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# On the existence of virtual exosystems for synchronized linear networks[☆]

Håvard Fjær Grip[a,1], Ali Saberi[b], Anton A. Stoorvogel[c]

[a] Department of Engineering Cybernetics, Norwegian University of Science and Technology, 7491 Trondheim, Norway

[b] School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164, USA

[c] Department of Electrical Engineering, Mathematics, and Computer Science, University of Twente, 7500 AE Enschede, The Netherlands

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ABSTRACT

When dealing with heterogeneous networks, where the agents are governed by non-identical models, interesting questions arise regarding the ability of the network to synchronize to a common non-trivial output trajectory, as well as the nature of such a trajectory. On this topic, Wieland, Allgöwer, and Sepulchre have recently derived results showing that for a class of heterogeneous networks of dynamically controlled linear agents, non-trivial output synchronization implies the existence of an observable *virtual exosystem* for which the regulator equations are solvable for each agent. Moreover, this virtual exosystem defines the output trajectories on the agreement manifold and is contained within each agent as an internal model. In this paper, we shed further light on this topic by showing that, under a more general set of assumptions, non-trivial output synchronization can occur in the absence of such a virtual exosystem. We propose a modified result for this case that specifies the existence of a possibly *unobservable* virtual exosystem for which the regulator equations are solvable, and for which the observable part defines the output trajectories on the agreement manifold. We also show that a variation of the virtual exosystem is contained within each agent as an internal model.



## 1. Introduction

In recent years, a large body of work has emerged on the topic of *synchronization*, where the goal is to secure agreement among networked agents on a common state or output trajectory. Although most of this work has been focused on state synchronization in *homogeneous* networks, a limited amount of work has also been done on *heterogeneous* networks, where the agents are governed by non-identical models (e.g., Bai, Arcak, & Wen, 2011; Chopra & Spong, 2008; Grip, Yang, Saberi, & Stoorvogel, 2012; Kim, Shim, & Seo, 2011; Xiang & Chen, 2007; Yang, Saberi, Stoorvogel, & Grip, in press; Zhao, Hill, & Liu, 2010). When dealing with heterogeneous networks, the goal is typically to achieve *output synchronization*—that is, agreement on some partial-state output. Non-identical models tend to produce outputs with different characteristics; thus, it is of interest to study when it is possible to achieve synchronization to a common non-trivial output trajectory, and what this output trajectory will look like.

Wieland, Allgöwer, and Sepulchre have recently derived results showing that, for a class of dynamically controlled, diffusively coupled networks with stabilizable and detectable agents, non-trivial output synchronization implies the existence of a *virtual exosystem* for which the regulator equations are solvable for each agent (Wieland, 2010; Wieland & Allgöwer, 2009; Wieland, Sepulchre, & Allgöwer, 2011). This virtual exosystem is described by an observable pair $(S, R)$, where the eigenvalues of $S$ are in the closed right-half complex plane. A consequence of the underlying analysis is that the virtual exosystem must be embedded within the dynamics of each agent together with its local controller. The result is therefore interpreted as an *internal model principle*, which is deemed necessary (as well as sufficient) for non-trivial output synchronization.

In this paper, we seek to shed further light on this topic by showing that, under a more general set of assumptions obtained by removing a detectability condition on the *combined agent–controller dynamics*, the existence of such a virtual exosystem is not guaranteed. We note that this detectability assumption was included by Wieland (2010) and Wieland and Allgöwer (2009), but it was left out of the more recent paper of Wieland et al. (2011). Thus,

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*E-mail addresses:* grip@itk.ntnu.no (H.F. Grip), saberi@eecs.wsu.edu (A. Saberi), A.A.Stoorvogel@utwente.nl (A.A. Stoorvogel).

[1] Tel.: +47 73 59 43 76; fax: +47 73 59 45 99.

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our intention is also to point out the significance of this apparent omission.

