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Discontinuous Galerkin Approximations for Computing Electromagnetic Bloch Modes in Photonic Crystals

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Abstract We analyze discontinuous Galerkin finite element discretizations of the Maxwell equations with periodic coefficients. These equations are used to model the behavior of light in photonic crystals, which are materials containing a spatially periodic variation of the refractive index commensurate with the wavelength of light. Depending on the geometry, material properties and lattice structure these materials exhibit a photonic band gap in which light of certain frequencies is completely prohibited inside the photonic crystal. By Bloch/Floquet theory, this problem is equivalent to a modified Maxwell eigenvalue problem with periodic boundary conditions, which is discretized with a mixed discontinuous Galerkin (DG) formulation using modified Nédélec basis functions. We also investigate an alternative primal DG interior penalty formulation and compare this method with the mixed DG formulation. To guarantee the non-pollution of the numerical spectrum, we prove a discrete compactness

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property for the corresponding DG space. The convergence rate of the numerical eigenvalues is twice the minimum of the order of the polynomial basis functions and the regularity of the solution of the Maxwell equations. We present both 2D and 3D numerical examples to verify the convergence rate of the mixed DG method and demonstrate its application to computing the band structure of photonic crystals.

Keywords Discontinuous Galerkin methods · Mixed finite element methods · Maxwell equations · Discrete compactness property · Eigenvalue problems · Photonic crystals · Band structure

1 Introduction

Photonic crystals are lattice-like nanostructures with periodic electric permittivity. For specific electric permittivities, they possess photonic band gaps in which the propagation of specific light frequencies through the crystal is prohibited. This can be used to control light propagation and emission thus making photonic crystals very important for a wide range of applications [29,43]. However, designing and fabricating photonic crystals requires the knowledge of the frequencies at which light waves are completely reflected, propagated only in desired directions, or contained within a specified region. Then a waveguide can be carved out of a photonic crystal.

The modeling of photonic crystals is done by Maxwell's equations with periodic electric permittivity in $\mathbb{R}^3$ (idealized to exist in the whole space). Bloch/Floquet Theory for periodic differential operators reduces the problem to a modified Maxwell cavity eigenproblem with periodic boundary conditions on a fundamental cell and the gaps in the spectrum of the underlying operator correspond to the band gaps of the photonic crystal.

Most of the time, band gaps cannot be identified analytically motivating the use of numerical methods. The finite-difference time-domain (FDTD) method [46] is probably the most popular one, but near material interfaces and singularities this method has a few disadvantages, one of which is the limited accuracy of the spatial approximation due to the use of Taylor expansions. Other commonly used methods are conforming and nonconforming finite element methods. Non-physical solutions may, however, arise in the discretizations due to the inability to correctly approximate the infinite dimensional null space of the operator [5,45]. Nédélec's curl-conforming elements were proven [6] to overcome this problem. These elements satisfy the discrete compactness property and therefore guarantee spurious free approximations.

Conforming finite element techniques have been widely used to solve the standard Maxwell eigenproblem and there is a complete theory on how to obtain spectrally correct approximations. More details can be found in [1,2,4,11,22,34,35]. Nonconforming, more specifically, DG methods are more capable, compared to the conforming methods, of handling singularities and discontinuities in the solution that may exist due to discontinuous problem coefficients.

In [41], an interior penalty DG method was proposed for high-frequency problems where the material coefficients were assumed to be smooth. This method deals with the divergence free constraint through the introduction of a Lagrange multiplier and includes an additional volume stabilization term. For piecewise constant material coefficients, this method was improved in [26] allowing the removal of the additional stabilization term. In both [41] and [26] optimal error estimates in the energy norm were proved. In [40], an *hp*-local discontinuous Galerkin method was proposed for the low-frequency regime for heterogeneous media. An hp-analysis was presented with optimal error estimates in the meshsize h and slightly sub-optimal estimates in the approximation degree p. These results were supported by numerical experiments in [25]. The LDG method in [21,47] uses nodal elements rather than Nédélec's curl-conforming elements and pushes the spurious modes out of the range of the remaining modes by taking the penalty term sufficiently large. In [9], hermitian and non-hermitian interior penalty DG formulations based on Nédélec elements of the second kind were presented with an asymptotic analysis; corresponding numerical experiments were presented in [8]. In [7] an interior penalty DG method with divergence-free basis functions is used for the two-dimensional curl-curl problem. This method is automatically free of spurious modes and using graded meshes optimal convergence estimates are satisfied. Two hybridizable discontinuous Galerkin (HDG) methods are presented in [38], one with a Lagrange multiplier to enforce the divergence constraint and one without, using polynomials of order k for each unknown. The review above is by no means complete and includes only a few selected works. A nonexhaustive list is [7–9,12,19–21,23,24,26–28,38,40–42,47].

In [15,16], to compute electromagnetic Bloch modes in 3D, the Mixed Finite Element Method (MFEM) based on the lowest order Nedéléc elements on cubes was used and analyzed. The convergence results therein were later extended in [3] to the case with Nedéléc's first family of elements on tetrahedrons/parallelepipeds requiring less assumptions on the regularity of the eigensolutions. For a more detailed review of DG methods related to nanophotonics, we refer the reader to [10,17].

In this paper, we consider the periodic Maxwell eigenproblem and analyze two discontinuous Galerkin (DG) methods; a stabilized mixed DG method as in [26] and a symmetric interior penalty DG method as in [9] using Nédélec elements of the first kind on tetrahedral meshes. Similar arguments can be used to extend the results to the second kind. We also provide extensive numerical results in 3D to validate our theoretical results and make a comparison of the two DG methods.

In Sect. 2, we introduce the eigenvalue problem with periodic coefficients and via the Bloch/Floquet theory we transform the problem to a bounded domain. In Sect. 3, we define functional spaces based on periodic Sobolev spaces and include theoretical results necessary for our analysis. The weak formulation is defined and using Hilbert-Schmidt theory we investigate the spectral properties of the problem. Section 4 introduces the discrete spaces together with their properties. In Sect. 5, we define our DG approximations. Pointwise and uniform convergence of the underlying operators are discussed in Sects. 5 and 6. In Sect. 7 for the mixed DG method and in Sect. 8 for the symmetric interior penalty DG method, we give the main theoretical results on the error estimates for eigenvalues and eigenpairs. In Sect. 9, we present our numerical results. Finally, in Sect. 10, we give some concluding remarks.

2 Maxwell Eigenproblem with Periodic Boundary Conditions

In this section, we consider the Maxwell equations to compute band structures of photonic crystals. Let d be the number of dimensions. The classical Maxwell eigenproblem in $\mathbb{R}^d$ is as follows:

$$\begin{aligned} \nabla \times (\epsilon^{-1} \nabla \times \boldsymbol{H}) &= \omega^2 \boldsymbol{H} \quad \text{in } \mathbb{R}^d, \\ \nabla \cdot \boldsymbol{H} &= 0 \qquad \text{in } \mathbb{R}^d, \end{aligned} \tag{2.1}$$

where the electric permittivity ϵ is periodic, and $\boldsymbol{H}$ is the magnetic field. This periodicity can be described mathematically by primitive lattice vectors $\{\boldsymbol{a}_i,\ i=1,\ldots,d\}$, which form a maximal set of linearly independent vectors in $\mathbb{R}^d$ as follows:

$$\epsilon(\boldsymbol{x}+\boldsymbol{a})=\epsilon(\boldsymbol{x}),\quad \forall \boldsymbol{x}\in\mathbb{R}^d,$$

for any $\boldsymbol{a}$ that belongs to the Bravais lattice

$$A:=\left\{\sum_{i=1}^{d} k_i\boldsymbol{a}_i,\ k_i\in\mathbb{Z},\ i=1,\ldots,d\right\}.$$

In the analysis of this paper, we only consider the orthogonal cases. The results can, however, be extended to general lattices by affine transform [30, Section 2.4]. The periodic solution is completely determined by its values on the primitive cell (fundamental domain) defined as

$$\Omega:=\left\{\boldsymbol{x}\in\mathbb{R}^d:\boldsymbol{x}=\sum_{i=1}^{d}x_i\boldsymbol{a}_i,\ x_i\in[0,1],\ i=1,\ldots,d\right\}.$$

Here we call $(\boldsymbol{H},\omega^2)$ an eigenpair of problem (2.1). The Bloch waves are quasi-periodic functions satisfying

$$\boldsymbol{H}(\boldsymbol{x})=e^{i\boldsymbol{\alpha}\cdot\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x}),$$

where $\boldsymbol{u}$ is periodic in $\boldsymbol{x}$, that is $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a})=\boldsymbol{u}(\boldsymbol{x})$, $\forall\ \boldsymbol{x}\in\mathbb{R}^d$, $\forall\ \boldsymbol{a}\in A$ and $\boldsymbol{\alpha}$ is in the associated first Brillouin zone K ([29]).

