Discrete Mathematics 312 (2012) 2800–2818

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Pairs of forbidden induced subgraphs for homogeneously traceable graphs

Binlong Li [a,b], Hajo Broersma [a,*], Shenggui Zhang [b]

[a] Faculty of EEMCS, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

[b] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China

ARTICLE INFO

Article history:
Received 9 January 2012
Received in revised form 25 May 2012
Accepted 27 May 2012

Keywords:
Induced subgraph
Forbidden subgraph
Homogeneously traceable graph
Hamiltonian graph

ABSTRACT

A graph G is called homogeneously traceable if for every vertex v of G, G contains a Hamilton path starting from v. For a graph H, we say that G is H-free if G contains no induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. Determining families of graphs $\mathcal{H}$ such that every $\mathcal{H}$-free graph G has some graph property has been a popular research topic for several decades, especially for Hamiltonian properties, and more recently for properties related to the existence of graph factors. In this paper we give a complete characterization of all pairs of connected graphs R, S such that every 2-connected $\{R, S\}$-free graph is homogeneously traceable.

1. Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or a subgraph of G induced by $V(G')$). For a given graph H, we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

The only graph on four vertices with degree sequence 1, 1, 1, 3 is denoted as $K_{1,3}$ and called a *claw*; the vertex with degree 3 is called the *center* of the claw. Instead of $K_{1,3}$-free, we say that a graph is *claw-free* if it does not contain a copy of $K_{1,3}$ as an induced subgraph. For a subgraph H of G, the vertices with degree 1 in H are called its *end vertices*.

Let P_i be the path on $i \geq 1$ vertices, and C_i the cycle on $i \geq 3$ vertices. We use Z_i to denote the graph obtained by identifying a vertex of a C_3 with an end vertex of a P_{i+1} $(i \geq 1)$, $B_{i,j}$ for the graph obtained by identifying two vertices of a C_3 with the end vertices of a P_{i+1} $(i \geq 1)$ and a P_{j+1} $(j \geq 1)$, respectively, and $N_{i,j,k}$ for the graph obtained by identifying the three vertices of a C_3 with the end vertices of a P_{i+1} $(i \geq 1)$, P_{j+1} $(j \geq 1)$ and P_{k+1} $(k \geq 1)$, respectively. In particular, we let $B = B_{1,1}$ (this graph is sometimes called a *bull*) and $N = N_{1,1,1}$ (this graph is sometimes called a *net*). The graphs $B_{1,4}$, $B_{2,3}$ and $N_{1,1,3}$ play a crucial role in the sequel, and are depicted in Fig. 1.

Adopting the terminology of [3], we call a graph G *Hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all its vertices, *traceable* if it contains a *Hamilton path*, i.e., a path containing all its vertices, and *Hamilton-connected* if for every pair of vertices x, y of G, G contains a Hamilton path starting from x and terminating in y. We say that G is *homogeneously traceable* if for every vertex x of G, G contains a Hamilton path starting from x. Homogeneously traceable graphs have been introduced by Skupień (see, e.g., [10]), but we do not know whether he is the original author of the concept.

* Corresponding author.
E-mail address: h.j.broersma@utwente.nl (H. Broersma).

0012-365X/$ – see front matter © 2012 Published by Elsevier B.V.
doi:10.1016/j.disc.2012.05.018

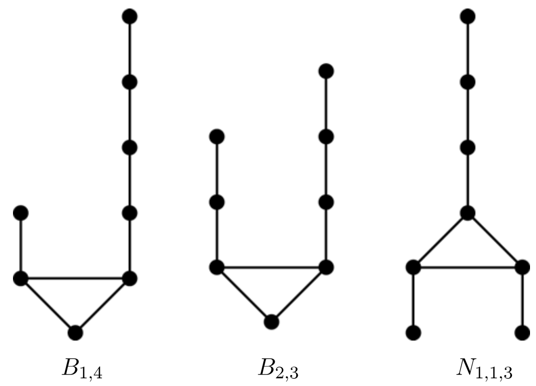

Fig. 1. The graphs $B_{1,4}$, $B_{2,3}$ and $N_{1,1,3}$.

Note that a Hamilton-connected graph (on at least three vertices) is Hamiltonian, that a Hamiltonian graph is homogeneously traceable, and that a homogeneously traceable graph is traceable, but that the reverse statements do not hold in general.

If a graph is connected and P_3-free, then it is a *complete graph*, i.e., its vertex set is a *clique*, i.e., all its vertices are mutually adjacent, and hence it is (homogeneously) traceable, and Hamiltonian if it has order at least 3. In fact, it is not hard to show that the statement 'every connected H-free graph is traceable' only holds if $H = P_3$ (or $H = P_2$, but in that case the statement is trivial). The case with pairs of forbidden subgraphs (different from P_2 and P_3) is much more interesting. For a connected graph to be traceable or Hamiltonian, the following theorem is one of the earliest of this kind.

Theorem 1 (*Duffus et al. [4]*). *Let G be a $\{K_{1,3}, N\}$-free graph.*

(1) *If G is connected, then G is traceable.*
(2) *If G is 2-connected, then G is Hamiltonian.*

Obviously, if H is an induced subgraph of N, then $\{K_{1,3}, H\}$-free instead of $\{K_{1,3}, N\}$-free yields the same conclusions in the above theorem. In particular, if we exclude P_2 as an induced subgraph, we consider graphs without edges, and we obtain trivial statements only. For this reason, throughout we assume that our forbidden subgraphs have at least three vertices. We also assume that our forbidden subgraphs are connected. A natural problem that, as far as we know, was considered for the first time in the Ph.D. Thesis of Bedrossian [2], is to characterize all pairs of forbidden subgraphs for hamiltonicity (and other graph properties). Faudree and Gould [6] later refined this approach by adding a lower bound on the number of vertices of the graph G in order to avoid small, more or less pathological, cases. Restricting our attention to traceability, they proved that (apart from trivial cases) the claw and any of the induced subgraphs of the net are the only forbidden pairs for the property of being traceable.

Theorem 2 (*Faudree and Gould [6]*). *Let R and S be connected graphs with $R, S \neq P_2, P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is P_4, C_3, Z_1, B or N.*

In the same paper, they discuss analogous results for other Hamiltonian properties. For many of these properties counterparts of Theorem 2 have been established, but for Hamilton-connectedness only partial results are known to date. We refer to [6] for more details. The property of being homogeneously traceable was not addressed in [6] and, as far as we are aware, has not been considered before. Recently, similar questions related to the existence of perfect matchings and 2-factors have been studied. We refer the interested reader to [8,9,1,5,7], respectively, for more details.

In the sequel we solve the analogous problem for homogeneously traceable graphs, so we are going to characterize the pairs of connected forbidden induced subgraphs that imply that a given graph is homogeneously traceable. Note that if a graph contains a cut vertex v, it cannot be homogeneously traceable since there exists no Hamilton path starting at v. So, apart from K_1 and K_2, all homogeneously traceable graphs are 2-connected. Thus we only consider 2-connected graphs. As noted before, if a connected graph G is P_3-free, then it is a complete graph, and hence trivially homogeneously traceable, and in fact it is easy to prove the following statement. We postpone the proof of the 'only-if' part of the next statement to Section 3.

Theorem 3. *Let $S \neq P_2$ be a connected graph and let G be a 2-connected graph. Then G being S-free implies G is homogeneously traceable if and only if $S = P_3$.*

A natural and more interesting problem is to consider pairs of forbidden subgraphs for this property. In this paper, we characterize all such pairs by proving the following result.

Theorem 4. *Let R and S be connected graphs with $R, S \neq P_2, P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$.*

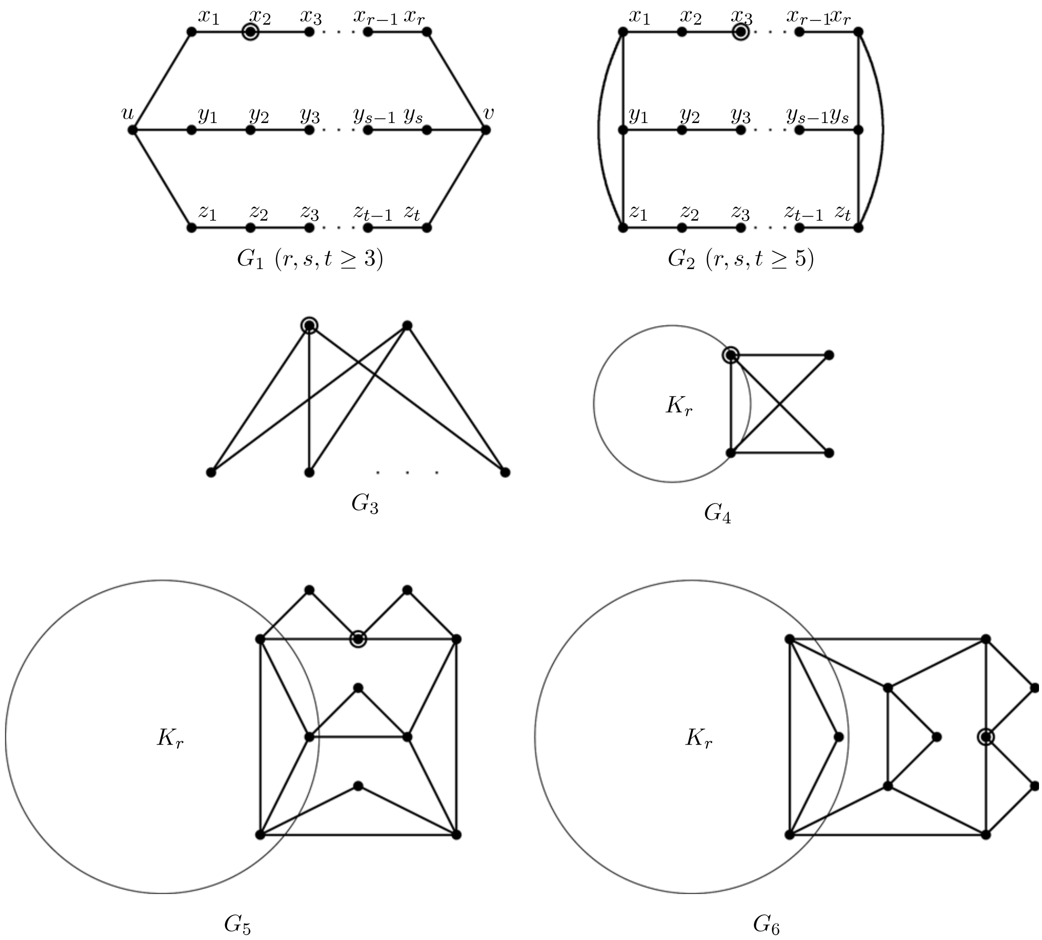

Fig. 2. Some graphs that are not homogeneously traceable.

In Section 2, we prove the 'only-if' part of the statements of Theorems 3 and 4, while the 'if' part of the statement of Theorem 4 is deduced from the following three theorems that will be proved in Sections 5–7, respectively.

Let G be a 2-connected graph.

Theorem 5. *If G is $\{K_{1,3}, B_{1,4}\}$-free, then G is homogeneously traceable.*

Theorem 6. *If G is $\{K_{1,3}, B_{2,3}\}$-free, then G is homogeneously traceable.*

Theorem 7. *If G is $\{K_{1,3}, N_{1,1,3}\}$-free, then G is homogeneously traceable.*

Section 4 contains the common set-up for the proofs of the above three theorems and some common preliminary observations. We present some general observations on claw-free graphs in Section 3.

2. The 'only-if' part of the statements of Theorems 3 and 4

We first sketch some families of graphs that are not homogeneously traceable (see Fig. 2). In each of the graphs in Fig. 2 we indicated one of the vertices by a double circle; it is easy to check that this vertex cannot be the starting vertex of a Hamilton path. When we say that a graph is of *type* G_i we mean that it is one particular, but arbitrarily chosen member of the family indicated by G_i in Fig. 2.

If $S \neq P_2$ is a connected graph such that every 2-connected S-free graph is homogeneously traceable, then S must be a common induced subgraph of all graphs of type G_1, G_2 and G_3. Note that the largest common induced connected subgraph of graphs of type G_1, G_2 and G_3 is a P_3, so we have that $S = P_3$. This completes the proof of the 'only-if' part of the statement of Theorem 3.

Let R and S be two connected graphs other than P_2, P_3 such that every 2-connected $\{R, S\}$-free graph is homogeneously traceable. Then R or S must be an induced subgraph of all graphs of type G_1. Without loss of generality, we assume that R is an induced subgraph of a graph of type G_1. If $R \neq K_{1,3}$, then R must contain an induced P_4. Note that the graphs of type G_3 and G_4 are all P_4-free, so they must contain S as an induced subgraph. Since the only common induced connected subgraph of the graphs of type G_3 and G_4 other than P_3 is a $K_{1,3}$, we have that $S = K_{1,3}$. This implies that R or S must be a $K_{1,3}$.

Let $R = K_{1,3}$. Note that the graphs of type G_2 are claw-free, so S must be an induced connected subgraph of all graphs of type G_2. The common induced connected subgraphs of such graphs have the form P_i, Z_i, $B_{i,j}$ or $N_{i,j,k}$. Note that graphs of type G_5 are claw-free and do not contain an induced P_8, Z_5 or $N_{1,1,4}$, and that graphs of type G_6 are claw-free and do not contain an induced $N_{1,2,2}$. So R must be an induced connected subgraph of P_7, Z_4, $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$. Since P_7 and Z_4 are induced subgraphs of $B_{1,4}$, R must be an induced connected subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$. This completes the proof of the 'only-if' part of the statement of Theorem 4.

3. Preliminaries and general observations

Let G be a graph. For a subgraph H of G, when no confusion can arise we also use H to denote the vertex set of H; and similarly, for a subset S of $V(G)$, we also use S to denote the subgraph of G induced by S. For two vertices u and v of G, we use $d_H(u, v)$ to denote the *distance* between u and v in H, i.e., the length of a shortest path between u and v with all edges in H.

We first prove some easy but useful observations on claw-free graphs.