We shall show by example that, in the absence of the above-mentioned detectability condition, there exist networks for which non-trivial output synchronization can be achieved even though the regulator equations are unsolvable for all observable virtual exosystems. We also present a modified result that specifies the existence of a possibly *unobservable* virtual exosystem for which the regulator equations are solvable, and for which the observable part defines the output trajectories on the agreement manifold. We furthermore show that a variation of the virtual exosystem is contained within each agent as an internal model.

### 1.1. Network description and previous results

We consider $N$ linear agents described by models

$$\dot{x}_k = A_k x_k + B_k u_k, \quad x_k \in \mathbb{R}^{n_k},\ u_k \in \mathbb{R}^{p_k}, \tag{1a}$$

$$y_k = C_k x_k, \quad y_k \in \mathbb{R}^q, \tag{1b}$$

where $(A_k, B_k)$ is stabilizable and $(A_k, C_k)$ is detectable. The agents are connected via diffusive couplings that allow the agents to exchange relative system and controller states; specifically, agent $k$ has access to the quantity

$$\upsilon_k = \sum_{j=1}^{N} a_{kj}(t)(\zeta_j - \zeta_k),$$

where $\zeta_k \in \mathbb{R}^{\mu}$ is a vector of system and controller states transmitted by agent $k$ over the network, and $a_{kj}(t)$ is element $(k, j)$ of the adjacency matrix describing the network's communication graph, which is assumed to be *uniformly connected* (see Wieland et al., 2011, Definition 2). The object of output synchronization is to achieve $(y_i - y_j) \to 0$ for each $i, j \in 1, \ldots, N$. Dynamic controllers for each agent are given in the general form

$$\dot{\xi}_k = D_k \xi_k + E_k y_k + F_k \upsilon_k, \quad \xi_k \in \mathbb{R}^{\upsilon_k} \tag{2a}$$

$$u_k = G_k \xi_k + M_k y_k + O_k \upsilon_k, \tag{2b}$$

$$\zeta_k = P_k \xi_k + Q y_k. \tag{2c}$$

The detectability condition discussed in Section 1 can be stated as follows.

**Assumption 1.** For each $k \in 1, \ldots, N$, the pair $(A_k^*, C_k^*)$, where

$$A_k^* = \begin{bmatrix} A_k + B_k M_k C_k & B_k G_k \\ E_k C_k & D_k \end{bmatrix}, \qquad C_k^* = \begin{bmatrix} C_k & 0 \end{bmatrix}, \tag{3}$$

is detectable.

Subject to Assumption 1, the following result from Wieland et al. (2011) can then be stated.

**Wieland et al., 2011, Theorem 3.** Consider $N$ linear state-space models (1) coupled through dynamic controllers (2). Assume the closed loop system has no asymptotically stable equilibrium set on which $y_k(t) = 0,\ k \in 1, \ldots, N$.

If $(y_i - y_j) \to 0$ and $(\zeta_i - \zeta_j) \to 0$ for $i, j \in 1, \ldots, N$ exponentially as $t \to \infty$, then there exist a scalar $m \in \mathbb{N}$, matrices $S \in \mathbb{R}^{m \times m}$ and $R \in \mathbb{R}^{q \times m}$, where the eigenvalues of $S$ are in the closed right-half complex plane and $(S, R)$ is observable, and matrices $\Pi_k \in \mathbb{R}^{n_k \times m}$, $\Gamma_k \in \mathbb{R}^{p_k \times m}$ for $k \in 1, \ldots, N$ such that

$$A_k \Pi_k + B_k \Gamma_k = \Pi_k S, \tag{4a}$$

$$C_k \Pi_k = R, \tag{4b}$$

for $k \in 1, \ldots, N$. Furthermore, there exists $z_0 \in \mathbb{R}^m$ such that $\lim_{t\to\infty} \|y_k(t) - R e^{St} z_0\| = 0$ for all $k \in 1, \ldots, N$.

## 2. Synchronization without virtual exosystem

Assumption 1 is not intrinsically related to the concept of synchronization; thus, it is of interest to investigate its significance to the results stated above. The following example shows that, in the absence of Assumption 1, synchronization can indeed occur without the existence of an observable virtual exosystem.

