \times_E J^1\pi$, being diffeomorphic to $\mathbb{R}^{n+1}$. And, secondly, a straightforward computation shows that

$$
\begin{aligned}
\left(\mathrm{pr}_2^* \omega_L\right)(x) &= \left(dq^A \wedge d\left(\frac{\partial L}{\partial \dot q^A}\right) + dE_L \wedge dt\right)(x) \\
&= \left(dq^A \wedge d\left(\frac{\partial L}{\partial \dot q^A}\right) + \dot q^A d\left(\frac{\partial L}{\partial \dot q^A}\right) \wedge dt \right.\\
&\quad \left. - \left(\frac{\partial L}{\partial q^A}\right) dq^A \wedge dt\right)(x) \\
&= \omega_{\mathcal{H}}(x),
\end{aligned}
$$

and, clearly, we also have $\mathrm{pr}_2^* \tilde\eta = \eta$. Herewith, the proof of Proposition 3.2 can be easily completed, following the same reasoning as in the autonomous case [2].

The solution generated by the constraint algorithm (if it exists) is not unique. On the other hand, we may observe that if the given dynamical problem (7) admits a consistent solution $Z$ on a final constraint submanifold $M_f$ then, by construction, its projection onto $J^1\pi$ will automatically verify the second-order equation condition along a submanifold of $P_f$. This is again in full analogy with the situation encountered in the autonomous case.

Next, assume that the given Lagrangian $L \in C^\infty(J^1\pi)$ is almost regular in the following sense: (i) $\mathrm{Leg}_L(J^1\pi)$ is a submanifold of $T^*E$, (ii) $\mathrm{Leg}_L$, regarded as a map from $J^1\pi$ onto its image, is a submersion with connected fibres, (iii) $\mathrm{leg}_L^{-1}(\mathrm{leg}_L(x))$ is a connected set for all $x \in J^1\pi$. In [15] it has been shown that, with these assumptions, one can develop a constraint algorithm on $J^1\pi^*$ which is equivalent to the one on $J^1\pi$. Again as in the autonomous case (see [2]), one can demonstrate that a solution of the constrained analysis on $J^1\pi^*$ can be related to a solution $Z$ of (7), defined on the final constraint submanifold $M_f$. This connection is established by choosing a suitable (local) section $\sigma$ of the projection $\nu \circ \mathrm{pr}_1 : T^*E \times_E J^1\pi \to J^1\pi^*$ and restricting $Z$ to $\mathrm{Im}(\sigma) \cap M_f$ (recall that $\nu$ is the canonical projection of $T^*E$ onto $J^1\pi^*$).

## 4. Conclusions

We have developed a non-autonomous version of the Skinner–Rusk approach to (Lagrangian) mechanics and have shown that, both in the regular and in the singular case, this yields a first-order system on the fibred product $T^*E \times_E J^1\pi$ which encodes all the information of the dynamics of the system under consideration. This approach to time-dependent mechanics possesses the same virtues as in the autonomous case, such as the fact that the "Hamiltonian" $\mathcal{H}$ is defined without having to solve the relations $p_A = \partial L/\partial \dot q^A$ for (some of) the velocities.

Within the above framework for the description of time-dependent mechanics, there are several lines of investigation that seem to be worth pursuing, such as: the role and the nature of gauge transformations in the case of singular Lagrangians and the general study of symmetries of (time-dependent) mechanical systems. In addition, it would certainly be of interest to use this formalism for establishing a geometric formulation of optimal control problems with an explicit time-dependence, such as in the case of time-dependent vakonomic dynamics, thereby generalizing the work presented in [7,8].

## Acknowledgements

This research has been partially supported by a FPI grant from the Spanish MCYT, by grant DGICYT PGC2000-2191-E and by the European Union through the Training and Mobility of Researchers Program ERB FMRXCT-970137. F.C. also wishes to acknowledge support from a research grant of the "Bijzonder Onderzoeksfonds" of Ghent University.

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