Lemma 1. *Let G be a 2-connected claw-free graph and let $\{x, y\}$ be a vertex cut of G. Then the following statements hold:*

(1) *$G - \{x, y\}$ has exactly two components;*
(2) *if x_1 and x_2 are two neighbors of x in the same component of $G - \{x, y\}$, then $x_1x_2 \in E(G)$.*

Proof. Note that each component H of $G - \{x, y\}$ contains a neighbor of x; otherwise y is a cut vertex of G, a contradiction.

If there are at least three components of $G - \{x, y\}$, then let H_1, H_2 and H_3 be three such components. Let x_1, x_2 and x_3 be neighbors of x in H_1, H_2 and H_3, respectively. Then the subgraph induced by $\{x, x_1, x_2, x_3\}$ is a claw, a contradiction. Thus we conclude that $G - \{x, y\}$ has exactly two components.

Let x_1 and x_2 be two neighbors of x in the same component of $G - \{x, y\}$. If $x_1x_2 \notin E(G)$, then let x' be a neighbor of x in the other component of $G - \{x, y\}$. Then the subgraph induced by $\{x, x_1, x_2, x'\}$ is a claw, a contradiction. Thus we have $x_1x_2 \in E(G)$. □

Throughout the remainder of this paper, by the word *cut* we will always refer to a vertex cut with exactly two vertices.

We say that two disjoint subsets or subgraphs S and T of G are *joined* if at least one vertex of S is adjacent to a vertex of T in G.

Let B and C be two subgraphs of G (possibly not disjoint), and let H be a subgraph of G that is disjoint from B and C. If P is a path with one end vertex x in B, one end vertex y in C, and its internal vertex set $V(P) \setminus \{x, y\} = V(H)$, then we call *$P$ a perfect path of H to B and C (in G)* and we say that H *supports* a perfect path to B and C; if $B = C$, then we call *P a perfect path of H to B (in G)* and we say that H *supports* a perfect path to B.

We will frequently use the following argumentation in the next sections. Let H be a 2-connected claw-free subgraph of G, and let r, s be a pair of distinct vertices of H. Then $H - s$ is a connected graph. We consider the neighborhood structure of r in $H - s$ by defining, for integers $i = 0, 1, \ldots,$

$$N_i(r) = \{u \in V(H - s) : d_{H-s}(u, r) = i\} \quad \text{and} \quad j = \max\{i : N_i(r) \neq \emptyset\}.$$

For a vertex $v \in N_i(r)$, the index i is referred to as the *level* of v. If these neighorhoods are complete or 'nearly' complete, we can deduce the existence of a Hamilton path of H between r and s, as follows.

Lemma 2. *Let H be a 2-connected claw-free graph, let r and s be a pair of distinct vertices of H, and let $N_i(r)$ and j be as defined above. Suppose there is an integer j' with $1 \le j' \le j$, such that*

(1) *for every i with $1 \le i \le j'$, $N_i(r)$ is a clique;*
(2) *$N(s) \setminus \{r\}$ is a clique; and*
(3) *$j' = j$, or for every component C of $\bigcup_{i=j'+1}^{j} N_i(r)$: if s is not adjacent to a vertex of C, then C supports a perfect path to $N_{j'}(r)$; if s is adjacent to a vertex of C, then C supports a perfect path to $N_{j'}(r)$ and s.*

Then there is a Hamilton path of H between r and s.

Proof. For convenience we let N_i denote $N_i(r)$ throughout this proof.

If $j' \le j - 1$, then let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be the set of components of $\bigcup_{i=j'+1}^{j} N_i$. For every i with $1 \le i \le k$, if s is not adjacent to a vertex of H_i, then let R_i be a perfect path of H_i to $N_{j'}$, and let y_i, y_i' be the two end vertices of R_i; if s is adjacent to a vertex of H_i, then let R_i be a perfect path of H_i to $N_{j'}$ and s, and let y_i be the end vertex of R_i other than s.

If two components H_i and $H_{i'}$ have a common neighbor y in $N_{j'}$, then let z be a neighbor of y in H_i, let z' be a neighbor of y in $H_{i'}$, and let x be a neighbor of y in $N_{j'-1}$. Then the subgraph induced by $\{y, x, z, z'\}$ is a claw, a contradiction. This implies that any two perfect paths R_i and $R_{i'}$ have no common end vertices in $N_{j'}$; since $N(s) \setminus \{r\}$ is a clique, R_i and $R_{i'}$ cannot have s as a common end vertex either.

Note that $N_0 = \{r\}$. Let $s' \in N_{j''} \setminus \{r\}$ be a neighbor of s such that its level j'' is as large as possible, where $1 \le j'' \le j$ (such a vertex exists since H is 2-connected).

We prove the following five claims in order to show that there is a Hamilton path of H between r and s.

Claim 1. If $j'' \le j' - 1$, then $\bigcup_{i=j'}^{j} N_i$ supports a perfect path to $N_{j'-1}$.

Proof. We first assume that $j' = j$. If N_j has only one vertex x, then by the 2-connectedness of H, x has at least two neighbors in N_{j-1}. Let w, w' be two neighbors of x in N_{j-1}. Then $R = wxw'$ is a perfect path of N_j to N_{j-1}.

If N_j has at least two vertices, then by the 2-connectedness of H, N_j is joined to N_{j-1} by (at least) two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in N_j$ and $w, w' \in N_{j-1}$. Let R' be a Hamilton path of (the clique) N_j from x to x'. Then $R = wxR'x'w'$ is a perfect path of N_j to N_{j-1}.

Thus we assume that $j' \leq j - 1$. By the 2-connectedness of H, $N_{j'}$ is joined to $N_{j'-1}$ by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in N_{j'}$ and $w, w' \in N_{j'-1}$.

We first assume that one vertex of x and x' is not an end vertex of some perfect path. Without loss of generality, we assume that x is not an end vertex of some perfect path. If x' is also not an end vertex of some perfect path, then let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\} \setminus \{x'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k'x'w'$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

If x' is an end vertex of some perfect path, then without loss of generality, we assume that $x' = y_k'$. Let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k'w'$ is a perfect path of $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

Suppose now that both x and x' are end vertices of some perfect paths. If there is a vertex x'' in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$, then let w'' be a neighbor of x'' in $N_{j'-1}$. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining $N_{j'}$ to $N_{j'-1}$ such that x'' is not an end vertex of some perfect path. By the previous arguments, we can find a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$. So we assume that there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$.

If x and x' are end vertices of two distinct perfect paths, then without loss of generality, we assume that $x = y_1$ and $x' = y_k'$. Then $R = wy_1R_1y_1' \cdots y_kR_ky_k'w'$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

Suppose now that x and x' are the two end vertices of a common perfect path. If there is a second perfect path, then let x'' be an end vertex of a second perfect path and w'' be a neighbor of x'' in $N_{j'-1}$. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining $N_{j'}$ to $N_{j'-1}$ such that x and x'' are end vertices of two distinct perfect paths. By the previous arguments, we can find a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$.

So finally we assume that there is only one perfect path R_1. Without loss of generality, we assume that $x = y_1$ and $x' = y_1'$. Then $R = wy_1R_1y_1'w'$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$. □

Claim 2. If $j'' \leq j' - 1$, then for every i with $j'' + 1 \leq i \leq j'$, $\bigcup_{i'=i}^{j} N_{i'}$ supports a perfect path to N_{i-1}.

Proof. We prove the claim by induction on $j' - i$.

If $i = j'$, then by Claim 1, $\bigcup_{i'=j'}^{j} N_{i'}$ supports a perfect path to $N_{j'-1}$. Thus we assume that $j'' + 1 \leq i \leq j' - 1$.

By the induction hypothesis, there is a perfect path R' supported by $\bigcup_{i'=i+1}^{j} N_{i'}$ to N_i. Let y and y' be the two end vertices of R'.

By the 2-connectedness of H, N_i is joined to N_{i-1} by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in N_i$ and $w, w' \in N_{i-1}$.

We first assume that x, x' and y, y' are two distinct pairs. Without loss of generality, we assume that $x \neq y, y'$. If $x' \neq y, y'$, then let T be a path of N_i from x to y passing through all the vertices in $N_i \setminus \{x', y'\}$. Then $R = wxTyR'y'x'w'$ is a perfect path supported by $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}; if $x' = y$ or y', then without loss of generality, we assume that $x' = y'$. Let T be a path of N_i from x to y passing through all the vertices in $N_i \setminus \{x'\}$. Then $R = wxTyR'x'w'$ is a perfect path supported by $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}.

Suppose now that x, x' and y, y' are the same pair.

If there is a third vertex x'' in N_i other that x and x', then let w'' be a neighbor of x'' in N_{i-1}. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining N_i to N_{i-1} such that x, x'' and y, y' are two distinct pairs. By the previous arguments, we can find a perfect path supported by $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}.

Finally we assume that there are only the two vertices x and x' in N_i. Then $R = wxR'x'w'$ is a perfect path supported by $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1}. □

Claim 3. If $j'' \leq j' - 1$, then $\bigcup_{i=j''}^{j} N_i$ supports a perfect path to $N_{j''-1}$ and s.

Proof. By Claim 2, there is a perfect path R' supported by $\bigcup_{i=j''+1}^{j} N_i$ to $N_{j''}$. Let y and y' be the two end vertices of R'.

We first assume that there is a vertex x in $N_{j''}$ other than y, y' and s'. Let w be a neighbor of x in $N_{j''-1}$. If $s' \neq y, y'$, then let T be a path of $N_{j''}$ from x to y passing through all the vertices in $N_{j''} \setminus \{y', s'\}$. Then $R = wxTyR'y's's$ is a perfect path supported by $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s; if $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let T be a path of $N_{j''}$ from x to y passing through all the vertices in $N_{j''} \setminus \{y'\}$. Then $R = wxTyR'y's$ is a perfect path supported by $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s.

Suppose now that there are no vertices in $N_{j''}$ other than y, y' and s'. If $s' \neq y, y'$, then let w be a neighbor of y in $N_{j''-1}$. Then $R = wyR'y's's$ is a perfect path supported by $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s; if $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let w be a neighbor of y in $N_{j''-1}$. Then $R = wyR'y's$ is a perfect path supported by $\bigcup_{i=j''}^{j} N_i$ to $N_{j''-1}$ and s. □

Claim 4. If $j'' \geq j'$, then $\bigcup_{i=j'}^{j} N_i$ supports a perfect path to $N_{j'-1}$ and s.

Proof. We first assume that $j' = j$, and thus $j'' = j$. If N_j consists of the vertex s', then let w be a neighbor of s' in N_{j-1}. Then $R = ws's$ is a perfect path supported by N_j to N_{j-1} and s; if N_j contains at least two vertices, then let x be a vertex in N_j other than s', let w be a neighbor of x in N_{j-1}, and let R' be a Hamilton path of N_j from x to s'. Then $R = wxR's's$ is a perfect path supported by N_j to N_{j-1} and s.

Next we assume that $j' \leq j - 1$.

First we assume that s is not adjacent to any vertex in $\mathcal{H}$. Then s' is a neighbor of s in $N_{j'}$.

We first treat the case that s' is not an end vertex of some perfect path. If there is a vertex x in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\} \cup \{s'\}$, then let w be a neighbor of x in $N_{j'-1}$, and let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\} \setminus \{s'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k's's$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s; if there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\} \cup \{s'\}$, then let w be a neighbor of y_1 in $N_{j'-1}$. Then $R = wy_1R_1y_1' \cdots y_kR_ky_k's's$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s.

Next we treat the case that s' is an end vertex of some perfect path. Without loss of generality, we assume that $s' = y_k'$. If there is a vertex x in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$, then let w be a neighbor of x in $N_{j'-1}$, and let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k}\{y_i, y_i'\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_ky_k's$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s; if there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k}\{y_i, y_i'\}$, then let w be a neighbor of y_1 in $N_{j'-1}$. Then $R = wy_1R_1y_1' \cdots y_kR_ky_k's$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s.

Suppose now that s is adjacent to a vertex of some component of $\mathcal{H}$. Note that $N(s) \setminus \{r\}$ is a clique and that s is adjacent to at most one component of $\mathcal{H}$. Without loss of generality, we assume that s is adjacent to a vertex of H_k, and thus s is the end vertex of R_k other than y_k. If there is a vertex x in $N_{j'}$ other than $\bigcup_{i=1}^{k-1}\{y_i, y_i'\} \cup \{y_k\}$, then let w be a neighbor of x in $N_{j'-1}$, and let T be a path of $N_{j'}$ from x to y_1 passing through all the vertices in $N_{j'} \setminus \bigcup_{i=1}^{k-1}\{y_i, y_i'\} \setminus \{y_k\}$. Then $R = wxTy_1R_1y_1' \cdots y_kR_k$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s; if there are no vertices in $N_{j'}$ other than $\bigcup_{i=1}^{k-1}\{y_i, y_i'\} \cup \{y_k\}$, then let w be a neighbor of y_1 in $N_{j'-1}$. Then $R = wy_1R_1y_1' \cdots y_kR_k$ is a perfect path supported by $\bigcup_{i=j'}^{j} N_i$ to $N_{j'-1}$ and s. $\square$

Claim 5. For every i with $1 \leq i \leq \min\{j', j''\}$, $\bigcup_{i'=i}^{j} N_{i'}$ supports a perfect path to N_{i-1} and s.

Proof. We prove the claim by induction on $\min\{j', j''\} - i$.

If $i = \min\{j', j''\}$, then by Claims 3 and 4, $\bigcup_{i'=i}^{j} N_{i'}$ supports a perfect path to N_{i-1} and s. Thus we assume that $1 \leq i \leq \min\{j', j''\} - 1$.

By the induction hypothesis, there is a perfect path R' supported by $\bigcup_{i'=i+1}^{j} N_{i'}$ to N_i and s. Let y be the end vertex of R' other than s.

If there is a second vertex x in N_i other than y, then let w be a neighbor of x in N_{i-1}, and let T be a Hamilton path of N_i from x to y. Then $R = wxTyR'$ is a perfect path supported by $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1} and s.

Thus we assume that N_i consists of the vertex y. Let w be a neighbor of y in N_{i-1}. Then $R = wyR'$ is a perfect path supported by $\bigcup_{i'=i}^{j} N_{i'}$ to N_{i-1} and s. $\square$

Taking $i = 1$ in Claim 5, we conclude that there exists a Hamilton path of H from r to s. This completes the proof of Lemma 2. $\square$

4. A common set-up for the proofs of Theorems 5–7

The three proofs are modeled along the same lines and use the same case distinctions. To avoid too much repetition of the arguments we give the generic set-up for all three proofs and treat some of the subcases simultaneously in this section.