**Example 1.** Consider a two-agent network with a uniformly connected communication graph described by $a_{11} = a_{12} = a_{22} = 0$ and $a_{21} = 1$, and with agent models given by

$$\text{Agent 1}: \begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = u_1, \\ y_{11} = x_{11}, \\ y_{12} = x_{12}, \end{cases} \qquad \text{Agent 2}: \begin{cases} \dot{x}_{21} = x_{22}, \\ \dot{x}_{22} = x_{21} + u_2, \\ \dot{x}_{23} = -x_{23}, \\ y_{21} = x_{22}, \\ y_{22} = x_{23}. \end{cases}$$

That is, the system matrices are given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that both $(A_1, B_1, C_1)$ and $(A_2, B_2, C_2)$ are stabilizable and detectable.

To design a synchronization protocol, we define the values $\zeta_1 = y_1$ and $\zeta_2 = y_2$ to be transmitted over the network. This means that agent 2 has access to $\upsilon_2 = y_1 - y_2$. For agent 1, we define the controller

$$u_1 = -\begin{bmatrix} 0 & 1 \end{bmatrix} y_1.$$

For agent 2, we define the dynamic controller

$$\dot{\hat{x}}_2 = A_2 \hat{x}_2 + B_2 u_2 + K_2 (y_2 - C_2 \hat{x}_2),$$

$$u_2 = -\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \hat{x}_2 + \begin{bmatrix} 1 & 0 \end{bmatrix} \upsilon_2,$$

where

$$K_2 = \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}$$

is chosen such that $A_2 - K_2 C_2$ is Hurwitz. It is easy to verify that these controllers are in the form of (2) with $Q = I$, $D_1, E_1, F_1, G_1$, and $P_1$ empty, $M_1 = -[0, 1]$, $O_1 = 0$, $D_2 = A_2 - B_2[1, 0, 0] - K_2 C_2$, $E_2 = K_2$, $F_2 = B_2[1, 0]$, $G_2 = -[1, 0, 0]$, $M_2 = 0$, $O_2 = [1, 0]$, and $P_2 = 0$.

The closed-loop dynamics of agent $x_1$ can be written as

$$\dot{x}_{11} = x_{12},$$

$$\dot{x}_{12} = -x_{12}.$$

It therefore follows that $y_{12}(t) = e^{-t} x_{12}(0) \to 0$ and $y_{11}(t) = x_{11}(0) + \int_0^t x_{12}(\tau)\, d\tau \to x_{11}(0) + x_{12}(0)$.

For agent 2, note that $\hat{x}_2$ is an observer estimate of $x_2$, with the dynamics of the estimation error $\tilde{x}_2 = x_2 - \hat{x}_2$ described by

$$\dot{\tilde{x}}_2 = (A_2 - K_2 C_2)\tilde{x}_2.$$

It follows that $\tilde{x}_2 \to 0$. The closed-loop dynamics of agent 2 can be written as

$$\dot{x}_{21} = x_{22},$$

$$\dot{x}_{22} = \tilde{x}_{21} + y_{11} - y_{21},$$

$$\dot{x}_{23} = -x_{23}.$$

We therefore see that $y_{22}(t) = e^{-t}x_{23}(0) \to 0$, and hence $(y_{22} - y_{12}) \to 0$. Furthermore, we have

$$\begin{aligned}\dot{y}_{21} - \dot{y}_{11} &= \dot{x}_{22} - \dot{x}_{11} \\ &= \tilde{x}_{21} + y_{11} - y_{21} - x_{12} \\ &= -(y_{21} - y_{11}) + \tilde{x}_{21} - x_{12}.\end{aligned}$$

Since $\tilde{x}_{21}$ and $x_{12}$ vanish exponentially, it follows that $(y_{21} - y_{11}) \to 0$. Hence, the agents synchronize to the (generally nonzero) output trajectory $[x_{11}(0) + x_{12}(0), 0]'$.