We assume that $\epsilon=\epsilon(\boldsymbol{x})$ is real and piecewise constant with respect to a partition of Ω, and there are real positive numbers ϵ_*, $\epsilon^*>0$ such that

$$0<\epsilon_*\le\epsilon(\boldsymbol{x})\le\epsilon^*<+\infty,\qquad \forall\ \boldsymbol{x}\in\overline{\Omega}. \tag{2.2}$$

By Bloch's theorem, we can transform the quasi-periodic problem (2.1) into a periodic problem. We introduce the following shifted differential operators:

$$\nabla_{\boldsymbol{\alpha}}=\nabla+\boldsymbol{\alpha}iI,$$

where I is the identity operator. To enforce the constraint $\nabla_{\boldsymbol{\alpha}}\cdot\boldsymbol{u}=0$, we introduce a new variable p as a Lagrange multiplier. Then the problem becomes: for all $\boldsymbol{\alpha}\in K$, find $(\boldsymbol{u},p,\omega^2)$, such that

$$\begin{aligned}\nabla_{\boldsymbol{\alpha}}\times(\epsilon^{-1}\nabla_{\boldsymbol{\alpha}}\times\boldsymbol{u})-\nabla_{\boldsymbol{\alpha}}p=\omega^2\boldsymbol{u}\quad&\text{in }\Omega,\\ \nabla_{\boldsymbol{\alpha}}\cdot\boldsymbol{u}=0\qquad&\text{in }\Omega,\end{aligned} \tag{2.3}$$

with periodic boundary conditions $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a})=\boldsymbol{u}(\boldsymbol{x})$ and $p(\boldsymbol{x}+\boldsymbol{a})=p(\boldsymbol{x})$ for all $\boldsymbol{x}\in\mathbb{R}^d$ and $\boldsymbol{a}\in A$.

3 Functional Spaces and Related Theoretical Results

We denote the complex conjugate of a vector by $\bar{\boldsymbol{v}}$ and introduce an inner product $(\boldsymbol{u},\boldsymbol{v})=\int_D\boldsymbol{u}\cdot\bar{\boldsymbol{v}}d\boldsymbol{x}$ on the space $\boldsymbol{L}^2(D)$ with $D\subseteq\mathbb{R}^d$. We define the norm $\|\cdot\|_{s,D}$ for the Sobolev space $H^s(D)$ as $\|\cdot\|_{s,D}=\left(\sum_{|\boldsymbol{\alpha}|\le s}\|D(\cdot)^{\boldsymbol{\alpha}}\|_{0,D}^2\right)^{1/2}$. We have the following standard inner products on the Sobolev spaces $H(\text{div};D)$ and $H(\text{curl};D)$:

$$\begin{aligned}(\boldsymbol{u},\boldsymbol{v})_{H(\text{div};D)}&=(\boldsymbol{u},\boldsymbol{v})+(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v}),\\ (\boldsymbol{u},\boldsymbol{v})_{H(\text{curl};D)}&=(\boldsymbol{u},\boldsymbol{v})+(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}).\end{aligned}$$

We denote the norms induced by these inner products as $\|\cdot\|_{H(\mathrm{div};D)}$ in $H(\mathrm{div}; D)$ and $\|\cdot\|_{H(\mathrm{curl};D)}$ in $H(\mathrm{curl}; D)$, respectively. Now let us define the periodic Sobolev spaces, which are needed for the weak formulation of (2.3). For a bounded, open and simply-connected Lipschitz domain D, we define

$$\begin{aligned}
\mathcal{C}^{\infty}(\overline{D}) &:= \{f : \overline{D} \to \mathbb{C} : D^{\boldsymbol{\alpha}} f \text{ exists, } \forall \text{ multi-indices } \boldsymbol{\alpha}\},\\
\mathcal{C}^{\infty}_{\mathrm{per}}(\overline{D}) &:= \{f \in \mathcal{C}^{\infty}(\overline{D}) : f(\boldsymbol{x}+\boldsymbol{a}) = f(\boldsymbol{x}),\ \ \boldsymbol{a} \in A,\ \ \boldsymbol{x},\ \boldsymbol{x}+\boldsymbol{a} \in \partial D\},\\
\mathcal{C}^{\infty}_{\mathrm{curl}}(\overline{D}) &:= \{\boldsymbol{f} \in [\mathcal{C}^{\infty}(\overline{D})]^d : \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{a}) \times \boldsymbol{n} = -\boldsymbol{f}(\boldsymbol{x}) \times \boldsymbol{n},\ \ \boldsymbol{a} \in A,\ \ \boldsymbol{x},\ \boldsymbol{x}+\boldsymbol{a} \in \partial D\},\\
\mathcal{C}^{\infty}_{\mathrm{div}}(\overline{D}) &:= \{\boldsymbol{f} \in [\mathcal{C}^{\infty}(\overline{D})]^d : \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{a}) \cdot \boldsymbol{n} = -\boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n},\ \ \boldsymbol{a} \in A,\ \ \boldsymbol{x},\ \boldsymbol{x}+\boldsymbol{a} \in \partial D\},
\end{aligned}$$

where $\boldsymbol{n}$ is the outward normal vector at the boundary of the domain D. The periodic Sobolev spaces are the closures of the above spaces with respect to the standard Sobolev space norms:

$$\begin{aligned}
L^2_{\mathrm{per}}(D) &= \overline{\mathcal{C}^{\infty}_{\mathrm{per}}(\overline{D})}^{\|\cdot\|_{0,D}}, & \boldsymbol{L}^2_{\mathrm{per}}(D) &= \overline{\left[\mathcal{C}^{\infty}_{\mathrm{per}}(\overline{D})\right]^d}^{\|\cdot\|_{0,D}},\\
H^1_{\mathrm{per}}(D) &= \overline{\mathcal{C}^{\infty}_{\mathrm{per}}(\overline{D})}^{\|\cdot\|_{1,D}}, & \boldsymbol{H}^1_{\mathrm{per}}(D) &= \left[H^1_{\mathrm{per}}(\overline{D})\right]^d,\\
\boldsymbol{H}_{\mathrm{per}}(\mathrm{curl}; D) &= \overline{\mathcal{C}^{\infty}_{\mathrm{curl}}(\overline{D})}^{\|\cdot\|_{H(\mathrm{curl};D)}}, & \boldsymbol{H}_{\mathrm{per}}(\mathrm{div}; D) &= \overline{\mathcal{C}^{\infty}_{\mathrm{div}}(\overline{D})}^{\|\cdot\|_{H(\mathrm{div};D)}}.
\end{aligned} \tag{3.1}$$

With the notation above, the spaces for our weak formulation are defined as follows:

$$\begin{aligned}
Q &:= H^1_{\mathrm{per}}(\Omega),\\
\boldsymbol{V} &:= \boldsymbol{H}_{\mathrm{per}}(\mathrm{curl}; \Omega),\\
\boldsymbol{V}^0 &:= \boldsymbol{H}_{\mathrm{per}}(\mathrm{curl}^0_{\boldsymbol{\alpha}}; \Omega) = \{\boldsymbol{v} \in \boldsymbol{V} : \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v} = \boldsymbol{0}\},\\
\boldsymbol{H}_{\mathrm{per}}(\mathrm{div}^0_{\boldsymbol{\alpha}}; \Omega) &:= \{\boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathrm{div}; \Omega) : \nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{v} = \boldsymbol{0}\},\\
\boldsymbol{W} &:= \boldsymbol{V} \cap \boldsymbol{H}_{\mathrm{per}}(\mathrm{div}^0_{\boldsymbol{\alpha}}; \Omega).
\end{aligned} \tag{3.2}$$

The space $\boldsymbol{V}$ is endowed with the following seminorm and inner product:

$$\begin{aligned}
|\boldsymbol{u}|_{\boldsymbol{V}} &= \|\epsilon^{-\frac{1}{2}} \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}\|_{\Omega},\\
(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}} &= (\epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}, \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}).
\end{aligned}$$