Let G be a 2-connected $\{K_{1,3}, F\}$-free graph, where $F = B_{1,4}, B_{2,3}$ or $N_{1,1,3}$. We are going to prove that G is homogeneously traceable by induction on $|V(G)|$. If $|V(G)| = 3$, the result is trivially true. So we assume that $|V(G)| \geq 4$ and that the statement holds for any 2-connected $\{K_{1,3}, F\}$-free graph with order $n < |V(G)|$.

Let v be an arbitrary vertex of G. It is sufficient to prove that G contains a Hamilton path starting from v.

If $G - v$ is 2-connected, then we consider a neighbor u of v in G. By the induction hypothesis, $G - v$ contains a Hamilton path P starting from u. Then vuP is a Hamilton path of G starting from v, and the statement holds.

So we assume that $G - v$ is *separable*, i.e., has a cut vertex. We consider the *blocks* of $G - v$, i.e., the maximal subgraphs of $G - v$ that do not have a cut vertex, so these blocks are either isomorphic to K_2 or 2-connected. We say that a block is *trivial* if it is isomorphic to K_2. An *end block* is a block containing exactly one cut vertex of $G - v$; the other blocks are called *inner blocks*. Except for the cut vertex, all other vertices of an end block are called *inner vertices*.

Note that every end block of $G - v$ contains an inner vertex adjacent to v, and that $G - v$ has at least two end blocks. Since G is claw-free, we deduce that there are exactly two end blocks of $G - v$. This implies that the $p + 1 \geq 2$ blocks of $G - v$ can be denoted as $B_0, B_1, B_2, \ldots, B_p$ with cut vertices s_i, $1 \leq i \leq p$, of $G - v$ common to B_{i-1} and B_i, and s_0 and s_{p+1} two neighbors of v contained in $B_0 - s_1$ and $B_p - s_p$, respectively.

We distinguish two main cases: there is a nontrivial inner block or all inner blocks are trivial. In the former case we need basically separate approaches except if we assume another nontrivial block. We complete this section by first treating the common subcase that there is a nontrivial inner block and another nontrivial block. We also give some generic observations for the other subcases and treat the subcase that all inner blocks are trivial simultaneously. The other subcases are treated in detail separately in Sections 5–7.

The case with a nontrivial inner block and another nontrivial block

Suppose B_q is a nontrivial inner block, where $1 \leq q \leq p-1$. Here we deal with the subcase that there is another nontrivial block B_r (either inner or end block). In this case, we only need the induction hypothesis. Let Q_q be a shortest path in B_q from s_q to s_{q+1}, and Q_r a shortest path in B_r from s_r to s_{r+1}. Since B_q (B_r) is nontrivial and 2-connected, Q_q (Q_r) must miss some vertices in B_q (B_r). Let G_q be the subgraph induced by $V(G-B_q) \cup V(Q_q)$, and let G_r be the subgraph induced by $V(G-B_r) \cup V(Q_r)$. By the induction hypothesis, G_r contains a Hamilton path H_r starting from v. Clearly s_q and s_{q+1} are two cut vertices of $G_r - v$, so the subpath Q'_q of H_r from s_q to s_{q+1} is a Hamilton path of B_q. Similarly, G_q contains a Hamilton path H_q starting from v, and Q_q is the subpath of H_q from s_q to s_{q+1}. Let P be the path obtained from H_q by replacing Q_q by Q'_q. Then P is a Hamilton path of G starting from v, and the statement holds.

This completes the proof for Theorems 5–7 in case $G - v$ contains a nontrivial inner block and another nontrivial (inner or end) block.

The case with one nontrivial inner block and all other blocks trivial

Next we assume that all the blocks of $G - v$ other than B_q are trivial. Then the structure of the blocks implies that it is sufficient to show that there exists a Hamilton path in B_q between s_q and s_{q+1}. The subcases can be treated by first analyzing the structure of the neighborhoods of s_q in $B_q - s_{q+1}$ and then using Lemma 2.

Set

$$N_i = \{u \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(u, s_q) = i\}, \quad \text{and} \quad j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

Recall that B_q is nontrivial, hence it is 2-connected. First we prove the following easy common observation.

Observation 1. *$N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique.*

Proof. If there are two neighbors x and x' of s_q in B_q such that $xx' \notin E(G)$, then the subgraph induced by $\{s_q, s_{q-1}, x, x'\}$ is a claw, a contradiction. Similarly we can prove that $N_{B_q}(s_{q+1})$ is a clique. □

Note that Observation 1 implies that N_1 is a clique. To analyze the structure of the other N_i we use slightly different arguments depending on the forbidden subgraph F. Although there is a lot of commonality, in Sections 5–7 we use the above set-up and notation, and treat the subcase that the inner block B_q is nontrivial and all other blocks are trivial separately for Theorems 5–7.

In the three different proofs for this subcase, we will implicitly prove the following technical lemma. We state it here already because we want to apply it in the next subcase as well. It will be clear from Sections 5–7 that the proof of this lemma is different for the different choices of the forbidden subgraph F, and that it would have been a bad idea to include the proof at this point.

Lemma 3. *Let G be a 2-connected $\{K_{1,3}, F\}$-free graph, where $F = B_{1,4}, B_{2,3}$ or $N_{1,1,3}$. Let H be an induced 2-connected subgraph of G, and let r, s be a pair of distinct vertices of H. Suppose:*

(1) *$N_H(r)$ is a clique;*
(2) *$N_H(s) \setminus \{r\}$ is a clique;*
(3) *there is an induced path P in G of length at least 3 with origin r, with $V(P) \cap V(H) = \{r\}$, and such that in G there are no edges joining $V(H) \setminus \{s\}$ and $V(P)$ except the first edge of P;*
(4) *if the distance between r and s in H is at least 4, there is a neighbor of r outside H that is nonadjacent to $V(H) \setminus \{r\}$.*

Then H has a Hamilton path between r and s.

The case that all inner blocks are trivial

In the final case we assume that all inner blocks of $G - v$ are trivial. If $p \geq 2$, we let Q be the (unique) path from s_1 to s_p with all internal vertices outside $B_0 \cup B_p$; if $p = 1$, we let Q consist of s_1. We recall that B_0 is either trivial or 2-connected. Using the induction hypothesis in the latter case, this implies that there is a Hamilton path in B_0 starting from s_1. Similarly, there is a Hamilton path in B_p starting from s_p. If there exists a Hamilton path in $B_0 \cup \{v\}$ from v to s_1, then combining it with Q (if $p \geq 2$) and the Hamilton path in B_p starting from s_p, we obtain a Hamilton path in G starting from v. By symmetry, it is sufficient to prove the claim that there is a Hamilton path in $B_0 \cup \{v\}$ from v to s_1 or a Hamilton path in $B_p \cup \{v\}$ from v to s_p.

If B_0 or B_p is trivial, then the claim clearly holds. So we assume that neither B_0 nor B_p is trivial.

If v has only one neighbor s_0 in B_0, then let $B_0' = B_0$ and $r_0 = s_0$; otherwise let B_0' be the subgraph induced by $B_0 \cup \{v\}$ and let $r_0 = v$. Analogously, if v has only one neighbor s_{p+1} in B_p, then let $B_p' = B_p$ and $r_{p+1} = s_{p+1}$; otherwise let B_p' be the subgraph induced by $B_p \cup \{v\}$ and let $r_{p+1} = v$. Now it is sufficient to prove that B_0' contains a Hamilton path from r_0 to s_1, or B_p' contains a Hamilton path from r_{p+1} to s_p.

By our choice of B_0' and B_p', we have that B_0' and B_p' are both 2-connected. Moreover, we can prove the following two observations by only using the claw-freeness of G.

Observation 2. *$N_{B_0'}(r_0) \setminus \{s_1\}$, $N_{B_0'}(s_1) \setminus \{r_0\}$, $N_{B_p'}(s_p) \setminus \{r_{p+1}\}$ and $N_{B_p'}(r_{p+1}) \setminus \{s_p\}$ are all cliques.*

Proof. Suppose that $N_{B_0'}(r_0) \setminus \{s_1\}$ is not a clique. Let x, x' be two neighbors of r_0 in $B_0' - s_1$ that are nonadjacent. If $r_0 = v$, then the subgraph induced by $\{v, s_{p+1}, x, x'\}$ is a claw, a contradiction. If $r_0 = s_0$, then the subgraph induced by $\{s_0, v, x, x'\}$ is a claw, a contradiction.

The other assertions can be proved in a similar way. □

Observation 3. *$N_{B_0'}(r_0)$ or $N_{B_p'}(r_{p+1})$ is a clique. Moreover, if $r_0 s_1 \notin E(G)$ or $r_{p+1} s_p \notin E(G)$, then both $N_{B_0'}(r_0)$ and $N_{B_p'}(r_{p+1})$ are cliques.*

Proof. Suppose that $N_{B_0'}(r_0)$ is not a clique. Let x, x' be two neighbors of r_0 in B_0' that are nonadjacent. By Observation 2, either $x = s_1$ or $x' = s_1$. Without loss of generality, we assume that $x' = s_1$.

If $r_0 = s_0$, then by our choice of B_0', $vs_1, vx \notin E(G)$ and the subgraph induced by $\{s_0, v, x, s_1\}$ is a claw, a contradiction. Thus $r_0 = v$. If $s_1 s_{p+1} \notin E(G)$, then the subgraph induced by $\{v, s_{p+1}, x, s_1\}$ is a claw, a contradiction. Thus we assume that $s_1 s_{p+1} \in E(G)$. This implies $s_1 \in B_p$, $p = 1$, and so there are only two blocks of $G - v$. Note that $vs_1 \in E(G)$, so by our choice of B_1', $r_2 = v$. Thus $r_0 s_1 \in E(G)$ and $r_{p+1} s_p \in E(G)$. In particular, if $r_0 s_1 \notin E(G)$ or $r_{p+1} s_p \notin E(G)$, then $N_{B_0'}(r_0)$ is a clique, and by symmetry $N_{B_p'}(r_{p+1})$ is a clique too, proving the second statement of the observation.

Similarly, if we assume $N_{B_p'}(r_{p+1})$ is not a clique, we also get that $r_0 = r_{p+1} = v$, $p = 1$ and $vs_1 \in E(G)$.

Moreover, if neither $N_{B_0'}(r_0)$ nor $N_{B_p'}(r_{p+1})$ is a clique, then there is a neighbor x of v in $B_0 - s_1$ that is nonadjacent to s_1 and a neighbor y of v in $B_1 - s_1$ that is nonadjacent to s_1. But in that case the subgraph induced by $\{v, x, y, s_1\}$ is a claw, a contradiction. □

By Observation 3 and symmetry arguments, without loss of generality we may assume that $N_{B_p'}(r_{p+1})$ is a clique, and that the distance between r_0 and s_1 in B_0' is at least as large as between r_{p+1} and s_p in B_p'.

Let Q' be the (unique) path from r_0 to r_{p+1} (possibly consisting of one vertex v only) outside $B_0' \cup B_p'$. Note that Q and Q' are disjoint. We prove one more common observation.

Observation 4. *If the distance between r_{p+1} and s_p in B_p' is at least 4, then there is a neighbor of r_{p+1} outside B_p' that is nonadjacent to s_p.*

Proof. By our assumption, the distance between r_0 and s_1 in B_0' is also at least 4. Let R' be a shortest path in B_0' from r_0 to s_1. Then $R = Q' r_0 R' s_1 Q$ is an induced path from r_{p+1} to s_p outside B_p' and of length at least 4. Let r_{p+1}' be the successor of r_{p+1} on R. Then $r_{p+1}' s_p \notin E(G)$. □

Now as in the set-up to Lemma 2, we set

$$N_i = \{u \in B_0' - s_1 : d_{B_0' - s_1}(u, r_0) = i\} \quad \text{and} \quad j = \max\{i : N_i \neq \emptyset\}.$$

By Observation 2, N_1 is a clique. We complete the proof by assuming that there is no Hamilton path in B_p' from r_{p+1} to s_p, and showing that this implies that there exists a Hamilton path in B_0' from r_0 to s_1. We start by proving the following claim on the structure of N_i.

Claim 1. $j \leq 2$ and N_2 is P_3-free.

Proof. If $j \geq 3$, then let x be a vertex in N_3, and let R' be a shortest path of $B_0' - s_1$ from x to r_0. Then $R = Q' r_0 R'$ is an induced path with origin r_{p+1} outside B_p' and of length at least 3. Using Lemma 3, we obtain a Hamilton path of B_p' from r_{p+1} to s_p. Hence $j \leq 2$.

Let $xx'x''$ be an induced P_3 in N_2. Let w be a neighbor of x' in N_1. Then either wx or $wx'' \notin E(G)$; otherwise the subgraph induced by $\{w, r_0, x, x''\}$ is a claw. Without loss of generality, we assume that $wx'' \notin E(G)$. Then $R = Q' r_0 w x' x''$ is an induced path with origin r_{p+1} outside B_p' and of length at least 3. Now Lemma 3 again implies that there is a Hamilton path of B_p' from r_{p+1} to s_p. Hence we conclude that N_2 is P_3-free. □

Claim 1 implies that every component of N_2 is a clique. To complete this subcase, we need one more observation on the existence of perfect paths.

Claim 2. Let H be a component of N_2. If s_1 is not adjacent to H, then H supports a perfect path to N_1; if s_1 is adjacent to H, then H supports a perfect path to N_1 and s_1.

Proof. We first assume that s_1 is not adjacent to H. If H contains only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in N_1. Let w and w' be two neighbors of x in N_1. Then $R = wxw'$ is a perfect path supported by H to N_1.

If H contains at least two vertices, then by the 2-connectedness of G, H is joined to N_1 by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in H$ and $w, w' \in N_1$. Let R' be a Hamilton path of H from x to x'. Then $R = wxR'x'w'$ is a perfect path supported by H to N_1.