The example system satisfies the conditions in the statement of Theorem 3 of Wieland et al. (2011), but not Assumption 1. We now show that there exists no observable matrix pair $(S, R)$ that solves the regulator equations. For the sake of establishing a contradiction, suppose that they do exist, and partition $R$, $\Pi_1$, and $\Pi_2$ row-wise as

$$R = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \qquad \Pi_1 = \begin{bmatrix} \pi_{11} \\ \pi_{12} \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} \pi_{21} \\ \pi_{22} \\ \pi_{23} \end{bmatrix}.$$

From the expression $C_1\Pi_1 = R$, we obtain that $\pi_{11} = r_1$ and $\pi_{12} = r_2$. From the first row of the expression $\Pi_1 S = A_1\Pi_1 + B_1\Gamma_1$, we obtain $\pi_{11}S = \pi_{12} \implies r_1 S = r_2$. From the expression $C_2\Pi_2 = R$, we obtain $\pi_{22} = r_1$ and $\pi_{23} = r_2$. From the first and third rows of the expression $\Pi_2 S = A_2\Pi_2 + B_2\Gamma_2$, we obtain $\pi_{21}S = \pi_{22} \implies \pi_{21}S = r_1$ and $\pi_{23}S = -\pi_{23} \implies r_2 S = -r_2$.

Now, since $r_1 S = r_2$ and $r_2 S = -r_2$, we have $\ker RS \supset \ker R$, and it follows that the unobservable subspace of $(S, R)$ is given by $\ker R$. Since $(S, R)$ is observable, it follows that either $m = 1$ or $m = 2$. In the latter case, $r_1$ and $r_2$ would have to be linearly independent, and we could therefore write $\pi_{21} = \alpha_1 r_1 + \alpha_2 r_2$ for some scalars $\alpha_1$ and $\alpha_2$. However, this would imply $\pi_{21}S = (\alpha_1 - \alpha_2)r_2$, which contradicts the expression $\pi_{21}S = r_1$. Hence, we must have $m = 1$, and either $r_1 \neq 0$ or $r_2 \neq 0$. If $r_2 \neq 0$, then it follows from $r_2 S = -r_2$ that $S = -1$, which cannot be the case since $S$ has all its eigenvalues in the closed right-half complex plane. We must therefore have $r_2 = 0$ and $r_1 \neq 0$. Then the expression $\pi_{21}S = r_1$ implies $S \neq 0$. But then $r_1 S = r_2$ implies $r_2 \neq 0$, which is a contradiction.

Note that the outputs in our example synchronize to a bounded trajectory $[x_{11}(0) + x_{12}(0), 0]'$. Nevertheless, the internal dynamics of agent 2 is not bounded, as can be seen by noting that $x_{21}$ is the integral of $y_{21}$. Since $y_{21}$ converges to a generally nonzero constant, this implies that $x_{21}$ will grow linearly. Indeed, such dynamics is necessary for synchronization. To see this, note first that $y_{22}$ is unaffected by inputs and converges to zero, which means that $y_{12}$ must also converge to zero in order to achieve synchronization in the second output channel. This implies that $y_{11}$ becomes the integral of a vanishing signal, and thus it converges to a (generally nonzero) constant. To achieve synchronization in the first output channel, $y_{21}$ must also converge to a nonzero constant; however, the transfer function from $u_2$ to $y_{21}$ is $s/(s^2 - 1)$, which means that $y_{21}$ can only sustain a nonzero constant if $u_2$ is an unbounded ramp signal.

## 3. A modified theorem

In view of the above observations, let us state a modified result in the absence of Assumption 1. For simplicity, we consider only fixed communication topologies (i.e., $a_{kj}(t)$ constant for all $k, j \in 1, \dots, N$).