Let $\mathcal{J} = \{2\pi \boldsymbol{I} : \boldsymbol{I} \in \mathbb{Z}^d\}$. Then $\boldsymbol{u} \in \boldsymbol{L}^2_{\mathrm{per}}(\Omega)$ and $p \in L^2_{\mathrm{per}}(\Omega)$ can be expanded as

$$\boldsymbol{u} = \sum_{\boldsymbol{I} \in \mathcal{J}} e^{i \boldsymbol{I} \cdot \boldsymbol{x}} \boldsymbol{C}_{\boldsymbol{I}} \quad \text{and} \quad p = \sum_{\boldsymbol{I} \in \mathcal{J}} c_{\boldsymbol{I}} e^{i \boldsymbol{I} \cdot \boldsymbol{x}},$$

where $\boldsymbol{C}_{\boldsymbol{I}} \in \mathbb{C}^d$ is a d-dimensional complex vector and $c_{\boldsymbol{I}} \in \mathbb{C}$ is a complex number. A consequence of [16, Theorem 3.1] is the following:

Lemma 3.1 *Assume that Ω is a periodic, simply-connected domain and $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$. Then $\phi = 0$ if and only if $\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{u} = 0$, and $\boldsymbol{w} = \boldsymbol{0}$ if and only if $\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u} = \boldsymbol{0}$. Furthermore, if $\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u} = \boldsymbol{0}$ and $\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{u} = 0$, we have $\boldsymbol{u} = \boldsymbol{0}$.*

Lemma 3.2 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$ and $p \in H^1_{\mathrm{per}}(\Omega)$, the following Poincaré inequality holds: $\|\nabla_{\boldsymbol{\alpha}} p\|_{0,\Omega} \geq C \|p\|_{0,\Omega}$ with C a positive constant independent of p, and $p = 0$ if and only if $\nabla_{\boldsymbol{\alpha}} p = \boldsymbol{0}$.*

Proof As $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$ and $\boldsymbol{I} \in \mathcal{J}$, $\boldsymbol{\alpha} + \boldsymbol{I}$ never vanishes and satisfies

$$|\boldsymbol{\alpha} + \boldsymbol{I}| > C,$$

where C is a positive constant, independent of $\boldsymbol{I}$.

From the expansion $p = \sum_{\boldsymbol{I}\in\mathcal{J}} c_{\boldsymbol{I}} e^{i\boldsymbol{I}\cdot\boldsymbol{x}}$ and $\nabla_{\boldsymbol{\alpha}} p = i \sum_{\boldsymbol{I}\in\mathcal{J}} (\boldsymbol{\alpha}+\boldsymbol{I}) c_{\boldsymbol{I}} e^{i\boldsymbol{I}\cdot\boldsymbol{x}}$, we know that the Poincaré inequality $\|\nabla_{\boldsymbol{\alpha}} p\|_{0,\Omega} \geq C\|p\|_{0,\Omega}$ holds, from which the conclusion directly follows. □

It is obvious that $\nabla_{\boldsymbol{\alpha}}$ obeys similar rules as ∇, viz.: $\nabla_{\boldsymbol{\alpha}} \cdot (\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v}) = 0$, and $\nabla_{\boldsymbol{\alpha}} \times (\nabla_{\boldsymbol{\alpha}} \phi) = \mathbf{0}$, which gives us the characterization: $\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega) \subset \boldsymbol{V}^0$. Now we consider the orthogonal complement $\left[\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)\right]^{\perp}$ of $\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)$ in $\boldsymbol{V}^0$. Let $\boldsymbol{v} \in \left[\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)\right]^{\perp}$, $0 = (\boldsymbol{v}, \nabla_{\boldsymbol{\alpha}} q) = -(\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{v}, q)$, for all $q \in H^1_{\text{per}}(\Omega)$, which means $\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{v} = 0$. Combining this with $\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v} = \mathbf{0}$, by Lemma 3.1, we know $\boldsymbol{v} = \mathbf{0}$. Then, we can conclude that

$$\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega) = \boldsymbol{V}^0.$$

Theorem 3.1 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$, the following L^2-decomposition holds:*

$$\boldsymbol{L}^2_{\text{per}}(\Omega) = \boldsymbol{H}_{\text{per}}(\text{div}^0_{\boldsymbol{\alpha}}; \Omega) \oplus \boldsymbol{V}^0. \tag{3.3}$$

Proof Firstly, we show that $\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)$ is a closed subspace in $\boldsymbol{L}^2_{\text{per}}(\Omega)$. As $\boldsymbol{L}^2_{\text{per}}(\Omega)$ is closed, for a Cauchy sequence $\{\nabla_{\boldsymbol{\alpha}} p_k\} \subset \nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)$, there is $\boldsymbol{w} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$ such that $\nabla_{\boldsymbol{\alpha}} p_k \to \boldsymbol{w}$ in $\boldsymbol{L}^2_{\text{per}}(\Omega)$. Using the Poincaré inequality in Lemma 3.2, for $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$, we know that $\{p_k\}$ is also a Cauchy sequence in $H^1_{\text{per}}(\Omega)$. By the definition of $H^1_{\text{per}}(\Omega)$ in (3.1), we know that $H^1_{\text{per}}(\Omega)$ is a closed space and then there exists some $p \in H^1_{\text{per}}(\Omega)$ such that $p_k \to p$ in $H^1_{\text{per}}(\Omega)$. Then we know $\boldsymbol{w} = \nabla_{\boldsymbol{\alpha}} p$, and $\nabla_{\boldsymbol{\alpha}} p_k \to \boldsymbol{w} \in \nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)$.

Let $\left[\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)\right]^{\perp}$ be the orthogonal complement of $\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)$ in $\boldsymbol{L}^2_{\text{per}}(\Omega)$ in the sense of the inner product $(\cdot, \cdot)$. It is enough to show that

$$\left[\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)\right]^{\perp} = \boldsymbol{H}_{\text{per}}(\text{div}^0_{\boldsymbol{\alpha}}; \Omega).$$

For any $\boldsymbol{v} \in \left[\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)\right]^{\perp}$,

$$0 = (\boldsymbol{v}, \nabla_{\boldsymbol{\alpha}} q) = -(\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{v}, q),$$

for all $q \in H^1_{\text{per}}(\Omega)$, which means $\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{v} = 0$, and $\boldsymbol{v} \in \boldsymbol{H}_{\text{per}}(\text{div}^0_{\boldsymbol{\alpha}}; \Omega)$.

For any $\boldsymbol{v} \in \boldsymbol{H}_{\text{per}}(\text{div}^0_{\boldsymbol{\alpha}}; \Omega)$, by going back the above procedure, it is easy to see that for all $q \in H^1_{\text{per}}(\Omega)$, $(\boldsymbol{v}, \nabla_{\boldsymbol{\alpha}} q) = 0$. Then $\boldsymbol{v} \in \left[\nabla_{\boldsymbol{\alpha}} H^1_{\text{per}}(\Omega)\right]^{\perp}$. □

Combined with the definition of $\boldsymbol{W}$ given in (3.2), we obtain the following corollary.

Corollary 3.1 *The following decomposition holds:*

$$\boldsymbol{V} = \boldsymbol{W} \oplus \boldsymbol{V}^0. \tag{3.4}$$

The variational form of (2.3) can now be stated as: find $(\boldsymbol{u}, p, \omega^2) \in \boldsymbol{V} \times Q \times \mathbb{C}$ with $(\boldsymbol{u}, p) \neq (\boldsymbol{0}, 0)$, such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \omega^2(\boldsymbol{u}, \boldsymbol{v}), \\ \overline{b(\boldsymbol{u}, q)} &= 0, \end{aligned} \tag{3.5}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q$, where

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u} \cdot \overline{\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v}} d\boldsymbol{x}, \\ b(\boldsymbol{v}, p) &= -\int_\Omega \overline{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{\alpha}} p d\boldsymbol{x}. \end{aligned} \tag{3.6}$$

For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, we define the operators (T, T_p) as: for $\boldsymbol{f} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$, $(T\boldsymbol{f}, T_p\boldsymbol{f}) \in \boldsymbol{L}^2_{\text{per}}(\Omega) \times L^2_{\text{per}}(\Omega)$ satisfies

$$\begin{aligned} a(T\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{v}, T_p\boldsymbol{f}) &= (\boldsymbol{f}, \boldsymbol{v}), \\ \overline{b(T\boldsymbol{f}, q)} &= 0, \end{aligned} \tag{3.7}$$

for all $\boldsymbol{f} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$ and $(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q$. From (3.3) and Corollary 3.1, we obtain $\boldsymbol{u} \in \boldsymbol{W}$. We have the following results as Lemma 2 in [3] and Theorem 3.2 in [16], respectively.

Theorem 3.2 *The operator T is compact and self-adjoint from $\boldsymbol{L}^2_{\text{per}}(\Omega)$ to itself. Moreover, $\boldsymbol{W}$ is compactly embedded in $\boldsymbol{L}^2_{\text{per}}(\Omega)$.*

Theorem 3.3 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, let $\boldsymbol{u}$ and p be the solution of (3.7). Then*

$$\|\boldsymbol{u}\|_{1,\Omega} + \|p\|_{1,\Omega} \leq C\|\boldsymbol{f}\|_{0,\Omega}.$$

Therefore, applying the Spectral Theory for compact and self-adjoint operators [34, Theorem 2.36 and Theorem 4.18], [37, Section VII.4], we obtain the following theorem.