Suppose now that s_1 is adjacent to H. Let s' be a neighbor of s_1 in H. If H consists of the vertex s', then let w be a neighbor of s' in N_1. Then $R = ws's_1$ is a perfect path supported by H to N_1 and s_1. If there are at least two vertices in H, then let x be a vertex in H other than s'. Let w be a neighbor of x in N_1, and let R' be a Hamilton path of H from x to s'. Then $R = wxR's's_1$ is a perfect path supported by H to N_1 and s_1. $\square$

Using Claim 2, by Lemma 2 we conclude that there exists a Hamilton path of B_0' from r_0 to s_1, completing this case.

By the arguments in this section, it remains to complete the proofs of the three theorems only for the subcase that there is exactly one nontrivial inner block B_q and all the other blocks of $G - v$ are trivial. We do this separately for the three theorems in the following three sections.

5. Proof of Theorem 5 ($F = B_{1,4}$)

Let G be a 2-connected $\{K_{1,3}, B_{1,4}\}$-free graph. Adopting the notation and set-up of the previous section we are going to prove that G has a Hamilton path starting from a vertex v, in case $G - v$ contains a nontrivial inner block B_q and all other inner and end blocks of $G - v$ are trivial, so here we assume that all the blocks other than B_q are trivial.

Recall that it is sufficient to prove that B_q contains a Hamilton path from s_q to s_{q+1}. Suppose to the contrary that there is no such path. Set

$$N_i = \{u \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(u, s_q) = i\}, \quad \text{and} \quad j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

We already know from Observation 1 that $N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique. In particular, this implies that N_1 is a clique. If $j = 1$, then let s' be a neighbor of s_{q+1} in N_1. If N_1 consists of the vertex s', then $R = s_q s' s_{q+1}$ is a Hamilton path of B_q from s_q to s_{q+1}, a contradiction. If N_1 contains at least two vertices, then let x be a vertex in N_1 other than s', and let R' be a Hamilton path of N_1 from x to s'. Then $R = s_q x R' s' s_{q+1}$ is a Hamilton path of B_q from s_q to s_{q+1}, a contradiction. So there is nothing to prove if $N_2 = \emptyset$. Hence we assume $N_2 \neq \emptyset$. We complete the proof of this case by first proving a number of claims.

Claim 1. $vs_q \in E(G)$ and $vs_{q+1} \in E(G)$.

Proof. Suppose that $vs_q \notin E(G)$. Let Q be a shortest path from s_q to s_{p+1} containing vs_{p+1} with all internal vertices outside B_q. Then Q is an induced path of length at least 3 containing v with all internal vertices outside B_q.

Recall that N_1 is a clique. We first prove the following claim on the structure of N_i.

Claim 1.1. If N_2 is a clique, then for every i with $2 \le i \le j$, N_i is a clique.

Proof. We use induction on i. For $i = 2$, the assertion is true by assumption. Thus we assume that N_2 is a clique and that $3 \le i \le j$.

Let x and x' be two vertices in N_i such that $xx' \notin E(G)$. If x and x' have a common neighbor in N_{i-1}, then let w be a common neighbor of x and x' in N_{i-1}, and y be a neighbor of w in N_{i-2}. Then the subgraph induced by $\{w, y, x, x'\}$ is a claw, a contradiction. Thus x and x' have no common neighbors in N_{i-1}.

Let w be a neighbor of x in N_{i-1} and w' be a neighbor of x' in N_{i-1}. Then from the above we conclude that $wx', w'x \notin E(G)$, and by the induction hypothesis, $ww' \in E(G)$. Let u be a neighbor of w in N_{i-2}. Then $uw' \in E(G)$; otherwise the subgraph induced by $\{w, u, w', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from u to s_q. Then the subgraph induced by $\{w', w, x', x\} \cup V(R) \cup V(Q)$ is an $N_{1,1,\ell}$ with $\ell \ge 4$, so it contains an induced $B_{1,4}$, a contradiction. $\square$

So, if N_2 is a clique, we can apply Lemma 2 and show the existence of a Hamilton path in B_q between s_q and s_{q+1}, a contradiction.

Hence, we assume next that N_2 is not a clique. We obtain more information on the structure of N_i by proving another set of claims.

Claim 1.2. If there is an induced P_3 in $\bigcup_{i=2}^{j} N_i$, then the level of the center vertex of the P_3 is larger than that of at least one of its end vertices.

Proof. Assuming the contrary, let $xx'x''$ be an induced P_3 in $\bigcup_{i=2}^{j} N_i$ such that x' is one of the vertices with the smallest level among the vertices in $\{x, x', x''\}$. Throughout the section, we call such a P_3 a *bad* P_3.

Suppose that $x' \in N_i$, where $i \ge 2$. Let w be a neighbor of x' in N_{i-1}. Then either wx or $wx'' \in E(G)$: otherwise the subgraph induced by $\{x', w, x, x''\}$ is a claw. Without loss of generality, we assume that $wx \in E(G)$. Then $wx'' \notin E(G)$; otherwise letting y be a neighbor of w in N_{i-2}, the subgraph induced by $\{w, y, x, x''\}$ is a claw.

Let R be a shortest path from w to s_q in $B_q - s_{q+1}$. Then the subgraph induced by $\{x, x', x''\} \cup V(R) \cup V(Q)$ is a $B_{1,\ell}$ with $\ell \ge 4$, a contradiction. $\square$

Claim 1.3. N_2 is P_3-free and $\bigcup_{i=3}^{j} N_i$ is P_3-free.

Proof. If there is an induced P_3 in N_2, then it is a bad P_3, a contradiction to Claim 1.2. Thus N_2 is P_3-free.

Let $xx'x''$ be an induced P_3 in $\bigcup_{i=3}^{j} N_i$. Then by Claim 1.2, x' is not a vertex with the smallest level in $\{x, x', x''\}$. Without loss of generality, we assume that x has the smallest level. Moreover, we choose the induced P_3 in $\bigcup_{i=3}^{j} N_i$ subject to the other assumptions in such a way that the level of x is as small as possible.

We claim that $x \in N_3$. Assuming the contrary, suppose that $x \in N_i$, where $i \geq 4$. Then $x' \in N_{i+1}$. Let w be a neighbor of x in N_{i-1}. Clearly $wx' \notin E(G)$. Thus wxx' is an induced P_3 in $\bigcup_{i=3}^{j} N_i$ such that w has a smaller level than x, a contradiction to our choice of $xx'x''$. Thus as we claimed, $x \in N_3$ and then $x' \in N_4$.

Now let w be a neighbor of x in N_2. Then $wx'' \notin E(G)$; otherwise letting y be a neighbor of w in N_1, the subgraph induced by $\{w, y, x, x''\}$ is a claw.

Let w' be a vertex in N_2 other than w. We claim that $ww' \in E(G)$. Assume the contrary. Note that w and w' have no common neighbors in N_1; otherwise letting y be a common neighbor of w and w' in N_1, the subgraph induced by $\{y, s_q, w, w'\}$ is a claw. Let now y be a neighbor of w in N_1 and y' be a neighbor of w' in N_1. Then $y'w \notin E(G)$ and the subgraph induced by $\{y', s_q, s_{q-1}, y, w, x, x', x''\}$ is a $B_{1,4}$, a contradiction. This implies that w is adjacent to all other vertices in N_2.

Let w', w'' be two vertices in N_2 other than w. We claim that $w'w'' \in E(G)$. Assume the contrary. If $w'x \in E(G)$, then by similar arguments as before we get that w' is adjacent to all other vertices in N_2, and then $w'w'' \in E(G)$. So we assume that $w'x \notin E(G)$ and similarly $w''x \notin E(G)$. Then the subgraph induced by $\{w, w', w'', x\}$ is a claw, a contradiction.

We conclude that N_2 is a clique, a contradiction. □

Claim 1.3 implies that every component of N_2 and $\bigcup_{i=3}^{j} N_i$ is a clique. Our next claims involve the connecting structure between such components.

Claim 1.4. Each component of N_2 is joined to at most one component of $\bigcup_{i=3}^{j} N_i$; each component of $\bigcup_{i=3}^{j} N_i$ is joined to at most two components of N_2.

Proof. Let C be a component of N_2 that is joined to at least two components D and D' of $\bigcup_{i=3}^{j} N_i$. Let R be a shortest path from D to D' with all internal vertices in C. Then R contains a bad P_3, a contradiction to Claim 1.2. Thus every component of N_2 is joined to at most one component of $\bigcup_{i=3}^{j} N_i$.

Let D be a component of $\bigcup_{i=3}^{j} N_i$ that is joined to at least three components C, C' and C'' of N_2. Let x, x' and x'' be three vertices of C, C' and C'', respectively, that are joined to D. Recall that any two vertices of $\{x, x', x''\}$ have no common neighbors in N_1. Let w, w' and w'' be the neighbors of x, x' and x'' in N_1, respectively.

If there is an induced path R of length at least 3 from x to x' with all internal vertices in D, then the subgraph induced by $\{w'', s_q, s_{q-1}, w\} \cup V(R)$ is an induced $B_{1,\ell}$ with $\ell \geq 4$, a contradiction. Thus we assume that all the induced paths from x to x' with all internal vertices in D have length 2. Hence x and x' have a common neighbor y in D. Similarly x' and x'' have a common neighbor y' in D.

If $x''y \in E(G)$, then the subgraph induced by $\{y, x, x', x''\}$ is a claw, a contradiction. So $x''y \notin E(G)$, and similarly $xy' \notin E(G)$, and the subgraph induced by $\{w, s_q, s_{q-1}, x, x', x'', y, y'\}$ is a $B_{1,4}$, a contradiction. □

Claim 1.5. Let H be a component of $\bigcup_{i=2}^{j} N_i$. If s_{q+1} is not joined to H, then H supports a perfect path to N_1; if s_{q+1} is joined to H, then H supports a perfect path to N_1 and s_{q+1}.

Proof. By Claim 1.4, one of the following situations applies to H:

(1) H consists of exactly one component C of N_2;

(2) H consists of one component C of N_2 and one component D of $\bigcup_{i=3}^{j} N_i$; or

(3) H consists of two components C and C' of N_2 and one component D of $\bigcup_{i=3}^{j} N_i$.

Case A. Situation (1) applies.

We first assume that s_{q+1} is not joined to H. If C has only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in N_1. Let w, w' be two neighbors of x in N_1. Then $R = wxw'$ is a perfect path supported by H to N_1.

If C has at least two vertices, then by the 2-connectedness of G, C is joined to N_1 by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in C$ and $w, w' \in N_1$. Let R' be a Hamilton path of C from x to x'. Then $R = wxR'x'w'$ is a perfect path supported by H to N_1.

Suppose now that s_{q+1} is joined to H. Let s' be a neighbor of s_{q+1} in C. If C contains only the vertex s', then let w be a neighbor of s' in N_1. Then $R = ws's_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}.

If C contains at least two vertices, then let x be a vertex in C other than s', let w be a neighbor of x in N_1, and let R' be a Hamilton path of C from x to s'. Then $R = wxR's's_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}.

Case B. Situation (2) applies.

We first assume that s_{q+1} is not joined to H. Similarly as in the proof of Case A, D supports a perfect path R' to C. Let y and y' be the two end vertices of R'. By the 2-connectedness of G, C is joined to N_1 by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in C$ and $w, w' \in N_1$.

If x, x' and y, y' are distinct pairs, then without loss of generality, we assume that $x \neq y, y'$. If $x' \neq y, y'$, then let T be a path of C from x to y passing through all the vertices in $C \setminus \{x', y'\}$. Then $R = wxTyR'y'x'w'$ is a perfect path supported by H

to N_1. If $x' = y$ or y', then without loss of generality, we assume that $x' = y'$. Let T be a path of C from x to y passing through all the vertices in $C \setminus \{x'\}$. Then $R = wxTyR'x'w'$ is a perfect path supported by H to N_1.

Now we assume that x, x' and y, y' are the same pair. If there is a third vertex x'' in C other that x and x', then let w'' be a neighbor of x'' in N_1. Without loss of generality, we assume that $w'' \neq w$. Then xw and $x''w''$ are two independent edges joining C to N_1 such that x, x'' and y, y' are distinct pairs. Then we can find a perfect path supported by H to N_1 in the same way as before. If we only have the vertices x and x' in C, then $R = wxR'x'w'$ is a perfect path supported by H to N_1.

Suppose now that s_{q+1} is joined to H. If s_{q+1} is joined to D, then let s' be a neighbor of s_{q+1} in D. If $|D| = 1$, the case is similar to Case A, hence we assume $|D| \geq 2$. By the 2-connectedness, not all vertices of C have the same common neighbor with s_{q+1} in D. This implies that we can choose s' in such a way that there is an edge zy with $z \in D \setminus \{s'\}$ and $y \in C$. Clearly, D supports a perfect path R' to C and s_{q+1} with end vertex y in C. If there is a second vertex x in C other than y, then let w be a neighbor of x in N_1 and let T be a Hamilton path of C from x to y. Then $R = wxTyR'$ is a perfect path supported by H to N_1 and s_{q+1}. If C has only one vertex y, then let w be a neighbor of y in N_1. Then $R = wyR'$ is a perfect path supported by H to N_1 and s_{q+1}.

Suppose now that s_{q+1} is not joined to D but joined to C. Let s' be a neighbor of s_{q+1} in C. Similarly as in the proof of Case A, D supports a perfect path R' to C. Let y and y' be the two end vertices of R'.

If there is a vertex x in C other than y, y' and s', then let w be a neighbor of x in N_1. If $s' \neq y, y'$, then let T be a path of C from x to y passing through all the vertices in $C \setminus \{y', s'\}$. Then $R = wxTyR'y's's_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}. If $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let T be a path of C from x to y passing through all the vertices in $C \setminus \{y'\}$. Then $R = wxTyR'y's_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}.

Now we assume that there are no vertices in C other than y, y' and s'. If $s' \neq y, y'$, then let w be a neighbor of y in N_1. Then $R = wyR'y's's_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}. If $s' = y$ or y', then without loss of generality, we assume that $s' = y'$. Let w be a neighbor of y in N_1. Then $R = wyR'y's_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}.

Case C. Situation (3) applies.

We first assume that s_{q+1} is not joined to H. If D contains only one vertex y, then y has a neighbor in both C and C'. Let x and x' be the neighbors of y in C and C', respectively. Then $R' = xyx'$ is a perfect path supported by D to C and C'.