**Theorem 1.** *Assume that the network topology is fixed, and consider $N$ linear state-space models (1) coupled through dynamic controllers (2). Assume the closed loop system has no asymptotically stable equilibrium set on which $y_k(t) = 0$, $k \in 1, \dots, N$. If $(y_i - y_j) \to 0$ and $(\zeta_i - \zeta_j) \to 0$ for $i, j \in 1, \dots, N$ as $t \to \infty$, then there exist a scalar $m \in \mathbb{N}$, matrices $S \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{q\times m}$, where the eigenvalues of $S$ are in the closed right-half complex plane and $(S, R)$ is observable, and a vector $z_0 \in \mathbb{R}^m$ such that $\lim_{t\to\infty} \|y_k(t) - Re^{St}z_0\| = 0$ for all $k \in 1, \dots, N$.*

*Furthermore, there exist a scalar $r \in \mathbb{N}$ and matrices $S_{21} \in \mathbb{R}^{r\times m}$ and $S_{22} \in \mathbb{R}^{r\times r}$, where the eigenvalues of $S_{22}$ are in the closed right-half complex plane, such that, for each $k \in 1, \dots, N$, there exist matrices $\Pi_k \in \mathbb{R}^{n_k\times(m+r)}$ and $\Gamma_k \in \mathbb{R}^{p_k\times(m+r)}$ satisfying*

$$A_k\Pi_k + B_k\Gamma_k = \Pi_k\bar{S}, \tag{5a}$$

$$C_k\Pi_k = \bar{R}, \tag{5b}$$

*where*

$$\bar{S} = \begin{bmatrix} S & 0 \\ S_{21} & S_{22} \end{bmatrix}, \qquad \bar{R} = \begin{bmatrix} R & 0 \end{bmatrix}.$$

**Proof.** Let $A^*$ and $C^*$ be defined as in the proof of Theorem 3 of Wieland et al. (2011), so that the diagonal elements are given by $A_k^*$ and $C_k^*$. Following the proof of Theorem 3 of Wieland et al. (2011) we can conclude that the trajectories converge to an invariant agreement manifold $\mathcal{S} = \operatorname{span}\tilde{\Psi}$ that contains no asymptotically stable modes. Thus, there exists a matrix $\tilde{S} \in \mathbb{R}^{\tilde{m}\times\tilde{m}}$ such that $A^*\tilde{\Psi} = \tilde{\Psi}\tilde{S}$, where $\tilde{S}$ represents the overall dynamics on the agreement manifold. Writing $\tilde{\Psi} = [\tilde{\Pi}_1', \tilde{\Sigma}_1', \dots, \tilde{\Pi}_N', \tilde{\Sigma}_N']'$, as in Wieland et al. (2011), we obtain $A_k\tilde{\Pi}_k + B_k(M_kC_k\tilde{\Pi}_k + G_k\tilde{\Sigma}_k) = \tilde{\Pi}_k\tilde{S}$, and hence $A_k\tilde{\Pi}_k + B_k\tilde{\Gamma}_k = \tilde{\Pi}_k\tilde{S}$ for $\tilde{\Gamma}_k = M_kC_k\tilde{\Pi}_k + G_k\tilde{\Sigma}_k$. Furthermore, since $y_i = y_j$ on $\mathcal{S}$ we have, as in Wieland et al. (2011), $C_k\tilde{\Pi}_k = \tilde{R}$ for some $\tilde{R}$.

Let $\Lambda$ be a nonsingular state transformation taking $(\tilde{S}, \tilde{R})$ to the Kalman observable canonical form. That is,

$$\Lambda^{-1}\tilde{S}\Lambda = \bar{S} = \begin{bmatrix} S & 0 \\ S_{21} & S_{22} \end{bmatrix}, \qquad \tilde{R}\Lambda = \bar{R} = \begin{bmatrix} R & 0 \end{bmatrix},$$

for some matrices $S \in \mathbb{R}^{m\times m}$, $S_{21} \in \mathbb{R}^{r\times m}$, $S_{22} \in \mathbb{R}^{r\times r}$, and $R \in \mathbb{R}^{q\times m}$, where $m = \tilde{m} - r$. The output trajectories on the agreement manifold are governed by the observable part of $(\tilde{S}, \tilde{R})$, given by $(S, R)$, and hence $\lim(y_k(t) - Re^{St}z_0) = 0$ for some $z_0 \in \mathbb{R}^m$. Furthermore, it is easily seen that the regulator equations (5) are satisfied with $\Pi_k = \tilde{\Pi}_k\Lambda$ and $\Gamma_k = \tilde{\Gamma}_k\Lambda$. Note that $m > 0$, because the closed-loop system is assumed to have no asymptotically stable subspace on which $y_k(t) = 0$. ■