Theorem 3.4 *For eigenproblem (3.5), there exists a sequence of eigenvalues $\{\omega^2_{\boldsymbol{\alpha},i}\}_{i\geq 1}$ satisfying*

$$0 < \omega^2_{\boldsymbol{\alpha},1} \leq \omega^2_{\boldsymbol{\alpha},2} \leq \dots .$$

The only possible accumulation point of the sequence $\{\omega^2_{\boldsymbol{\alpha},i}\}_{i\geq 1}$ is at $+\infty$. The eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the $\boldsymbol{L}^2_{\text{per}}(\Omega)$ inner product. The eigenspaces of nonzero eigenvalues are finite dimensional.

4 Discrete Periodic Spaces

In this section, we introduce the discontinuous Galerkin spaces. The Nédélec elements of the first family [36] are considered here, while the second family can be studied similarly. Let $\mathcal{T}_h$ be a mesh consisting of triangles or quadrilaterals in two dimensions, or tetrahedra or parallelepipeds in three dimensions. Here, the analysis is restricted to tetrahedra, but computations will also be performed for cubic elements. Because of the discontinuity of DG spaces, we take for each element $K \in \mathcal{T}_h$ its corresponding Nédélec element without enforcing tangential continuity. Next, we introduce some scalar and vector periodic discontinuous finite elements spaces:

$$\begin{aligned} Q_h^{\boldsymbol{\alpha}} &= \{\phi \in L^2(\Omega) : \phi|_K = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{x}} \tilde{\phi} \text{ for some } \tilde{\phi} \in \mathcal{P}_{k+1}(K) \quad \forall K \in \mathcal{T}_h\}, \\ \boldsymbol{V}_h^{\boldsymbol{\alpha}} &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_K = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{x}} \tilde{\boldsymbol{v}} \text{ for some } \tilde{\boldsymbol{v}} \in \mathcal{S}_k(K) \quad \forall K \in \mathcal{T}_h\}, \end{aligned}$$

where $\mathcal{P}_k(K)$ is the set local polynomials of degree less than or equal to k on K; the elements in $\mathcal{S}_k(K)$ have the forms $\boldsymbol{a}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}) \times \boldsymbol{x}$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{P}_k(K)^3$. We also define periodic conforming versions of these spaces as $\boldsymbol{V}_h^{\boldsymbol{\alpha},c} := \boldsymbol{V}_h^{\boldsymbol{\alpha}} \cap \boldsymbol{V}$, $Q_h^{\boldsymbol{\alpha},c} := Q_h^{\boldsymbol{\alpha}} \cap Q$. We define $\boldsymbol{V}_h^{\boldsymbol{\alpha},c,0}$ as the divergence-free subspace of $\boldsymbol{V}_h^{\boldsymbol{\alpha},c}$:

$$\boldsymbol{V}_h^{\boldsymbol{\alpha},c,0} = \{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{\alpha},c} | (\boldsymbol{v}_h, \nabla_{\boldsymbol{\alpha}} q_h) = 0 \quad \text{for all } q_h \in Q_h^{\boldsymbol{\alpha},c}\}.$$

The discrete Helmholtz decomposition given in Lemma 4.5 in [34] can be changed to the periodic case:

$$\boldsymbol{V}_h^{\boldsymbol{\alpha},c} = \boldsymbol{V}_h^{\boldsymbol{\alpha},c,0} \oplus \nabla_{\boldsymbol{\alpha}} Q_h^{\boldsymbol{\alpha},c}. \tag{4.1}$$

Next we state an important property for $\boldsymbol{V}_h^{\boldsymbol{\alpha},c,0}$ in [3, Lemma 8].

Lemma 4.1 *For all h small enough and any $\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{\alpha},c,0}$, there exist a $\boldsymbol{v} \in \boldsymbol{W}$ such that*

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,\Omega} \le \eta_h \|\boldsymbol{v}_h\|_{\boldsymbol{H}_{\mathrm{per}}(\mathrm{curl};\Omega)}$$

with $\eta_h \to 0$ as $h \to 0$.

Lemmas 4 and 5 in [3] give the following estimates for the conforming interpolation operators $\Pi_{Q_h^{\boldsymbol{\alpha},c}}$ and $\Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}}$.

Lemma 4.2 *Suppose that $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$, $\nabla \times \boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$, $q \in H^s(\Omega)$. Then there exist a constant $C > 0$ independent of the mesh size h, such that the following interpolation error estimates hold true for $k \ge 0$:*

$$\begin{aligned}
\|q - \Pi_{Q_h^{\boldsymbol{\alpha},c}} q\|_{1,\Omega} &\le C h^{s-1} \|q\|_{s,\Omega} && 1 \le s \le k+2,\\
\|\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v} - \nabla_{\boldsymbol{\alpha}} \times \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}} \boldsymbol{v}\|_{0,\Omega} &\le C h^s |\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v}|_{s,\Omega} && 0 < s \le k+1,\\
\|\boldsymbol{v} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}} \boldsymbol{v}\|_{0,\Omega} &\le C h^s \left(|\boldsymbol{v}|_{s,\Omega} + \|\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \right) && 1/2 < s \le 1,\ p > 2,\\
\|\boldsymbol{v} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}} \boldsymbol{v}\|_{0,\Omega} &\le C h^s |\boldsymbol{v}|_{s,\Omega} && 1 < s \le k+1,\ k > 0.
\end{aligned}$$

5 Discontinuous Galerkin Approximation

Let $\mathcal{T}_h$ be a periodic, shape-regular, conformal mesh on Ω aligned with the possible discontinuities of ϵ, where the word "periodic" means that the meshes are the same on each pair of periodic parts of the boundary. Let $h = \max_{K \in \mathcal{T}_h} h_K$, where h_K is the diameter of $K \in \mathcal{T}_h$. We denote the set of all faces of $\mathcal{T}_h$ by $\mathcal{F}_h$, the set of boundary faces $\mathcal{F}_h^b = \mathcal{F}_h \cap \partial\Omega$ and the set of interior faces $\mathcal{F}_h^i = \mathcal{F}_h \backslash \mathcal{F}_h^b$. Here, we use the shifted discontinuous finite element spaces $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$ and $Q_h^{\boldsymbol{\alpha}}$. For functions that are discontinuous on element faces, we define jumps and averages across a face $f \in \mathcal{F}_h$ as follows: If $f \in \mathcal{F}_h$ is shared by tetrahedra $K_\pm \in \mathcal{T}_h$ with unit outward normal vectors $\boldsymbol{n}_\pm$, we define, respectively, the tangential and normal jumps and the average of $\boldsymbol{v}$ across the interior face $f \in \mathcal{F}_h^i$ as:

$$[\![\boldsymbol{v}]\!]_T := \boldsymbol{n}_+ \times \boldsymbol{v}_+ + \boldsymbol{n}_- \times \boldsymbol{v}_-, \quad [\![\boldsymbol{v}]\!]_N := \boldsymbol{n}_+ \cdot \boldsymbol{v}_+ + \boldsymbol{n}_- \cdot \boldsymbol{v}_-, \quad \{\!\{\boldsymbol{v}\}\!\} := \frac{1}{2}(\boldsymbol{v}_+ + \boldsymbol{v}_-),$$

and we define the normal jump and average of q as:

$$[\![q]\!]_N = \boldsymbol{n}_+ q_+ + \boldsymbol{n}_- q_-, \qquad \{\!\{q\}\!\} = \frac{1}{2}(q_+ + q_-),$$

where $\boldsymbol{v}_\pm$ denote the traces of $\boldsymbol{v}$ taken from within $K_\pm$, that is $\boldsymbol{v}_\pm := \boldsymbol{v}|_{\partial K_\pm}$. At the boundary, we define the jumps and means in a periodic way. Next, we define the mesh function h as $h(\boldsymbol{x}) := h_f, \boldsymbol{x} \in f, f \in \mathcal{F}_h$, where h_f is the diameter of the face f. If $\boldsymbol{x}$ is in the interior of $\partial K_+ \cap \partial K_-$,

$$e(\boldsymbol{x}) = \min\{\epsilon_+(\boldsymbol{x}), \epsilon_-(\boldsymbol{x})\},$$