If D contains at least two vertices, then we claim that D is joined to C and C' by two independent edges. Let x and x' be two vertices in C and C', respectively, that are joined to D. If x and x' are joined to D by two independent edges, then clearly D is joined to C and C' by two independent edges. Thus we assume that x and x' are adjacent to only one common vertex y in D. Let y' be a neighbor of y in D. Then the subgraph induced by $\{y, x, x', y'\}$ is a claw, a contradiction. Thus, as we claimed, D is joined to C and C' by two independent edges. Let yx, $y'x'$ be two such edges, where $y, y' \in D$, $x \in C$ and $x' \in C'$. Let R'' be a Hamilton path of D from y to y'. Then $R' = xyR''y'x'$ is a perfect path supported by D to C and C'. Thus in any case, D supports a perfect path R' to C and C'. Let x and x' be the two end vertices of R', where $x \in C$ and $x' \in C'$.

If C contains only the vertex x, then let $w = x$; otherwise let w be a vertex in C other than x. Let y be a neighbor of w in N_1, and let T be a Hamilton path of C from w to x. If C' contains only the vertex x', then let $w' = x'$; otherwise let w' be a vertex in C' other than x'. Let y' be a neighbor of w' in N_1, and let T' be a Hamilton path of C' from x' to w'. Note that C and C' have no common neighbors in N_1, so we have $y \neq y'$. Now $R = ywTxR'x'T'w'y'$ is a perfect path supported by H to N_1.

Suppose next that s_{q+1} is joined to H. If s_{q+1} is joined to C or C', then without loss of generality, we assume that s_{q+1} is joined to C', and that s' is a neighbor of s_{q+1} in C'. By similar arguments as before, there is a perfect path R' supported by D to C and C'. Let x and x' be the two end vertices of R', where $x \in C$ and $x' \in C'$. If C contains only the vertex x, then let $w = x$; otherwise let w be a vertex in C other than x. Let y be a neighbor of w in N_1, and let T be a Hamilton path of C from w to x. If $s' \neq x'$, then let T' be a Hamilton path of C' from x' to s'. Then $R = ywTxR'x'T's's_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}. Now we assume that $s' = x'$. If C' contains only the vertex x', then $R = ywTxR'x's'_{q+1}$ is a perfect path supported by H to N_1 and s_{q+1}. Thus we assume that C' contains a second vertex x'' other than x'. Let y' be a neighbor of x' in R'. Then we have that $x''y' \in E(G)$; otherwise $x''x'y'$ is a bad P_3, a contradiction to Claim 1.2. Thus $R' - y'x' \cup y'x''$ is a perfect path supported by D to C and C' such that $s' \neq x''$. Then we can find a perfect path supported by H to N_1 and s_{q+1} by similar arguments as before.

Suppose now that s_{q+1} is not joined to C and C', but that it is joined to D. Then s_{q+1} has no neighbors in any components of N_2 since $N_{B_q}(s_{q+1}) \setminus \{s_q\}$ is a clique and D cannot be joined to three components of N_2. Let x be a vertex in C joined to D, let w be a neighbor of x in N_1, let x' be a vertex in C' joined to D, and let w' be a neighbor of x' in N_1. Note that s_{q+1} has no neighbors in N_1 since $N_{B_q}(s_{q+1}) \setminus \{s_q\}$ is a clique, and $s_qs_{q+1} \notin E(G)$ since $N_{B_q}(s_q)$ is a clique. Thus the distance between s_q and s_{q+1} in B_q is at least 4. Note that s_{q-1} is a neighbor of s_q outside B_q and $s_{q-1}s_{q+1} \notin E(G)$. If the distance between x and s_{q+1} in $D \cup \{x, s_{q+1}\}$ is at least 3, then let R be a shortest path from x to s_{q+1} with all internal vertices in D. Then the subgraph induced by $\{w', s_q, s_{q-1}, w\} \cup V(R)$ is a $B_{1,\ell}$ with $\ell \geq 4$, a contradiction. Thus we assume that x and s_{q+1} have a common neighbor y in D. Similarly, x' and s_{q+1} have a common neighbor y' in D. If $x'y \in E(G)$, then the subgraph induced by $\{y, x, x', s_{q+1}\}$ is a claw, a contradiction. Thus we assume that $x'y \notin E(G)$ and similarly $xy' \notin E(G)$. Then the subgraph induced by $\{s_{q+1}, y', x', y, x, w, s_q, s_{q-1}\}$ is a $B_{1,4}$, a contradiction. □

By Claim 1.5 we can apply Lemma 2 to obtain a Hamilton path of B_q from s_q to s_{q+1}, a contradiction.
Thus, we have $vs_q \in E(G)$. The second assertion follows by symmetry. □

We note here that in the above argumentation we have implicitly proved Lemma 3 in case $F = B_{1,4}$.

By Claim 1, $vs_q, vs_{q+1} \in E(G)$. If $p \geq 3$, G contains a claw centered at v, a contradiction. So we have that $p = 2, q = 1$, and $G - v$ consists of three blocks. Recall that the two end blocks B_0 and B_2 are both trivial, so we have that vs_0s_1v and vs_2s_3v are two triangles. We again obtain more information on the structure of N_i by proving the following claim.

Claim 2. $j \leq 3$, and N_3 is P_3-free.

Proof. If $j \geq 4$, then let x be a vertex in N_4, and let R be a shortest path from x to s_1 in $B_1 - s_2$. Then the subgraph induced by $\{s_0, v, s_3\} \cup V(R)$ is a $B_{1,4}$, a contradiction. Thus $j \leq 3$.

Let $xx'x''$ be an induced P_3 in N_3. Let w be a neighbor of x' in N_2, and let y be a neighbor of w in N_1. Then either wx or $wx'' \notin E(G)$; otherwise the subgraph induced by $\{w, y, x, x''\}$ is a claw. Without loss of generality, we assume that $wx'' \notin E(G)$. Then the subgraph induced by $\{s_0, v, s_3, s_1, y, w, x', x''\}$ is a $B_{1,4}$, a contradiction. $\square$

The next claim shows that s_1 and s_2 are neighbors in B_1.

Claim 3. $s_1s_2 \in E(G)$.

Proof. Assuming the contrary, let d be the distance between s_1 and s_2 in B_1, and let Q be a shortest path from s_1 to s_2 in B_1. Then $d \geq 2$ and, since $j \leq 3$, we have $d \leq 4$. We distinguish three cases according to the value of d.

Case A. $d = 2$.

Let $Q = s_1xs_2$. If $G - x$ is 2-connected, then by the induction hypothesis, $G - x$ contains a Hamilton path P' starting from v. Clearly s_1 and s_2 are two cut vertices of $G - v$. Thus the subpath R' of P' from s_1 to s_2 is a Hamilton path of $B_1 - x$. Let s' be the neighbor of s_1 in R'. Then $xs' \in E(G)$ and $R = R' - s_1s' \cup s_1xs'$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction. Thus there is another vertex y such that $\{x, y\}$ is a cut.

First note that $\{x, v\}$ is not a cut, since the only cut vertices of $G - v$ are s_1 and s_2. Thus $y \neq v$. Recalling that $s_1s_2 \notin E(G)$, by Lemma 1, s_1 and s_2 are not in a common component of $G - \{x, y\}$. Since s_1vs_2 is a path from s_1 to s_2 not passing through x, we have that either $y = s_1$ or $y = s_2$. Without loss of generality, we assume that $y = s_1$. Let H and H' be the two components of $G - \{x, s_1\}$, where $v \in H$. Let u be a vertex in H', and let R be an arbitrary path of G from u to s_3. Then R will pass through either x or s_1. Note that s_1 has only two neighbors v and s_0 in H. If R does not pass through x, then it will pass through either the edge s_1v or the subpath s_1s_0v. This implies that $\{x, v\}$ is a cut, a contradiction.

Case B. $d = 3$.

Let $Q = s_1xus_2$. Similarly as in Case A, we can prove that there is a vertex y such that $\{x, y\}$ is a cut, and $y \neq v, s_1$ or s_2. Since s_1 and u are both neighbors of x but $s_1u \notin E(G)$, they are not contained in the same component of $G - \{x, y\}$. Since s_1vs_2u is a path from s_1 to u not passing through x, we get that $y = u$. Note that the vertices v, s_0, s_1, s_2, s_3 and all vertices of $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are in a common component of $G - x, u$.

Let H be the component of $G - \{x, u\}$ not containing v. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint; otherwise we have $d = 2$. If x has a neighbor z outside $\{s_1\} \cup N_{B_1}(s_1) \cup H$, then let z' be a neighbor of x in H; in this case the subgraph induced by $\{x, s_1, z, z'\}$ is a claw, a contradiction. Thus all the neighbors of x are in $\{s_1\} \cup N_{B_1}(s_1) \cup H$, and similarly, all the neighbors of u are in $\{s_2\} \cup N_{B_1}(s_2) \cup H$. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let u' be a vertex in $N_{B_1}(s_2)$ other than u. Then $u' \notin H$, hence $u'x \notin E(G)$.

If there is a vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$, then without loss of generality, we assume that z is such a vertex and $zx' \in E(G)$. Then the subgraph induced by $\{s_3, v, s_0, s_2, u, x, x', z\}$ is a $B_{1,4}$, a contradiction. Thus we assume that there are no vertices in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$.

If H contains a vertex that is nonadjacent to x, then let z' be a vertex with distance 2 from x in H, and let z be a common neighbor of x and z' in H. Then the subgraph induced by $\{s_3, s_2, u', v, s_1, x, z, z'\}$ is a $B_{1,4}$, a contradiction. Thus we assume that every vertex in H is adjacent to x. Then by Lemma 1, H is a clique.

Let R' be a Hamilton path of $H \cup \{x, u\}$ from x to u, let T be a Hamilton path of $N_{B_1}(s_1)$ from x to x', and let T' be a Hamilton path of $N_{B_1}(s_2)$ from u to u'. Then $R = s_1x'TxR'uT'u's_2$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction.

Case C. $d = 4$.

Let $Q = s_1xyzs_2$. Similarly as in Case B, we have that either $\{x, y\}$ or $\{x, z\}$ is a cut. We claim that $\{x, z\}$ is a cut. Assuming the contrary, we have that $\{x, y\}$ is a cut, and similarly $\{y, z\}$ is a cut. Let H be the component of $G - \{x, y\}$ not containing v, and let H' be the component of $G - \{y, z\}$ not containing v. If H and H' share a common vertex h, then there is a path between x and z through h with all internal vertices in $H \cup H'$, implying that v is in the same component of $G - \{y, z\}$ as h, a contradiction. So H and H' are disjoint. Then every neighbor of y is in either $H \cup \{x\}$ or $H' \cup \{z\}$. Thus every path of G from y to v passes through either x or z, and then $\{x, z\}$ is a cut, a contradiction.

Let x' be a vertex in $N_{B_1}(s_1)$ other than x. Then $x'y \notin E(G)$ and the subgraph induced by $\{s_3, v, s_0, s_2, z, y, x, x'\}$ is a $B_{1,4}$, a contradiction. $\square$

By Observation 1 and Claim 3, $N_{B_1}(s_2) \setminus \{s_1\} = N_{B_1}(s_1) \setminus \{s_2\} = N_1$.

Our next claim shows that the vertices of N_1 can be paired into vertex cuts, as follows.

Claim 4. For every vertex $x \in N_1$, there is a unique vertex $x' \in N_1 \setminus \{x\}$ such that $\{x, x'\}$ is a cut.

Proof. Assume the contrary. Similarly as in the proof of Claim 3, x is contained in a cut $\{x, y\}$ with $y \neq v, s_1$ or s_2. It is easy to check that $y \neq s_0$ or s_3. Thus $y \in \bigcup_{i=2}^{j} N_i$. Let H be the component of $G - \{x, y\}$ not containing v, and let Q be a shortest path from x to y with all internal vertices in H.

Let R be a shortest path in $G - x$ from y to N_1, and let x' be the end vertex of R other than y. Similarly as in the proof of Claim 3, x' is contained in a cut $\{x', y'\}$ with a vertex $y' \neq s_1$. Let z' be the neighbor of x' in R. Note that s_1 and z' are not contained in a common component of $G - \{x', y'\}$. Note that $s_1xQ \cup R - z'x'$ is a path from s_1 to z' not passing through x'. We conclude that y' must be a vertex in $V(Q) \cup V(R) \setminus \{x'\}$. By our assumption $y' \neq x$. If $y' \in H \cup \{y\}$, then let H' be the component of $G - \{x', y'\}$ not containing v. Then every neighbor of y will be either in $H \cup \{x\}$ or in $H' \cup \{x'\}$. Hence every path from y to v passes through either x or x', a contradiction. Thus $y' \in V(R) \setminus \{x', y\}$.

Let T be the subpath of R from y to y', let H' be the component of $G - \{x', y'\}$ not containing v, and let z' be a neighbor of y' in H'. Then the subgraph induced by $\{s_0, v, s_3\} \cup V(Q) \cup V(T) \cup \{z'\}$ is a $B_{1,\ell}$ with $\ell \geq 4$, a contradiction.

Thus we conclude that there is a vertex $x' \in N_1$ such that $\{x, x'\}$ is a cut.

Let H be the component of $G - \{x, x'\}$ not containing v. Then all the neighbors of x in $\bigcup_{i=2}^{j} N_i$ are in H; otherwise, let y be a neighbor of x in H, and let y' be a neighbor of x in $\bigcup_{i=2}^{j} N_i \setminus H$. Then the subgraph induced by $\{x, s_1, y, y'\}$ is a claw. This implies that for any vertex x'' in $N_1 \setminus \{x, x'\}$, the pair $\{x, x''\}$ is not a cut. □

By Claim 4, we can partition N_1 into pairs such that each pair is a cut. The next claim shows how we can pick up the vertices of components in paths between the pairs.

Claim 5. Let $\{t, t'\}$ be a cut of G such that $t, t' \in N_1$, and let H be the component of $G - \{t, t'\}$ not containing v. Then there is a perfect path supported by H to $\{t, t'\}$.

Proof. If $H \cap N_2$ contains only one vertex x, then by the 2-connectedness of G, $H \cap N_3 = \emptyset$ and $xt, xt' \in E(G)$. Then $R = txt'$ is a perfect path supported by H to $\{t, t'\}$. Next we assume that $H \cap N_2$ contains at least two vertices. Note that both t and t' are adjacent to some vertices in $H \cap N_2$. We can divide $H \cap N_2$ into two nonempty subsets C and C' such that every vertex in C is adjacent to t, and every vertex in C' is adjacent to t'.