## 4. Internal model principle

The results of Wieland (2010); Wieland and Allgöwer (2009); Wieland et al. (2011) can be interpreted as an internal model principle, because they imply that the observable pair $(S, R)$, which dictates the outputs on the agreement manifold, is embedded within each agent's dynamics together with its local controller. That is,

$$T_k^{-1}A_k^*T_k = \begin{bmatrix} S & \star \\ 0 & \star \end{bmatrix}, \qquad C_k^*T_k = \begin{bmatrix} R & \star \end{bmatrix},$$

for some nonsingular matrix $T_k$, where $\star$ denotes a block of no importance in this context. In the absence of Assumption 1, this internal model principle may fail to hold, both for $(S, R)$ and $(\bar{S}, \bar{R})$. We can, however, state the following result.

**Theorem 2.** *Assume that the conditions of Theorem 1 hold. For each $k \in 1, \dots, N$, there exist a $T_k \in \mathbb{R}^{(n_k+\upsilon_k)\times(n_k+\upsilon_k)}$ and matrices $S_{k21} \in \mathbb{R}^{r_k\times m}$ and $S_{k22} \in \mathbb{R}^{r_k\times r_k}$, where $r_k \leq r$, such that*

$$T_k^{-1}A_k^*T_k = \begin{bmatrix} \bar{S}_k & \star \\ 0 & \star \end{bmatrix}, \qquad C_k^*T_k = \begin{bmatrix} \bar{R}_k & \star \end{bmatrix},$$

*where*

$$\bar{S}_k = \begin{bmatrix} S & 0 \\ S_{k21} & S_{k22} \end{bmatrix}, \qquad \bar{R}_k = \begin{bmatrix} R & 0 \end{bmatrix}.$$

**Proof.** Referring back to the proof of Theorem 1, it is clear that $A^*\Psi = \Psi\bar{S}$, where $\Psi = \tilde{\Psi}\Lambda$. Let $\Psi = [\Psi_1', \dots, \Psi_N']'$, where $\Psi_k \in \mathbb{R}^{(n_k+\upsilon_k)\times\tilde{m}}$. Then $A_k^*\Psi_k = \Psi_k\bar{S}$ and $C_k^*\Psi_k = \bar{R}$. Let $\Psi_k = [\Psi_{k1}, \Psi_{k2}]$, where $\Psi_{k1} \in \mathbb{R}^{(n_k+\upsilon_k)\times m}$ and $\Psi_{k2} \in \mathbb{R}^{(n_k+\upsilon_k)\times r}$. Then $\Psi_{k1}$ is of full column rank $m$ and $\operatorname{im}\Psi_{k1} \cap \operatorname{im}\Psi_{k2} = \{0\}$. To see this, let $O(A, C) := [C', \dots, (CA^{\tilde{m}-1})']'$ for any matrices $A$ and $C$ of compatible dimensions, and note that $O(\bar{S}, \bar{R}) = O(A_k^*, C_k^*)\Psi_k$. Moreover, $O(\bar{S}, \bar{R}) = [O(S, R), 0]$, and hence we have $O(A_k^*, C_k^*)\Psi_{k1} = O(S, R)$ and $O(A_k^*, C_k^*)\Psi_{k2} = 0$. Since $O(S, R)$ is the observability matrix of the observable pair $(S, R)$, we see that $\operatorname{rank} O(A_k^*, C_k^*)\Psi_{k1} = m$, which implies $\operatorname{rank}\Psi_{k1} = m$. Moreover, $\operatorname{im}\Psi_{k1} \cap \operatorname{im}\Psi_{k2} = \{0\}$, since otherwise there would be an $\eta_1 \neq 0$ such that $\Psi_{k1}\eta_1 = \Psi_{k2}\eta_2$. This would imply $O(A_k^*, C_k^*)\Psi_{k1}\eta_1 = O(A_k^*, C_k^*)\Psi_{k2}\eta_2 = 0$, which contradicts the fact that $\operatorname{rank} O(A_k^*, C_k^*)\Psi_{k1} = m$.