where $\epsilon_\pm$ are the extensions of $\epsilon|_{K_\pm}$ up to $\partial K_\pm$. Note that $h, e \in L^\infty(\mathcal{F}_h)$. Let $\boldsymbol{V}(h) := \boldsymbol{V} + \boldsymbol{V}_h^{\boldsymbol{\alpha}}$ and $Q(h) = Q + Q_h^{\boldsymbol{\alpha}}$. We introduce the stabilization parameters as follows:

$$a_F = \mathfrak{a} e^{-1} h^{-1}, \quad b_F = \mathfrak{b} h, \quad c_F = \mathfrak{c} h^{-1},$$

where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are positive parameters, independent of the mesh size and the coefficient ϵ. With these definitions in mind, we define the seminorms and norms on $\boldsymbol{V}(h)$ and $Q(h)$ as follows:

$$\begin{aligned}
|\boldsymbol{v}|^2_{\boldsymbol{V}(h)} &= \|\epsilon^{-\frac{1}{2}} \nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{v}\|^2_{0,\Omega} + \|e^{-\frac{1}{2}} h^{-\frac{1}{2}} [\![\boldsymbol{v}]\!]_T\|^2_{0,\mathcal{F}_h} + \|h^{\frac{1}{2}} [\![\boldsymbol{v}]\!]_N\|^2_{0,\mathcal{F}_h},\\
\|\boldsymbol{v}\|^2_{\boldsymbol{V}(h)} &= \|\boldsymbol{v}\|^2_{0,\Omega} + |\boldsymbol{v}|^2_{\boldsymbol{V}(h)},\\
\|q\|^2_{Q(h)} &= \|\nabla_{\boldsymbol{\alpha},h} q\|^2_{0,\Omega} + \|h^{-\frac{1}{2}} [\![q]\!]_N\|^2_{0,\mathcal{F}_h},
\end{aligned}$$

where $\nabla_{\boldsymbol{\alpha},h}$ stands for the element wise application of the differential operator $\nabla_{\boldsymbol{\alpha}}$.

Remark 5.1 As we define the normal jump $[\![\cdot]\!]_N$ of a scalar function in a periodic way, for a constant $0 \neq C \equiv q \in H^1_{\text{per}}(\Omega)$, the jump $[\![q]\!]_N = 0$. But, since $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$ and by Lemma 3.2, $\|q\|_{Q(h)} = \|\nabla_{\boldsymbol{\alpha}} C\|_{0,\Omega} \neq 0$, hence the definition of $\|\cdot\|_{Q(h)}$ is still proper.

We also introduce an auxiliary discrete space:

$$M_h^{\boldsymbol{\alpha}} = \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\mathcal{F}_h) : \boldsymbol{\eta}|_f = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{x}} \tilde{\boldsymbol{\eta}} \text{ for some } \tilde{\boldsymbol{\eta}} \in \mathcal{S}_k(f) \quad \forall f \in \mathcal{F}_h\},$$

with norm $\|\boldsymbol{\eta}\|_{M_h^{\boldsymbol{\alpha}}} = \|h^{-\frac{1}{2}} \boldsymbol{\eta}\|_{0,\mathcal{F}_h}$. Let $\boldsymbol{U}_h^{\boldsymbol{\alpha}} := \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times M_h^{\boldsymbol{\alpha}}$ and $\boldsymbol{U}(h) := \boldsymbol{V}(h) \times M_h^{\boldsymbol{\alpha}}$ with semi-norm and norm defined as:

$$|(\boldsymbol{v}, \boldsymbol{\eta})|^2_{\boldsymbol{U}(h)} = |\boldsymbol{v}|^2_{\boldsymbol{V}(h)} + \|\boldsymbol{\eta}\|^2_{M_h^{\boldsymbol{\alpha}}}, \quad \|(\boldsymbol{v}, \boldsymbol{\eta})\|^2_{\boldsymbol{U}(h)} = \|\boldsymbol{v}\|^2_{\boldsymbol{V}(h)} + \|\boldsymbol{\eta}\|^2_{M_h^{\boldsymbol{\alpha}}}.$$

From Lemma 23 in [41], we have the following estimate.

Lemma 5.1 *Under the assumptions of Lemma 4.2, we have*

$$\|h^{\frac{1}{2}} [\![\boldsymbol{v} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}} \boldsymbol{v}]\!]_N\|_{0,\mathcal{F}_h} \le C h^s \left(\|\boldsymbol{v}\|_{s,\Omega} + \|\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v}\|_{s,\Omega}\right) \quad 1/2 < s \le k+1.$$

As $\Pi_{Q_h^{\boldsymbol{\alpha},c}}$ and $\Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}}$ are conforming interpolations, we have $[\![\Pi_{Q_h^{\boldsymbol{\alpha},c}} q]\!]_N = \boldsymbol{0}$ and $[\![\Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}} \boldsymbol{v}]\!]_T = \boldsymbol{0}$ on $\mathcal{F}_h$, for $q \in Q$ and $\boldsymbol{v} \in \boldsymbol{V}$. From the facts that $Q_h^{\boldsymbol{\alpha},c} \subset Q_h^{\boldsymbol{\alpha}}$, $\boldsymbol{V}_h^{\boldsymbol{\alpha},c} \subset \boldsymbol{V}_h^{\boldsymbol{\alpha}}$ and ϵ is piecewise constant, we obtain a corollary of Lemmas 4.2 and 5.1.

Corollary 5.1 *Suppose that $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$, $\epsilon^{-1}\nabla \times \boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$, $p \in H^s(\Omega)$. Then there exists a constant $C > 0$ independent of h, such that the following interpolation error estimates hold true for $k \ge 0$:*

$$\inf_{q \in Q} \|p - q\|_{Q(h)} \le \|p - \Pi_{Q_h^{\boldsymbol{\alpha},c}} p\|_{Q(h)} \le C h^{\min\{s,k+1\}} \|p\|_{s+1,\Omega}, \quad \textit{for} \quad s \ge 0,$$

$$\inf_{\boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{V}(h)} \le \|\boldsymbol{u} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha},c}} \boldsymbol{u}\|_{\boldsymbol{V}(h)} \le C h^{\min\{s,k+1\}} \left(\|\boldsymbol{u}\|_{s,\Omega} + \|\epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}\|_{s,\Omega}\right),$$

$$\textit{for} \quad s > 1.$$

5.1 Mixed DG Form

For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$, we introduce the following mixed DG method: find $(\boldsymbol{u}_h, p_h, \omega_h^2) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}} \times \mathbb{C}$ with $(\boldsymbol{u}_h, p_h) \neq (\mathbf{0}, 0)$, such that

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}) + b_h(\boldsymbol{v}, p_h) &= \omega_h^2(\boldsymbol{u}_h, \boldsymbol{v}), \\ \overline{b_h(\boldsymbol{u}_h, q)} - c_h(p_h, q) &= 0, \end{aligned} \tag{5.1}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$, where the discrete forms a_h, b_h and c_h are defined as follows:

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) =& \int_\Omega \epsilon^{-1} \nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u} \cdot \overline{\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{v}} d\boldsymbol{x} - \int_{\mathcal{F}_h} [\![\boldsymbol{u}]\!]_T \cdot \overline{\{\!\{\epsilon^{-1} \nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{v}\}\!\}} ds \\ &- \int_{\mathcal{F}_h} \overline{[\![\boldsymbol{v}]\!]}_T \cdot \{\!\{\epsilon^{-1} \nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u}\}\!\} ds + \int_{\mathcal{F}_h} \mathfrak{a}_F [\![\boldsymbol{u}]\!]_T \cdot \overline{[\![\boldsymbol{v}]\!]}_T ds + \int_{\mathcal{F}_h} \mathfrak{b}_F [\![\boldsymbol{u}]\!]_N \overline{[\![\boldsymbol{v}]\!]}_N ds, \\ b_h(\boldsymbol{v}, p) =& -\int_\Omega \overline{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{\alpha},h} p d\boldsymbol{x} + \int_{\mathcal{F}_h} \overline{\{\!\{\boldsymbol{v}\}\!\}} \cdot [\![p]\!]_N ds, \\ c_h(p, q) =& \int_{\mathcal{F}_h} \mathfrak{c}_F [\![p]\!]_N \cdot \overline{[\![q]\!]}_N ds. \end{aligned}$$

We define the DG operators T_h and $T_{p,h}$ as follows: for $\boldsymbol{f} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$, $(\boldsymbol{u}_h, p_h) = (T_h \boldsymbol{f}, T_{p,h} \boldsymbol{f}) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$ satisfies

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) + b_h(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}), \tag{5.2}$$

$$\overline{b_h(\boldsymbol{u}_h, q)} - c_h(p_h, q) = 0, \tag{5.3}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$.