Recall that $j \leq 3$ and N_3 is P_3-free, so every component of $H \cap N_3$ is a clique.

Claim 5.1. Let D be a component of $H \cap N_3$. If D is joined to C but not to C', then D supports a perfect path to C; if D is joined to C' but not to C, then D supports a perfect path to C'; and if D is joined to both C and C', then D supports a perfect path to C and C'.

Proof. *Case* A. D is joined to C but not to C'.

If D contains only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in C. Let w, w' be two neighbors of x in C. Then $R = wxw'$ is a perfect path supported by D to C.

Now we assume that D contains at least two vertices. By the 2-connectedness of G, D is joined to C by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in D$ and $w, w' \in C$. Let R' be a Hamilton path of D from x to x'. Then $R = wxR'x'w'$ is a perfect path supported by D to C.

Case B. D is joined to C' but not to C.

This case can be treated in a similar way as Case A.

Case C. D is joined to both C and C'.

If D consists of the vertex x, then x has at least one neighbor in C and in C'. Let w be a neighbor of x in C, and let w' be a neighbor of x in C'. Then $R = wxw'$ is a perfect path supported by D to C and C'.

Now we assume that D contains at least two vertices. Clearly D is joined to C and C' by two independent edges. Let xw and $x'w'$ be two such edges, where $x, x' \in D$, $w \in C$ and $w' \in C'$. Let R' be a Hamilton path of D from x to x'. Then $R = wxR'x'w'$ is a perfect path supported by D to C and C'. □

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of components in $H \cap N_3$ that are joined to C but not to C', let R_i $(1 \leq i \leq k)$ be a perfect path supported by D_i to C, and let x_i, y_i be the two end vertices of R_i; let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the set of components in $H \cap N_3$ that are joined to C' but not to C, let R'_i $(1 \leq i \leq k')$ be a perfect path supported by D'_i to C', and let x'_i, y'_i be the two end vertices of R'_i; let $\mathcal{D}'' = \{D''_1, D''_2, \ldots, D''_{k''}\}$ be the set of components in $H \cap N_3$ that are joined to both C and C', let R''_i $(1 \leq i \leq k'')$ be a perfect path supported by D''_i to C and C', and let x''_i, y''_i be the two end vertices of R''_i, where $x''_i \in C$ and $y''_i \in C'$.

We first assume that k'' is odd. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise let $w = x''_1$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k}\{x_i, y_i\} \setminus \bigcup_{i=1}^{k''}\{x''_i\}$. If $\mathcal{D}' \neq \emptyset$, then let $w' = y'_{k'}$; otherwise let $w' = y''_{k''}$. Let T' be a path from t' to w' passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'}\{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''}\{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1y''_2R''_2x''_2 \cdots x''_{k''}R''_{k''}y''_{k''}x'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path supported by H to $\{t, t'\}$.

Next we assume that k'' is even. If there is an edge joining C to C' such that its two vertices are not the two end vertices of a common perfect path supported by some component in $\mathcal{D}''$ (we call such an edge a *good edge*), then let zz' be a good edge, where $z \in C$ and $z' \in C'$. Note that z is possibly an end vertex of a perfect path supported by some component in $\mathcal{D}$ or $\mathcal{D}''$, or that it is not such an end vertex, and that z' is possibly an end vertex of a perfect path supported by some component in $\mathcal{D}'$ or $\mathcal{D}''$, or that it is not such an end vertex. So there are nine different cases to consider. Here we only discuss two of the cases; for the other cases, a perfect path supported by H to $\{t, t'\}$ can be found in a similar way.

If z is not an end vertex of a perfect path supported by some component in $\mathcal{D}$ or $\mathcal{D}''$, and z' is an end vertex of a perfect path supported by some component in $\mathcal{D}'$, then without loss of generality, we assume that $z' = x'_1$. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise, if $\mathcal{D}'' \neq \emptyset$, then let $w = x''_1$; otherwise let $w = z$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k}\{x_i, y_i\} \setminus \bigcup_{i=1}^{k''}\{x''_i\} \setminus \{z\}$. Let T' be a path from t' to $y'_{k'}$ passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'}\{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''}\{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1y''_2R''_2x''_2 \cdots y''_{k''}R''_{k''}x''_{k''}\, zx'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path supported by H to $\{t, t'\}$.

If both z and z' are end vertices of perfect paths supported by some components in $\mathcal{D}''$, then note that zz' is a good edge, so these vertices are not the end vertices of a common perfect path. Without loss of generality, we assume that $z = x''_2$ and $z' = y''_1$. If $\mathcal{D} \neq \emptyset$, then let $w = x_1$; otherwise let $w = x''_1$. Let T be a path from t to w passing through all the vertices in $C \setminus \bigcup_{i=1}^{k}\{x_i, y_i\} \setminus \bigcup_{i=1}^{k''}\{x''_i\}$. If $\mathcal{D}' \neq \emptyset$, then let $w' = y'_{k'}$; otherwise let $w' = y''_{k''}$. Let T' be a path from t' to w' passing through all the vertices in $C' \setminus \bigcup_{i=1}^{k'}\{x'_i, y'_i\} \setminus \bigcup_{i=1}^{k''}\{y''_i\}$. Then $R = Tx_1R_1y_1 \cdots x_kR_ky_kx''_1R''_1y''_1x''_2R''_2y''_2 \cdots x''_{k''}R''_{k''}y''_{k''}x'_1R'_1y'_1 \cdots x'_{k'}R'_{k'}y'_{k'}T'$ is a perfect path supported by H to $\{t, t'\}$.

Next we assume that each edge joining C to C' is not a good edge.

If C is not joined to C', then $\mathcal{D}'' \neq \emptyset$; otherwise t will be a cut vertex of G. If C is joined to C', then we also have $\mathcal{D}'' \neq \emptyset$, since every edge joining C to C' is not good. Recall that we assume that k'' is even, so $k'' \geq 2$.

Note that $x''_1y''_2, x''_2y''_1 \notin E(G)$; otherwise they are good edges. Thus $ty''_1, ty''_2 \notin E(G)$; otherwise the subgraph induced by $\{t, s_1, x''_2, y''_1\}$ or $\{t, s_1, x''_1, y''_2\}$ is a claw. Let R be a shortest path from x''_1 to y''_1 with all internal vertices in D''_1 (possibly of length 1). Then the subgraph induced by $\{s_0, v, s_3, s_1, t\} \cup V(R) \cup \{y''_2\}$ is a $B_{1,\ell}$ with $\ell \geq 4$, a contradiction. □

Let $N_1 = \{x_i, x'_i : 1 \leq i \leq k\}$ such that for every i with $1 \leq i \leq k$, $\{x_i, x'_i\}$ is a cut. Let H_i be the component of $G - \{x_i, x'_i\}$ not containing v, and let R_i be a perfect path supported by H_i to $\{x_i, x'_i\}$. Then $R = s_1x_1R_1x'_1 \cdots x_kR_kx'_ks_2$ is a Hamilton path of B_1 from s_1 to s_2, our final contradiction.

6. Proof of Theorem 6 ($F = B_{2,3}$)

Let G be a 2-connected $\{K_{1,3}, B_{2,3}\}$-free graph. Adopting the notation and set-up of Section 4 we are going to prove that G has a Hamilton path starting from a vertex v, in case $G - v$ contains a nontrivial inner block B_q and all other inner and end blocks of $G - v$ are trivial. Recall that it is sufficient to prove that B_q contains a Hamilton path from s_q to s_{q+1}. Suppose to the contrary that there is no such path. Set

$$N_i = \{u \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(u, s_q) = i\}, \quad \text{and} \quad j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

We already know from Observation 1 that $N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique. In particular, this implies that N_1 is a clique. If $N_2 = \emptyset$, there is nothing to prove, so we assume $N_2 \neq \emptyset$. We complete the proof of this case by first proving a number of claims.

Claim 1. $vs_q \in E(G)$ and $vs_{q+1} \in E(G)$.

Proof. Suppose that $vs_q \notin E(G)$. Let Q be a shortest path from s_q to s_{p+1} containing vs_{p+1} and all internal vertices outside B_q. Then Q is an induced path containing v with all internal vertices outside B_q and of length at least 3.

We consider the structure of N_i and prove the following claim.

Claim 1.1. For every i with $1 \leq i \leq j - 1$, N_i is a clique, and N_j is P_4-free.

Proof. We use induction on i. We already know that N_1 is a clique, so we assume that $2 \leq i \leq j - 1$.

Let x be a vertex in N_i that has a neighbor y in N_{i+1}. Let x' be a vertex in N_i other than x. We first claim that $xx' \in E(G)$. Assume the contrary. Then x and x' have no common neighbors in N_{i-1}. Let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. Then $wx', w'x \notin E(G)$, and by the induction hypothesis, $ww' \in E(G)$. Let u be a neighbor of w in N_{i-2}. Then $uw' \in E(G)$; otherwise the subgraph induced by $\{w, u, w', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from u to s_q. Then the subgraph induced by $\{w', w, x, y\} \cup V(R) \cup V(Q)$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction. Thus, as we claimed, x is adjacent to all other vertices in N_i.

Let x', x'' be two arbitrary vertices in N_i other than x. We claim that $x'x'' \in E(G)$. Assume the contrary. If $x'y \in E(G)$, then similarly as before, x' is adjacent to all other vertices in N_i and $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$ and similarly $x''y \notin E(G)$. Then the subgraph induced by $\{x, x', x'', y\}$ is a claw, a contradiction.

Thus we have that N_i is a clique.

Let $xx'x''x'''$ be an induced P_4 in N_j. Let w be a neighbor of x in N_{j-1}, and let w''' be a neighbor of x''' in N_{j-1}. Then $wx'' \notin E(G)$; otherwise let u be a neighbor of w in N_{j-2}. Then the subgraph induced by $\{w, u, x, x''\}$ is a claw. Similarly, $wx''', w'''x, w'''x' \notin E(G)$. If $wx' \in E(G)$, then let R be a shortest path of $B_q - s_{q+1}$ from w to s_q. Then the subgraph induced by $\{x, x', x'', x'''\} \cup V(R) \cup V(Q)$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction. Thus we assume that $wx' \notin E(G)$, and similarly $w'''x'' \notin E(G)$. Let u be a neighbor of w in N_{j-2}. Then $w'''u \in E(G)$; otherwise the subgraph induced by $\{w, u, w''', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from u to s_q. Then the subgraph induced by $\{w''', w, x, x'\} \cup V(R) \cup V(Q)$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction.

Thus N_j is P_4-free. □

We next prove the following claim on the existence of perfect paths.

Claim 1.2. Let H be a component of N_j. If s_{q+1} is not adjacent to a vertex of H, then H supports a perfect path to N_{j-1}; if s_{q+1} is adjacent to a vertex of H, then H supports a perfect path to N_{j-1} and s_{q+1}.

Proof. We distinguish three cases.

Case A. H contains only one or two vertices.

We first assume that s_{q+1} is not adjacent to H. If H contains only one vertex x, then by the 2-connectedness of G, x has at least two neighbors in N_{j-1}. Let w and w' be two neighbors of x in N_{j-1}. Then $R = wxw'$ is a perfect path supported by H to N_{j-1}. If H contains two vertices x and x', then by the 2-connectedness of G, x and x' are joined to N_{j-1} by two independent edges. Let xw and $x'w'$ be two such edges. Then $R = wxx'w'$ is a perfect path supported by H to N_{j-1}.

Suppose now that s_{q+1} is adjacent to H. If H contains only one vertex x, then x is adjacent to s_{q+1}. Let w be a neighbor of x in N_{j-1}. Then $R = wxs_{q+1}$ is a perfect path supported by H to N_{j-1} and s_{q+1}. If H contains two vertices x and x', then without loss of generality, we assume that $x's_{q+1} \in E(G)$. Let w be a neighbor of x in N_{j-1}. Then $R = wxx's_{q+1}$ is a perfect path supported by H to N_{j-1} and s_{q+1}.

Case B. H is 2-connected.

We use that N_j is P_4-free, and thus H is P_4-free and also N-free. By Theorem 1, H contains a Hamilton cycle C.

We first assume that s_{q+1} is not adjacent to H. By the 2-connectedness of G, not all the vertices of H are adjacent to only one common vertex in N_{j-1}. Thus there are two vertices x and x' of H that are adjacent on C such that x and x' are joined to N_{j-1} by two independent edges. Let w and w' be the neighbors of x and x' in N_{j-1} such that $w \neq w'$. Then $R = C - xx' \cup \{xw, x'w'\}$ is a perfect path supported by H to N_{j-1}.

Suppose now that s_{q+1} is adjacent to H. Let s' be a neighbor of s_{q+1} in H, let x be a vertex in H that is adjacent to s' on C, and let w be a neighbor of x in N_{j-1}. Then $R = C - xs' \cup \{xw, s's_{q+1}\}$ is a perfect path supported by H to N_{j-1} and s_{q+1}.

Case C. H has a cut vertex.

Let x be a cut vertex of H. Obviously, $H - x$ has exactly two components. Let C and C' be the two components of $H - x$. If there is a vertex in C that is nonadjacent to x, then let z be a vertex in C with distance 2 from x in C, let y be a common neighbor of x and z in C, and let y' be a neighbor of x in C'. Then $zyxy'$ is an induced P_4 in H, a contradiction. This implies that x is adjacent to every vertex in C. If there are two vertices y, z in C that are nonadjacent, then let y' be a neighbor of x in C'; then the subgraph induced by $\{x, y, z, y'\}$ is a claw, a contradiction. Thus $C \cup \{x\}$ is a clique and similarly $C' \cup \{x\}$ is a clique.

We first assume that s_{q+1} is not adjacent to H. Let y be a vertex in C and let y' be a vertex in C'. Let T be a Hamilton path of $C \cup \{x\}$ from x to y, let w be a neighbor of y in N_{j-1}, let T' be a Hamilton path of $C' \cup \{x\}$ from x to y', and let w' be a neighbor of y' in N_{j-1}. Then $R = wyTxT'y'w'$ is a perfect path supported by H to N_{j-1}.