Define $T_k = [\Psi_{k1}, \Psi_{k2}V_k, T_{k3}]$, where $V_{k2}$ is chosen such that the columns of $\Psi_{k2}V_k$ form a nonsingular basis for $\operatorname{im}\Psi_{k2}$, and $T_{k3}$ is chosen to make $T_k$ nonsingular. Then

$$\begin{aligned} A_k^*T_k &= \begin{bmatrix} A_k^*\Psi_{k1} & A_k^*\Psi_{k2}V_k & A_k^*T_{k3} \end{bmatrix} \\ &= \begin{bmatrix} \Psi_{k1}S + \Psi_{k2}S_{21} & \Psi_{k2}S_{22}V_k & A_k^*T_{k3} \end{bmatrix}. \end{aligned}$$

Let $S_{k21}$ be defined such that $\Psi_{k2}S_{21} = \Psi_{k2}V_kS_{k21}$, and let $S_{k22}$ be defined such that $\Psi_{k2}S_{22}V_k = \Psi_{k2}V_kS_{k22}$. Then

$$\begin{aligned} A_k^*T_k &= \begin{bmatrix} \Psi_{k1} & \Psi_{k2}V_k & T_{k3} \end{bmatrix} \begin{bmatrix} S & 0 & \star \\ S_{k21} & S_{k22} & \star \\ 0 & 0 & \star \end{bmatrix} \\ &= T_k \begin{bmatrix} \bar{S}_k & \star \\ 0 & \star \end{bmatrix}. \end{aligned}$$

Furthermore,

$$\begin{aligned} C_k^*T_k &= \begin{bmatrix} C_k^*\Psi_{k1} & C_k^*\Psi_{k2}V_k & C_k^*T_{k3} \end{bmatrix} = \begin{bmatrix} R & 0 & \star \end{bmatrix} \\ &= \begin{bmatrix} \bar{R}_k & \star \end{bmatrix}. \quad \blacksquare \end{aligned}$$

We now confirm that the above results are consistent with our example.

**Example 1** (*Continued*)**.** The outputs of the agents synchronize to a trajectory $Re^{St}z_0$, where the observable pair $(S, R)$ is given by

$$S = 0, \qquad R = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It can be confirmed that the regulator equations (5) are solvable for the pair

$$\bar{S} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \bar{R} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

by choosing

$$\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Gamma_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

This is consistent with Theorem 1 with $S_{21} = 1$ and $S_{22} = 0$.

The matrices $A_1^*$ and $C_1^*$ are given by

$$A_1^* = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \qquad C_1^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and hence we can choose $T_1 = I$ to satisfy Theorem 2 with $S_{121}$ and $S_{122}$ empty. The matrices $A_2^*$ and $C_2^*$ are given by

$$A_2^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -2 & 0 \\ 0 & 3 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$C_2^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Defining

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

satisfies Theorem 2 with $S_{221} = 1$ and $S_{222} = 0$. Note that there is no internal model describing the output trajectories that works both for $(A_1^*, C_1^*)$ and $(A_2^*, C_2^*)$.

## 5. Concluding remarks

The above analysis shows that, even though the synchronized outputs converge to trajectories defined by an observable pair $(S, R)$, the regulator equations are not necessarily solvable with respect to this pair. This may seem to contradict the notion that output regulation is equivalent to solvability of the regulator equations. Note, however, that classical output regulation as defined by Francis and Wonham (1975) assumes *loop stability* – that is, stability of the closed-loop system when the state of the exosystem is zero – as an intrinsic part of the problem formulation. In the absence of loop stability, feasibility of the regulation problem is not equivalent to solvability of the regulator equations.

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