From Theorem 3.3, we know that the exact solution $(\boldsymbol{u}, p)$ belongs to $\boldsymbol{H}^1_{\text{per}}(\Omega) \times H^1_{\text{per}}(\Omega)$, which means that all jumps of $\boldsymbol{u}$ and p vanish.

Lemma 5.2 *For any $(\boldsymbol{v}, q) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$, the solution $(\boldsymbol{u}, p)$ of (3.7) satisfies the DG formulation (5.2)–(5.3).*

By adding the term $\mathfrak{b} \| h^{\frac{1}{2}} [\![\boldsymbol{v}]\!]_N \|^2_{0,\mathcal{F}_h}$, it is straightforward to obtain the continuity and semi-norm ellipticity of $a_h(\cdot, \cdot)$ from the result in [42, Lemma 4 and Lemma 6].

Lemma 5.3 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$, there exist a parameter $a_{stab} > 0$, independent of the mesh size and the coefficient ϵ, such that for $\mathfrak{a} \geq a_{stab} > 0$ and $\mathfrak{b} > 0$,*

$$a_h(\boldsymbol{u}, \boldsymbol{v}) \leq C_1 \|\boldsymbol{u}\|_{\boldsymbol{V}(h)} \|\boldsymbol{v}\|_{\boldsymbol{V}(h)} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}},$$

$$\text{and} \quad a_h(\boldsymbol{u}, \boldsymbol{u}) \geq C_2 |\boldsymbol{u}|^2_{\boldsymbol{V}(h)} \quad \forall\, \boldsymbol{u} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}},$$

with constants $C_1, C_2 > 0$ independent of the mesh size and the coefficient ϵ.

Theorem 5.1 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$, $\mathfrak{a} \geq a_{stab}$, $\mathfrak{b} > 0$ and $\mathfrak{c} > 0$, the mixed DG method defined by (5.2) and (5.3) has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$.*

Proof It is enough to show that $\boldsymbol{f} = \mathbf{0}$ implies $\boldsymbol{u}_h = \mathbf{0}$ and $p_h = 0$. Taking $\boldsymbol{v} = \boldsymbol{u}_h$ in (5.2) and $q = p_h$ in (5.3) and subtracting the complex conjugate of (5.3) from (5.2), we have $a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + c_h(p_h, p_h) = 0$. From Lemma 5.3, we know $\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u}_h = \mathbf{0}$ in Ω, $[\![\boldsymbol{u}_h]\!]_T = \mathbf{0}$ on $\mathcal{F}_h$, $[\![\boldsymbol{u}_h]\!]_N = 0$ on $\mathcal{F}_h$ and $[\![p_h]\!]_N = \mathbf{0}$ on $\mathcal{F}_h$. Then $\boldsymbol{u}_h \in \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega) \cap \boldsymbol{H}_{\text{per}}(\text{div}; \Omega)$ with $\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}_h = \mathbf{0}$ and $p_h \in H^1_{\text{per}}(\Omega)$. From (5.3), we have that $\int_\Omega (\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{u}_h) \overline{q} d\boldsymbol{x} = 0$ for any $q \in Q_h^{\boldsymbol{\alpha}}$, and $\nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{u}_h = 0$. From Lemma 3.1, we have $\boldsymbol{u}_h = \mathbf{0}$.

Equation (5.2) for $\boldsymbol{f} = \mathbf{0}$ becomes $\int_\Omega \boldsymbol{v} \cdot \overline{\nabla_{\boldsymbol{\alpha}} p_h} d\boldsymbol{x} = 0$, for any $\boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}$, hence $\nabla_{\boldsymbol{\alpha}} p_h = \mathbf{0}$. From Lemma 3.2, we obtain $p_h = 0$. □

Similar to [25], we introduce an auxiliary mixed formulation, which can be analyzed using classical theory. Firstly, we introduce the lifting operators $\mathcal{L}$ and $\mathcal{M}$. For $\boldsymbol{v} \in \boldsymbol{V}(h)$ and $q \in Q(h)$, we define $\mathcal{L}(\boldsymbol{v}) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}$ and $\mathcal{M}(q) \in [Q_h^{\boldsymbol{\alpha}}]^d$ by

$$\int_\Omega \mathcal{L}(\boldsymbol{v}) \cdot \overline{\boldsymbol{w}} d\boldsymbol{x} = \int_{\mathcal{F}_h} [\![\boldsymbol{v}]\!]_T \cdot \overline{\{\!\{\boldsymbol{w}\}\!\}} ds, \qquad \int_\Omega \mathcal{M}(q) \cdot \overline{\boldsymbol{w}} d\boldsymbol{x} = \int_{\mathcal{F}_h} \overline{\{\!\{\boldsymbol{w}\}\!\}} \cdot [\![q]\!]_N ds$$

for all $\boldsymbol{w} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}$. Next, we define the perturbed forms:

$$\begin{aligned}\tilde{a}_h(\boldsymbol{u}, \boldsymbol{v}) = &\int_\Omega \epsilon^{-1} \nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u} \cdot \overline{\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{v}} d\boldsymbol{x} - \int_\Omega \mathcal{L}(\boldsymbol{u}) \cdot \overline{(\epsilon^{-1} \nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{v})} d\boldsymbol{x} \\ &- \int_\Omega \overline{\mathcal{L}(\boldsymbol{v})} \cdot (\epsilon^{-1} \nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u}) d\boldsymbol{x} + \int_{\mathcal{F}_h} a_F [\![\boldsymbol{u}]\!]_T \cdot \overline{[\![\boldsymbol{v}]\!]}_T ds + \int_{\mathcal{F}_h} b_F [\![\boldsymbol{u}]\!]_N \overline{[\![\boldsymbol{v}]\!]}_N ds, \\ \tilde{b}_h(\boldsymbol{v}, p) = &-\int_\Omega \overline{\boldsymbol{v}} \cdot [\nabla_{\boldsymbol{\alpha},h} p - \mathcal{M}(p)] d\boldsymbol{x}.\end{aligned}$$

We obtain the following auxiliary mixed formulation of (5.2) and (5.3): find $(\boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times M_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$ such that

$$\begin{aligned}A_h(\boldsymbol{u}_h, \boldsymbol{\lambda}_h; \boldsymbol{v}, \boldsymbol{\eta}) + B_h(\boldsymbol{v}, \boldsymbol{\eta}; p_h) &= (\boldsymbol{f}, \boldsymbol{v}), \\ \overline{B_h(\boldsymbol{u}_h, \boldsymbol{\lambda}_h; q)} &= 0,\end{aligned} \tag{5.4}$$

for all $(\boldsymbol{v}, \boldsymbol{\eta}, q) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times M_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$, where A_h and B_h are given by

$$A_h(\boldsymbol{u}, \boldsymbol{\lambda}; \boldsymbol{v}, \boldsymbol{\eta}) = \tilde{a}_h(\boldsymbol{u}, \boldsymbol{v}) + \int_{\mathcal{F}_h} c_F \boldsymbol{\lambda} \cdot \overline{\boldsymbol{\eta}} ds, \quad B_h(\boldsymbol{v}, \boldsymbol{\eta}; p) = \tilde{b}_h(\boldsymbol{v}, p) - \int_{\mathcal{F}_h} c_F [\![p]\!]_N \cdot \overline{\boldsymbol{\eta}} ds.$$

Lemma 5.4 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$, the form (5.4) has a unique solution $(\boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times M_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$, where $(\boldsymbol{u}_h, p_h)$ is the solution to the DG problem (5.2)–(5.3) and $\boldsymbol{\lambda}_h = [\![p_h]\!]_N$.*

Proof Choosing the test function $\boldsymbol{v} = \mathbf{0}$ in the first equation of the form (5.4), we have

$$\int_{\mathcal{F}_h} c_F \boldsymbol{\lambda}_h \cdot \overline{\boldsymbol{\eta}} ds = \int_{\mathcal{F}_h} c_F [\![p_h]\!]_N \cdot \overline{\boldsymbol{\eta}} ds \qquad \forall\, \boldsymbol{\eta} \in M_h^{\boldsymbol{\alpha}}.$$

Since c_F is a positive constant on each $f \in \mathcal{F}_h$, we have $\boldsymbol{\lambda}_h = [\![p_h]\!]_N$. Then (5.4) coincides with (5.2) and (5.3). Then we conclude that if $(\boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times M_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$ is a solution to (5.4), and $\boldsymbol{\lambda}_h = [\![p_h]\!]_N$, then $(\boldsymbol{u}_h, p_h)$ is the solution to (5.2) and (5.3). Next, using Theorem 5.1, we obtain that the solution to (5.4) is also unique. □