Suppose now that s_{q+1} is adjacent to H. We claim that s_{q+1} must be adjacent to C or C'. Assuming the contrary, s_{q+1} has only one neighbor x in H. Let y be a vertex in C, and let y' be a vertex in C'. Then the subgraph induced by $\{x, y, y', s_{q+1}\}$ is a claw, a contradiction. Without loss of generality, we assume that s_{q+1} is adjacent to C'. Let s' be a neighbor of s_{q+1} in C', and let y be a vertex in C. Let T be a Hamilton path of $C \cup \{x\}$ from x to y, let w be a neighbor of y in N_{j-1}, and let T' be a Hamilton path of $C' \cup \{x\}$ from x to s'. Then $R = wyTxT's's_{q+1}$ is a perfect path supported by H to N_{j-1} and s_{q+1}. $\square$

The above claims and Lemma 2 imply that there exists a Hamilton path of B_q from s_q to s_{q+1}, a contradiction. Thus we conclude that $vs_q \in E(G)$. The second assertion follows by symmetry. $\square$

We note here that in the above argumentation we have implicitly proved Lemma 3 in case $F = B_{2,3}$.

By Claim 1, $vs_q, vs_{q+1} \in E(G)$. If $p \geq 3$, G contains a claw centered at v, a contradiction. So $p = 2, q = 1$, and $G - v$ consists of three blocks. Recall that the two end blocks B_0 and B_2 are both trivial, so vs_0s_1v and vs_2s_3v are two triangles. We again obtain more information on the structure of N_i by proving the following claims.

Claim 2. $j \leq 3$, and N_3 is P_3-free.

Proof. The proofs of the following implications are completely analogous to the proofs of Claims 1.1 and 1.2, and the application of Lemma 2, and are therefore omitted.

Claim 2.1. If N_2 is a clique, then for every i with $2 \leq i \leq j - 1$, N_i is a clique and N_j is P_4-free.

Claim 2.2. If for every i with $1 \leq i \leq j - 1$, N_i is a clique and N_j is P_4-free, then B_1 contains a Hamilton path from s_1 to s_2.

Thus if N_2 is a clique, then by Claims 2.1 and 2.2, there is a Hamilton path of B_1 from s_1 to s_2, a contradiction. So we assume that N_2 is not a clique.

If $j \geq 4$, then let x be a vertex in N_2, let y be a neighbor of x in N_3, and let z be a neighbor of y in N_4. Let x' be a vertex in N_2 other than x. We claim that $xx' \in E(G)$. Assume the contrary. Then x and x' have no common neighbors in N_1. Let w be a neighbor of x in N_1, and let w' be a neighbor of x' in N_1. Then $w'x \notin E(G)$, and the subgraph induced by $\{w', s_1, r, s_3, w, x, y, z\}$ is a $B_{2,3}$, a contradiction. This implies that x is adjacent to all the other vertices in N_2.

Now let x' and x'' be two vertices in N_2 other than x. We claim that $x'x'' \in E(G)$. Assume the contrary. If $x'y \in E(G)$, then similarly as before, x' is adjacent to all the other vertices in N_2, and then $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$, and similarly $x''y \notin E(G)$. Then the subgraph induced by $\{x, x', x'', y\}$ is a claw, a contradiction.

This implies that N_2 is a clique, a contradiction. Thus $j \leq 3$.

Let $yy'y''$ be an induced P_3 in N_3. Let x be a neighbor of y' in N_2. Then x is nonadjacent to y or y''; otherwise, let w be a neighbor of x in N_1; then the subgraph induced by $\{x, w, y, y''\}$ is a claw. Without loss of generality, we assume that $xy'' \notin E(G)$. Then similarly as before, we can prove that N_2 is a clique, a contradiction. Thus N_3 is P_3-free. $\square$

We next show that s_1 and s_2 are neighbors in B_1.

Claim 3. $s_1s_2 \in E(G)$.

Proof. Assume the contrary. Let d be the distance between s_1 and s_2 in B_1 and let Q be a shortest path from s_1 to s_2 in B_1. Then $d \geq 2$ and, since $j \leq 3$, we have $d \leq 4$. We distinguish three cases according to the value of d.

Case A. $d = 2$.

Noting that we have not used $B_{1,4}$-freeness in Case A of the proof of Claim 3 in Section 5, this case can be proved completely analogously.

Case B. $d = 3$.

Let $Q = s_1xys_2$. Similarly as in Case B of the proof of Claim 3 in Section 5, we can prove that $\{x, y\}$ is a cut of G. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint; otherwise $d = 2$. Let H be the component of $G - \{x, y\}$ not containing v. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let y' be a vertex in $N_{B_1}(s_2)$ other than y.

If there is a vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$, then without loss of generality, we assume that z is such a vertex and $zx' \in E(G)$. Let z' be a neighbor of y in H. Then the subgraph induced by $\{s_0, s_1, x', z, v, s_2, y, z'\}$ is a $B_{2,3}$, a contradiction. Thus we assume that there are no vertices in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$.

If H contains a vertex nonadjacent with x, then let z' be a vertex with distance 2 from x in H, and let z be a common neighbor of x and z' in H. Then the subgraph induced by $\{s_0, v, s_2, y', s_1, x, z, z'\}$ is a $B_{2,3}$, a contradiction. Thus we assume that every vertex in H is adjacent to x. Then by Lemma 1, H is a clique.

Let R' be a Hamilton path of $H \cup \{x, y\}$ from x to y, let T be a Hamilton path of $N_{B_1}(s_1)$ from x to x', and let T' be a Hamilton path of $N_{B_1}(s_2)$ from y to y'. Then $R = s_1x'TxR'yT'y's_2$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction.

Case C. $d = 4$.

Let $Q = s_1xyzs_2$. Similarly as in Case C of the proof of Claim 3 in Section 5, we can prove that $\{x, z\}$ is a cut of G. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint and not adjacent; otherwise $d \leq 3$. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let z' be a vertex in $N_{B_1}(s_2)$ other than z. Then the subgraph induced by $\{x's_1, v, s_3, x, y, z, z'\}$ is a $B_{2,3}$, a contradiction. □

By Observation 1 and Claim 3, $N_{B_1}(s_2) \setminus \{s_1\} = N_{B_1}(s_1) \setminus \{s_2\} = N_1$. Our next observation shows that N_1 can be partitioned into cut pairs.

Claim 4. For every vertex $x \in N_1$, there is a unique vertex $x' \in N_1 \setminus \{x\}$ such that $\{x, x'\}$ is a cut.

Proof. Assume the contrary. Similarly as in the proof of Claim 4 in Section 5, we have that there is a vertex $y \in \bigcup_{i=2}^{j} N_i$ such that $\{x, y\}$ is a cut. Let H be the component of $G - \{x, y\}$ not containing v, and let R be a shortest path from x to y with all internal vertices in H.

Let R' be a shortest path in $G - x$ from y to N_1, and let x' be the end vertex of R' other than y. Similarly as in the proof of Claim 4 in Section 5, x' is contained in a cut $\{x', y'\}$, and with the other vertex $y' \in V(R') \setminus \{x', y\}$. Let T' be the subpath of R' from y to y', and let H' be the component of $G - \{x', y'\}$ not containing v.

Note that $\{x, x'\}$ is not a cut by our assumption. Let R'' be a shortest path of $G - \{x, x'\}$ from T' to N_1, and let x'' be the end vertex of R'' in N_1. Similarly as before, we have that x'' is contained in a cut $\{x'', y''\}$, and with the other vertex $y'' \in V(R'') \setminus \{x'', y, y'\}$. Let H'' be the component of $G - \{x'', y''\}$ not containing v.

If T' passes through y'', then let z and z' be the two neighbors of y'' on T', and let z'' be a neighbor of y'' in H''. Then the subgraph induced by $\{y'', z, z', z''\}$ is a claw, a contradiction. Thus we assume that T' does not pass through y''.

Let z' be a neighbor of y' in H'. Then the subgraph induced by $\{x'', s_1, r, s_3\} \cup V(R) \cup V(T') \cup \{z'\}$ is a $B_{2,\ell}$ with $\ell \geq 3$, a contradiction.

Thus there is a vertex $x' \in N_1$ such that $\{x, x'\}$ is a cut.

One can prove the uniqueness similarly as in the proof of Claim 4 in Section 5. □

By Claim 4, we can partition N_1 into pairs such that each pair is a cut. These pairs have a nice property with respect to perfect paths, as follows.

Claim 5. Let $\{t, t'\}$ be a cut of G such that $t, t' \in N_1$, and let H be the component of $G - \{t, t'\}$ not containing v. Then there is a perfect path supported by H to $\{t, t'\}$.

Proof. If $H \cap N_2$ contains only one vertex x, then by the 2-connectedness of G, $H \cap N_3 = \emptyset$ and $xt, xt' \in E(G)$. Then $R = txt'$ is a perfect path supported by H to $\{t, t'\}$. Thus we assume that $H \cap N_2$ contains at least two vertices. Note that both t and t' are adjacent to some vertices in $H \cap N_2$. We can divide $H \cap N_2$ into two nonempty subsets C and C' such that every vertex of C is adjacent to t and every vertex of C' is adjacent to t'.

Recall that $j \leq 3$ and that N_3 is P_3-free, so every component of $H \cap N_3$ is a clique. The proof of the next observations is completely analogous to the proof of Claim 5.1 in Section 5.

Claim 5.1. Let D be a component of $H \cap N_3$. If D is joined to C but not to C', then D supports a perfect path to C; if D is joined to C' but not to C, then D supports a perfect path to C'; and if D is joined to both C and C', then D supports a perfect path to C and C'.

We proceed similarly as in Section 5.

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of components in $H \cap N_3$ that are joined to C but not to C', let R_i $(1 \leq i \leq k)$ be a perfect path supported by D_i to C, and let x_i, y_i be the two end vertices of R_i; let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the set of components in $H \cap N_3$ that are joined to C' but not to C, let R'_i $(1 \le i \le k')$ be a perfect path supported by D'_i to C', and let x'_i, y'_i be the two end vertices of R'_i; and let $\mathcal{D}'' = \{D''_1, D''_2, \dots, D''_{k''}\}$ be the components in $H \cap N_3$ that are joined to both C and C', let R''_i $(1 \le i \le k'')$ be a perfect path supported by D''_i to C and C', and let x''_i, y''_i be the two end vertices of R''_i, where $x''_i \in C$ and $y''_i \in C'$.

If k'' is odd, or k'' is even and there is a good edge joining C to C', then we can prove the assertion similarly as in Section 5. Thus we assume that k'' is even and that every edge joining C to C' is not good. Similarly as in Section 5, note that $k'' \ge 2$.

If C is joined to C', then without loss of generality, we assume that $x''_1y''_1 \in E(G)$. Let z be a neighbor of x''_1 in D''_1, and let z' be a neighbor of y''_2 in D''_2. Then $x''_1y''_2, x''_2, y''_1, ty''_1, ty''_2 \notin E(G)$. Besides, $y''_1z \in E(G)$; otherwise the subgraph induced by $\{x''_1, t, y''_1, z\}$ is a claw. Thus the subgraph induced by $\{z, y''_1, y''_2, z', x''_1, t, s_1, s_0\}$ is a $B_{2,3}$, a contradiction.

Now we assume that C is not joined to C'. Let R be a shortest path from x''_1 to y''_1 with all internal vertices in D''_1. Then the subgraph induced by $\{x''_2, t, s_1, s_0\} \cup V(R) \cup \{y''_2\}$ is a $B_{2,l}$ with $l \ge 3$, a contradiction. □

We complete the proof of this case by reaching our final contradiction, as follows.

Let $N_1 = \{x_i, x'_i : 1 \le i \le k\}$ such that for every i with $1 \le i \le k$, $\{x_i, x'_i\}$ is a cut. Let H_i be the component of $G - \{x_i, x'_i\}$ not containing v, and let R_i be a perfect path supported by H_i to $\{x_i, x'_i\}$. Then $R = s_1x_1R_1x'_1 \cdots x_kR_kx'_ks_2$ is a Hamilton path of B_1 from s_1 to s_2, our final contradiction.

7. Proof of Theorem 7 ($F = N_{1,1,3}$)

Let G be a 2-connected $\{K_{1,3}, N_{1,1,3}\}$-free graph. Adopting the notation and set-up of Section 4 we are going to prove that G has a Hamilton path starting from a vertex v, in case $G - v$ contains a nontrivial inner block B_q and all other inner and end blocks of $G - v$ are trivial. Recall that it is sufficient to prove that B_q contains a Hamilton path from s_q to s_{q+1}. Suppose to the contrary that there is no such path. Set

$$N_i = \{u \in B_q - s_{q+1} : d_{B_q - s_{q+1}}(u, s_q) = i\}, \quad \text{and} \quad j = \max\{i : N_i \ne \emptyset\}.$$

Note that $N_0 = \{s_q\}$ and $N_1 = N_{B_q}(s_q) \setminus \{s_{q+1}\}$.

We already know from Observation 1 that $N_{B_q}(s_q)$ is a clique and $N_{B_q}(s_{q+1})$ is a clique. In particular, this implies that N_1 is a clique. There is nothing to prove if $N_2 = \emptyset$, so we assume $N_2 \ne \emptyset$. We complete the proof of this case by first proving a number of claims.

Claim 1. $vs_q \in E(G)$; $vs_{q+1} \in E(G)$.

Proof. Suppose that $vs_q \notin E(G)$. Let Q be a shortest path from s_q to s_{p+1} containing vs_{p+1} and with all internal vertices outside B_q. Then Q is an induced path with origin s_q and internal vertices outside B_q and of length at least 3.

Note that N_1 is a clique. We first show that all N_i are cliques.

Claim 1.1. For every i with $1 \le i \le j$, N_i is a clique.

Proof. We use induction on i. The result is true for $i = 1$. Thus we assume that $2 \le i \le j$.