Assuming that $(\boldsymbol{u}, p)$ is the analytical solution to (3.7), we define the following residuals:

$$\begin{aligned}R_h^1(\boldsymbol{u}, p; \boldsymbol{v}, \boldsymbol{\nu}) &= A_h(\boldsymbol{u}, \mathbf{0}; \boldsymbol{v}, \boldsymbol{\nu}) + B_h(\boldsymbol{v}, \boldsymbol{\nu}; p) - a_h(\boldsymbol{u}, \boldsymbol{v}) - b_h(\boldsymbol{v}, p), \\ R_h^2(\boldsymbol{u}; q) &= \overline{B_h(\boldsymbol{u}, \mathbf{0}; q)} - \overline{b_h(\boldsymbol{u}, q)} + c_h(p, q),\end{aligned}$$

for all $(\boldsymbol{v}, \boldsymbol{\nu}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}}$ and $q \in Q_h^{\boldsymbol{\alpha}}$ and set

$$\begin{aligned}\mathcal{R}_h^1(\boldsymbol{u}, p) &= \sup_{(\mathbf{0},\mathbf{0}) \neq (\boldsymbol{v},\boldsymbol{\nu}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}}} \frac{\left|R_h^1(\boldsymbol{u}, p; \boldsymbol{v}, \boldsymbol{\nu})\right|}{\|(\boldsymbol{v}, \boldsymbol{\nu})\|_{\boldsymbol{U}(h)}}, \\ \mathcal{R}_h^2(\boldsymbol{u}) &= \sup_{0 \neq q \in Q_h^{\boldsymbol{\alpha}}} \frac{\left|R_h^2(\boldsymbol{u}; q)\right|}{\|q\|_{Q(h)}}.\end{aligned}$$

We also define the kernel of B_h:

$$\mathrm{Ker}(B_h) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}} \mid B_h(\boldsymbol{u}, \boldsymbol{v}; q) = 0, \forall q \in Q_h^{\boldsymbol{\alpha}}\}.$$

To analyze the convergence of T_h, we need the ellipticity of A_h on $\mathrm{Ker}(B_h)$, which is proven in Theorem 5.3 and the inf-sup condition of B_h, given in Theorem 5.2. The proofs are similar to those in [26], but we need to extend them to the periodic case.

For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, let $Q_h^{\boldsymbol{\alpha},\perp}$ be the orthogonal complement of $Q_h^{\boldsymbol{\alpha},c}$ in $Q_h^{\boldsymbol{\alpha}}$. Then $\|q\|_{Q_h^{\boldsymbol{\alpha},\perp}} = \|h^{-\frac{1}{2}}[\![q]\!]_N\|_{0,\mathcal{F}_h}$ can be a norm in $Q_h^{\boldsymbol{\alpha},\perp}$. In fact, if $\|q\|_{Q_h^{\boldsymbol{\alpha},\perp}} = 0$, $q \in Q_h^{\boldsymbol{\alpha},\perp} \cap Q_h^{\boldsymbol{\alpha},c} = \{0\}$. The following lemma follows Theorem 2.2 in [31]. Here we extend it to periodic case.

Lemma 5.5 *Let $\mathcal{T}_h$ be a conforming mesh consisting of triangles in 2D and tetrahedra in 3D. Then for $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$ and any $q_h \in Q_h^{\boldsymbol{\alpha}}$,*

$$\inf_{q_h^c \in Q_h^{\boldsymbol{\alpha},c}} \|\nabla_{\boldsymbol{\alpha},h}(q_h - q_h^c)\|_{0,\Omega} \leq C\|h^{-\frac{1}{2}}[\![q_h]\!]_N\|_{0,\mathcal{F}_h},$$

with a constant $C > 0$ independent of the mesh size.

Using the lemma above, we obtain the equivalence of norms $\|\cdot\|_{Q(h)}$ and $\|\cdot\|_{Q_h^{\boldsymbol{\alpha},\perp}}$ in $Q_h^{\boldsymbol{\alpha},\perp}$ as Theorem 5.3 in [26].

Lemma 5.6 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, there exist $C_1, C_2 \geq 0$ independent of the mesh size, such that $C_1\|q\|_{Q(h)} \leq \|q\|_{Q_h^{\perp}} \leq C_2\|q\|_{Q(h)}$ for any $q \in Q_h^{\boldsymbol{\alpha},\perp}$.*

With the two lemmas above, similar to Proposition 5.4 in [26], we obtain the inf-sup condition for the periodic case. The proofs of Lemma 5.5 and Theorem 5.2 can be found in "Appendix B".

Theorem 5.2 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$,*

$$\inf_{0 \neq q \in Q_h^{\boldsymbol{\alpha}}} \sup_{(\boldsymbol{0},\boldsymbol{0}) \neq (\boldsymbol{v},\boldsymbol{\nu}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}}} \frac{B_h(\boldsymbol{v}, \boldsymbol{\nu}; q)}{\|(\boldsymbol{v}, \boldsymbol{\nu})\|_{\boldsymbol{U}(h)}\|q\|_{Q(h)}} \geq k > 0,$$

for a constant k, independent of the mesh size and the coefficient ϵ.

From Lemmas 3.1, 3.2 and Theorem 5.1, for $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, we obtain that the solutions to both equation (2.3) and the DG formulation (5.2)–(5.3), and also to the auxiliary form (5.4) in the DG spaces used in Lemma 5.4, are unique in periodic domains. This implies that the ellipticity of A_h on $\mathrm{Ker}(B_h)$ is also satisfied in periodic domains under the proper condition on $\boldsymbol{\alpha}$. In the proof of the ellipticity property of A_h, we need to make sure that the map in the following lemma is continuous, similar to Corollary 7.2 in [18]. The proofs of Lemma 5.7 and Theorem 5.3 are given in "Appendix C".

Lemma 5.7 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, there exists a continuous linear map R : $\boldsymbol{H}_{\mathrm{per}}(\mathrm{curl}; \Omega) \to \boldsymbol{H}^1_{\mathrm{per}}(\Omega)$ such that $\nabla_{\boldsymbol{\alpha}} \times (R\boldsymbol{w}) = \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{w}$ for any $\boldsymbol{w} \in \boldsymbol{H}_{\mathrm{per}}(\mathrm{curl}; \Omega)$.*

Theorem 5.3 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, $\mathfrak{a} > 0$ large enough, $\mathfrak{b} > 0$ and $\mathfrak{c} > 0$, there is a constant $b > 0$, independent of the mesh size, such that*

$$A_h(\boldsymbol{u}, \boldsymbol{\nu}; \boldsymbol{u}, \boldsymbol{\nu}) \geq b\|(\boldsymbol{u}, \boldsymbol{\nu})\|^2_{\boldsymbol{U}(h)} \qquad \forall\ (\boldsymbol{u}, \boldsymbol{\nu}) \in \mathrm{Ker}(B_h).$$

With the ellipticity of A_h on Ker(B_h) and the inf-sup condition of B_h, the procedure to obtain abstract error estimates for the numerical solution to (5.4) is standard. The detailed proofs of Theorem 5.4 and Lemma 5.8 are in "Appendix D".

Theorem 5.4 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$, let $(\boldsymbol{u}, p)$ be the analytical solution of (3.7) satisfying $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$, $\epsilon^{-1}\nabla \times \boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$ and $p \in H^{s+1}(\Omega)$ for a regularity exponent $s > \frac{1}{2}$. Let $(\boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times M_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$ be the numerical solution of (5.4). Then the error can be estimated as*

$$\begin{aligned}
\|(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)} \leq C\Big(&\inf_{\boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{V}(h)} + \inf_{q \in Q_h^{\boldsymbol{\alpha}}} \|p - q\|_{Q(h)} \\
&+ \mathcal{R}_h^1(\boldsymbol{u}, p) + \mathcal{R}_h^2(\boldsymbol{u})\Big), \\
\|p - p_h\|_{Q(h)} \leq C\Big(&\inf_{q \in Q_h^{\boldsymbol{\alpha}}} \|p - q\|_{Q(h)} + \|(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)} + \mathcal{R}_h^1(\boldsymbol{u}, p)\Big).
\end{aligned}$$

We estimate a bound of the residuals.

Lemma 5.8 *Assume that $(\boldsymbol{u}, p)$ is the analytical solution to (3.7) such that $\boldsymbol{u} \in \boldsymbol{H}^s(\mathcal{T}_h)$ and $\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u} \in \boldsymbol{H}^s(\mathcal{T}_h)$ for $s > \frac{1}{2}$. Then we have*

$$\mathcal{R}_h^1(\boldsymbol{u}, p) + \mathcal{R}_h^2(\boldsymbol{u}) \leq C h^{\min\{s,k+1\}} \left(\|\boldsymbol{u}\|_{s,\Omega} + \|\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}\|_{s,\Omega}\right),$$

with a constant $C > 0$ independent of the mesh size and the coefficient ϵ.