Let x and x' be two vertices in N_i. Suppose $xx' \notin E(G)$. Then x and x' have no common neighbors in N_{i-1}. Let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. By the induction hypothesis, $ww' \in E(G)$. Let u be a neighbor of w in N_{i-2}. Then $w'u \in E(G)$; otherwise the subgraph induced by $\{w, u, w', x\}$ is a claw. Let R be a shortest path of $B_q - s_{q+1}$ from u to s_q. Then the subgraph induced by $\{w, x, w', x'\} \cup V(R) \cup V(Q)$ is an $N_{1,1,\ell}$ with $\ell \ge 3$, a contradiction. Thus $xx' \in E(G)$, completing the proof. □

Using the above observations and Lemma 2, we conclude that B_q contains a Hamilton path from s_q to s_{q+1}, a contradiction. Hence we get that $vs_q \in E(G)$. The second assertion follows by symmetry. □

We note here that in the above argumentation we have implicitly proved Lemma 3 in case $F = N_{1,1,3}$.

By Claim 1, $vs_q, vs_{q+1} \in E(G)$. If $p \ge 3$, G contains a claw centered at v, a contradiction. So $p = 2$, $q = 1$, and $G - v$ consists of three blocks. Recall that the two end blocks B_0 and B_2 are both trivial, so vs_0s_1v and vs_2s_3v are two triangles. We again obtain more information on the structure of N_i by proving the following claims.

Claim 2. $j \le 3$, and if $s_1s_2 \in E(G)$, then N_3 is P_3-free.

Proof. We first deduce that N_2 is not a clique by showing the following.

Claim 2.1. If N_2 is a clique, then for every i with $2 \le i \le j$, N_i is a clique.

Proof. Let $Q = s_1vs_3$. Then Q is an induced path with origin s_1 and internal vertices outside B_1 and of length 2.

For $i = 2$, the assertion is true by our assumption. So let $i \ge 3$, and let x and x' be two vertices in N_i. If $xx' \notin E(G)$, then x and x' have no common neighbors in N_{i-1}. Let w be a neighbor of x in N_{i-1}, and let w' be a neighbor of x' in N_{i-1}. By the induction hypothesis, $ww' \in E(G)$. Let u be a neighbor of w in N_{i-2}. Then $w'u \in E(G)$; otherwise the subgraph induced by $\{w, u, w', x\}$ is a claw. Let R be a shortest path of $B_1 - s_2$ from u to s_1. Then the subgraph induced by $\{w, x, w', x'\} \cup V(R) \cup V(Q)$ is an $N_{1,1,\ell}$ with $\ell \ge 3$, a contradiction. Thus $xx' \in E(G)$, completing the proof. □

If for every i with $1 \le i \le j$, N_i is a clique, then Lemma 2 implies that B_1 contains a Hamilton path from s_1 to s_2, a contradiction. So we assume that N_2 is not a clique.

Next suppose $j \geq 4$. Let z be a vertex in N_4, let y be a neighbor of z in N_3, and let x be a neighbor of y in N_2. Let x' be a vertex in N_2 other than x. We claim that $xx' \in E(G)$. Assume the contrary. Then $x'y \notin E(G)$; otherwise the subgraph induced by $\{y, x, x', z\}$ is a claw. Besides, x and x' have no common neighbors in N_1. Let w be a neighbor of x in N_1, and let w' be a neighbor of x' in N_1. Then $wx', w'x \notin E(G)$, and the subgraph induced by $\{s_1, s_0, w', x', w, x, y, z\}$ is an $N_{1,1,3}$, a contradiction. This implies that x is adjacent to all other vertices in N_2. Now letting x' and x'' be two vertices in N_2 other than x, we claim that $x'x'' \in E(G)$. Assume the contrary. If $x'y \in E(G)$, then similarly as before, x' is adjacent to all the other vertices in N_1, and then $x'x'' \in E(G)$. Thus we assume that $x'y \notin E(G)$, and similarly $x''y \notin E(G)$. Then the subgraph induced by $\{x, x', x'', y\}$ is a claw, a contradiction. This implies that N_2 is a clique, a contradiction. Thus we get that $j \leq 3$.

Suppose now that $s_1s_2 \in E(G)$, and that $yy'y''$ is an induced P_3 in N_3. Let x be a neighbor of y' in N_2. Then either xy or $xy'' \notin E(G)$. Without loss of generality, we assume that $xy'' \notin E(G)$. Let w be a neighbor of x in N_1. Then $s_2w \in E(G)$; otherwise the subgraph induced by $\{s_1, s_0, s_2, w\}$ is a claw. Now the subgraph induced by $\{s_1, s_0, s_2, s_3, w, x, y', y''\}$ is an $N_{1,1,3}$, a contradiction. $\square$

We next show that s_1 and s_2 are neighbors in B_1.

Claim 3. $s_1s_2 \in E(G)$.

Proof. Assume the contrary. Let d be the distance between s_1 and s_2 in B_1, and let Q be a shortest path from s_1 to s_2 in B_1. Then $d \geq 2$ and, since $j \leq 3$, we have $d \leq 4$. We distinguish three cases according to the value of d.

Case A. $d = 2$.

Noting that we have not used $B_{1,4}$-freeness in Case A of the proof of Claim 3 in Section 5, this case can be proved completely analogously.

Case B. $d = 3$.

Let $Q = s_1xys_2$. Similarly as in Case B of the proof of Claim 3 in Section 5, we can prove that $\{x, y\}$ is a cut of G. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint; otherwise $d = 2$. Let H be the component of $G - \{x, y\}$ not containing v. Let x' be a vertex in $N_{B_1}(s_1)$ other than x, and let y' be a vertex in $N_{B_1}(s_2)$ other than y.

If there is a vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$, then without loss of generality, we assume that z is such a vertex and $zx' \in E(G)$. Then the subgraph induced by $\{s_1, s_0, x', z, x, y, s_2, s_3\}$ is an $N_{1,1,3}$, a contradiction. Thus we assume that there are no vertices in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$.

If H contains a vertex nonadjacent with x, then let z' be a vertex with distance 2 from x in H, and let z be a common neighbor of x and z' in H. Then $yz \in E(G)$; otherwise the subgraph induced by $\{x, s_1, y, z\}$ is a claw. $yz' \notin E(G)$; otherwise the subgraph induced by $\{y, x, z', s_2\}$ is a claw. Now the subgraph induced by $\{y, y', z, z', x, s_1, v, s_3\}$ is an $N_{1,1,3}$, a contradiction. Thus we assume that every vertex in H is adjacent to x. Then by Lemma 1, H is a clique.

Let R' be a Hamilton path of $H \cup \{x, y\}$ from x to y, let T be a Hamilton path of $N_{B_1}(s_1)$ from x to x', and let T' be a Hamilton path of $N_{B_1}(s_2)$ from y to y'. Then $R = s_1x'TxR'yT'y's_2$ is a Hamilton path of B_1 from s_1 to s_2, a contradiction.

Case C. $d = 4$.

Let $Q = s_1xyzs_2$. Similarly as in Case C of the proof of Claim 3 in Section 5, we can prove that $\{x, z\}$ is a cut of G. Let x' be a vertex in $N_{B_1}(s_1)$ other than x. Note that $N_{B_1}(s_1)$ and $N_{B_1}(s_2)$ are disjoint and not adjacent; otherwise $d \leq 3$. There must be some vertex in B_1 other than $\{s_1, s_2\} \cup N_{B_1}(s_1) \cup N_{B_1}(s_2) \cup H$; otherwise $\{v, x\}$ is a cut. Without loss of generality, we assume that y' is such a vertex, and $x'y' \in E(G)$. Recall that $y \in H$ and that x is only adjacent to $\{s_1\} \cup N_{B_1}(s_1) \cup H$. Then the subgraph induced by $\{s_1, s_0, x', y', x, y, z, s_2\}$ is an $N_{1,1,3}$, a contradiction. $\square$

By Observation 1 and Claim 3, $N_{B_1}(s_2) \setminus \{s_1\} = N_{B_1}(s_1) \setminus \{s_2\} = N_1$, and by Claims 2 and 3, N_3 is P_3-free. Our next observation shows that N_1 can be partitioned into cut pairs.

Claim 4. For every vertex $x \in N_1$, there is a unique vertex $x' \in N_1 \setminus \{x\}$ such that $\{x, x'\}$ is a cut.

Proof. Assume the contrary. Similarly as in the proof of Claim 4 in Section 5, there is a vertex $y \in \bigcup_{i=2}^{j} N_i$ such that $\{x, y\}$ is a cut. Let H be the component of $G - \{x, y\}$ not containing v, and let R be a shortest path from x to y with all internal vertices in H.

Let R' be a shortest path in $G - x$ from y to N_1, and let x' be the end vertex of R' other than y. Similarly as in Section 5, x' is contained in a cut $\{x', y'\}$ with $y' \in V(R') \setminus \{x', y\}$. Let T' be the subpath of R' from y to y', let H' be the component of $G - \{x', y'\}$ not containing v, and let z' be a neighbor of y' in H'. Then the subgraph induced by $\{s_1, s_0, s_2, s_3\} \cup V(R) \cup V(T') \cup \{z'\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. Thus there is a vertex $x' \in N_1$ such that $\{x, x'\}$ is a cut.

One can prove the uniqueness similarly as in the proof of Claim 4 in Section 5. $\square$

By Claim 4, we can partition N_1 into pairs such that each pair is a cut. These pairs have a nice property with respect to perfect paths, as follows.

Claim 5. Let $\{t, t'\}$ be a cut of G such that $t, t' \in N_1$, and let H be the component of $G - \{t, t'\}$ not containing v. Then there is a perfect path supported by H to $\{t, t'\}$.

Proof. If $H \cap N_2$ contains only one vertex x, then by the 2-connectedness of G, $H \cap N_3 = \emptyset$ and $xt, xt' \in E(G)$. Then $R = txt'$ is a perfect path supported by H to $\{t, t'\}$. Thus we assume that $H \cap N_2$ contains at least two vertices. Note that both t and t' are adjacent to some vertices in $H \cap N_2$. We can divide $H \cap N_2$ into two nonempty subset C and C' such that every vertex of C is adjacent to t and every vertex of C' is adjacent to t'.

Recall that $j \leq 3$ and that N_3 is P_3-free, so every component of $H \cap N_3$ is a clique. The proof of the next observations is completely analogous to the proof of Claim 5.1 in Section 5.

Claim 5.1. Let D be a component of $H \cap N_3$. If D is joined to C but not to C', then D supports a perfect path to C; if D is joined to C' but not to C, then D supports a perfect path to C'; and if D is joined to both C and C', then D supports a perfect path to C and C'.

We proceed similarly as in Section 5.

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of components in $H \cap N_3$ that are joined to C but not to C', let R_i $(1 \leq i \leq k)$ be a perfect path supported by D_i to C, and let x_i, y_i be the two end vertices of R_i; let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{k'}\}$ be the set of components in $H \cap N_3$ that are joined to C' but not to C, let R'_i $(1 \leq i \leq k')$ be a perfect path supported by D'_i to C', and let x'_i, y'_i be the two end vertices of R'_i; let $\mathcal{D}'' = \{D''_1, D''_2, \ldots, D''_{k''}\}$ be the set of components in $H \cap N_3$ that are joined to both C and C', let R''_i $(1 \leq i \leq k'')$ be a perfect path supported by D''_i to C and C', and let x''_i, y''_i be the two end vertices of R''_i, where $x''_i \in C$ and $y''_i \in C'$.

If k'' is odd, or k'' is even and there is a good edge joining C to C', then we can prove the assertion similarly as in Section 5. Thus we assume that k'' is even and that every edge joining C to C' is not good. Similarly as in Section 5, note that $k'' \geq 2$.

Let R be a shortest path from x''_1 to y''_1 with all internal vertices in D''_1. Then the subgraph induced by $\{s_1, s_0, s_2, s_3, t\} \cup V(R) \cup \{y''_2\}$ is an $N_{1,1,\ell}$ with $\ell \geq 3$, a contradiction. $\square$

We complete the proof of this case by reaching our final contradiction, as follows.

Let $N_1 = \{x_i, x'_i : 1 \leq i \leq k\}$ such that for every i with $1 \leq i \leq k$, $\{x_i, x'_i\}$ is a cut. Let H_i be the component of $G - \{x_i, x'_i\}$ not containing v, and let R_i be a perfect path supported by H_i to $\{x_i, x'_i\}$. Then $R = s_1x_1R_1x'_1 \cdots x_kR_kx'_ks_2$ is a Hamilton path of B_1 from s_1 to s_2, our final contradiction.

Acknowledgments

The first author was supported by the Doctorate Foundation of Northwestern Polytechnical University (No. cx201202).

References

[1] R. Aldred, J. Fujisawa, A. Saito, Forbidden subgraphs and the existence of a 2-factor, J. Graph Theory 64 (2010) 250–266.
[2] P. Bedrossian, Forbidden subgraph and minimum degree conditions for Hamiltonicity, Ph.D. Thesis, Memphis State University, 1991.
[3] J.A. Bondy, U.S.R. Murty, Graduate Texts in Mathematics, vol. 244, Springer, 2008.
[4] D. Duffus, R.J. Gould, M.S. Jacobson, Forbidden subgraphs and the Hamiltonian theme, in: The Theory and Applications of Graphs, Kalamazoo, Michigan, 1980, Wiley, New York, 1981, pp. 297–316.
[5] J.R. Faudree, R.J. Faudree, Z. Ryjáček, Forbidden subgraphs that imply 2-factors, Discrete Math. 308 (2008) 1571–1582.
[6] R.J. Faudree, R.J. Gould, Characterizing forbidden pairs for Hamiltonian properties, Discrete Math. 173 (1997) 45–60.
[7] J. Fujisawa, A. Saito, A pair of forbidden subgraphs and 2-factors, Combin. Probab. Comput., in press (Published online: http://dx.doi.org/10.1017/S0963548311000514).
[8] S. Fujita, K. Kawarabayashi, C.L. Lucchesi, K. Ota, M.D. Plummer, A. Saito, A pair of forbidden subgraphs and perfect matchings, J. Combin. Theory Ser. B 96 (2006) 315–324.
[9] K. Ota, M.D. Plummer, A. Saito, Forbidden triples for perfect matchings, J. Graph Theory 67 (2011) 250–259.
[10] Z. Skupień, Problem 17 on homogeneously traceable graphs, in: Beiträge zur Graphentheorie und deren Anwendungen, Proc. Conf. Held in Oberhof GDR, 1977, Math. Gesellschaft der DDR-Techn. Hochschule Ilmenau, 1977, pp. 317–318.