From the error bound in Theorem 5.4, the estimates of the residuals given in Lemma 5.8 and Corollary 5.1, we obtain the error estimate of the numerical solution to (5.2).

Theorem 5.5 *Let $(\boldsymbol{u}, p)$ be the analytical solution of (3.7) satisfying $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$, $\epsilon^{-1}\nabla \times \boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$ and $p \in H^{s+1}(\Omega)$ for a regularity exponent $s > \frac{1}{2}$. Let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times Q_h^{\boldsymbol{\alpha}}$ be the numerical solution of (5.2)–(5.3). We then have the following a priori error bound*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}(h)} + \|p - p_h\|_{Q(h)} \leq C h^{\min\{s,k+1\}} \left(\|\boldsymbol{u}\|_{s,\Omega} + \|\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}\|_{s,\Omega} + \|p\|_{s+1,\Omega}\right),$$

with $C > 0$ independent of the mesh size h.

6 Uniform Convergence

In the previous section, we obtained an error estimate for the numerical solution operator T_h, i.e., for $\boldsymbol{f} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$, $\|T\boldsymbol{f} - T_h\boldsymbol{f}\|_{\boldsymbol{V}(h)} \to 0$ as $h \to 0$. However, this is just pointwise convergence (Definition 2.48 in [34]). To analyse the convergence of the eigenproblem of the numerical operator, we need a uniform convergence result, that is $\|T - T_h\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)} \to 0$ as $h \to 0$. Here the operator norm is defined as: for $T : \mathcal{X} \to \mathcal{X}$, $\|T\|_{\mathcal{X}} = \sup_{\mathbf{0} \neq \boldsymbol{v} \in \mathcal{X}} \frac{\|T\boldsymbol{v}\|_{\mathcal{X}}}{\|\boldsymbol{v}\|_{\mathcal{X}}}$. To prove this kind of convergence, the discrete compactness property plays an important role. In [32], [33] and [34], the discrete compactness property is proven for the conforming case. Here we prove this property for the discontinuous and periodic case.

On the space $\boldsymbol{V}(h)$, we introduce another seminorm and norm:

$$\begin{aligned}
|\boldsymbol{v}|^2_{\boldsymbol{V}(h),-} &= \|\epsilon^{-\frac{1}{2}}\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{v}\|^2_{0,\Omega}, + \|e^{-\frac{1}{2}}h^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\|^2_{0,\mathcal{F}_h}, \\
\|\boldsymbol{v}\|^2_{\boldsymbol{V}(h),-} &= \|\boldsymbol{v}\|^2_{0,\Omega} + |\boldsymbol{v}|^2_{\boldsymbol{V}(h)}.
\end{aligned} \tag{6.1}$$

From [9], the seminorm $|\cdot|_{\boldsymbol{V}(h),-}$ and norm $\|\cdot\|_{\boldsymbol{V}(h),-}$ are also well-posed in $\boldsymbol{V}(h)$. It is obvious that $\|\boldsymbol{v}\|^2_{\boldsymbol{V}(h)} = \|\boldsymbol{v}\|^2_{\boldsymbol{V}(h),-} + \|h^{\frac{1}{2}}[\![\boldsymbol{v}]\!]_N\|^2_{0,\mathcal{F}_h}$.

Lemma 6.1 *There exists an operator* $\Pi_h^c : \boldsymbol{V}_h^{\boldsymbol{\alpha}} \to \boldsymbol{V}_h^{\boldsymbol{\alpha},c}$ *such that*

$$\begin{aligned} \|\boldsymbol{v} - \Pi_h^c \boldsymbol{v}\|_{0,\Omega}^2 &\le C \int_{\mathcal{F}_h} h\, |[\![\boldsymbol{v}]\!]_T|^2\, ds, \\ \|\boldsymbol{v} - \Pi_h^c \boldsymbol{v}\|_{\boldsymbol{V}(h),-}^2 &\le C \int_{\mathcal{F}_h} h^{-1}\, |[\![\boldsymbol{v}]\!]_T|^2\, ds. \end{aligned} \tag{6.2}$$

The lemma above follows Proposition 4.5 in [23], which can easily be turned into the periodic case similarly as for Lemma 5.5 and its proof is in the "Appendix". Using this lemma, we can prove the following decomposition, which is similar to Proposition 7.5 in [9].

Lemma 6.2 *There exists a complement* $\boldsymbol{V}_h^{\boldsymbol{\alpha},\perp}$ *of* $\boldsymbol{V}_h^{\boldsymbol{\alpha},c}$ *in* $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$ *such that the decomposition* $\boldsymbol{V}_h^{\boldsymbol{\alpha}} = \boldsymbol{V}_h^{\boldsymbol{\alpha},c} \oplus \boldsymbol{V}_h^{\boldsymbol{\alpha},\perp}$ *is stable in* $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$*, i.e.,*

$$\boldsymbol{v}_h = \boldsymbol{v}_h^c + \boldsymbol{v}_h^\perp, \qquad \|\boldsymbol{v}_h^c\|_{\boldsymbol{V}(h),-} + \|\boldsymbol{v}_h^\perp\|_{\boldsymbol{V}(h),-} \le C \|\boldsymbol{v}_h\|_{\boldsymbol{V}(h),-}.$$

Moreover, it holds that

$$\|\boldsymbol{v}_h^\perp\|_{\boldsymbol{V}(h),-} \le Ch |\boldsymbol{v}_h^\perp|_{\boldsymbol{V}(h),-} \qquad \forall\, \boldsymbol{v}_h^\perp \in \boldsymbol{V}_h^{\boldsymbol{\alpha},\perp}.$$

The constant $C > 0$ *is independent of the mesh size.*

We define the space

$$\boldsymbol{K}_h^{\boldsymbol{\alpha}} = \{\boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \mid (\boldsymbol{v}, \boldsymbol{\lambda}) \in \mathrm{Ker}(B_h) \text{ for some } \boldsymbol{\lambda} \in M_h^{\boldsymbol{\alpha}}\}.$$

By definition, $\boldsymbol{K}_h^{\boldsymbol{\alpha}}$ contains the range of T_h. Similar to Lemma 4.1, we prove the following theorem for discontinuous spaces.

Theorem 6.1 *For all* h *small enough and any* $\boldsymbol{v}_h \in \boldsymbol{K}_h^{\boldsymbol{\alpha}}$*, there exist a* $\boldsymbol{v} \in \boldsymbol{W}$*, such that*

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,\Omega} \le \eta_h \|\boldsymbol{v}_h\|_{\boldsymbol{V}(h)},$$

with $\eta_h \to 0$ *as* $h \to 0$.

Proof From the definition of the norm $\|\cdot\|_{\boldsymbol{V}(h),-}$, it is enough to prove $\|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,\Omega} \le \eta_h \|\boldsymbol{v}_h\|_{\boldsymbol{V}(h),-}$. For any $\boldsymbol{v}_h \in \boldsymbol{K}_h^{\boldsymbol{\alpha}}$, using Lemma 6.2 we can decompose $\boldsymbol{v}_h$ as $\boldsymbol{v}_h = \boldsymbol{v}_h^c + \boldsymbol{v}_h^\perp$. Furthermore, using the Helmholtz decomposition (4.1), $\boldsymbol{v}_h^c = \boldsymbol{v}_h^0 + \nabla_{\boldsymbol{\alpha}} p_h$ with $\boldsymbol{v}_h^0 \in \boldsymbol{V}_h^{\boldsymbol{\alpha},c,0}$ and $p_h \in Q_h^{\boldsymbol{\alpha},c}$.

From Lemma 4.1, we know that there exists a $\boldsymbol{v} \in \boldsymbol{W}$ satisfying

$$\|\boldsymbol{v} - \boldsymbol{v}_h^0\|_{0,\Omega} \le \eta_h \|\boldsymbol{v}_h^0\|_{\boldsymbol{V}(h),-}.$$

With the decomposition of $\boldsymbol{v}_h$, we have

$$\begin{aligned} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,\Omega}^2 &= (\boldsymbol{v} - \boldsymbol{v}_h, \boldsymbol{v} - \boldsymbol{v}_h) \\ &= (\boldsymbol{v} - \boldsymbol{v}_h, \boldsymbol{v} - \boldsymbol{v}_h^0) - (\boldsymbol{v} - \boldsymbol{v}_h, \boldsymbol{v}_h^\perp) - (\boldsymbol{v} - \boldsymbol{v}_h, \nabla_{\boldsymbol{\alpha}} p_h) \\ &\le \|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,\Omega} \left[\|\boldsymbol{v} - \boldsymbol{v}_h^0\|_{0,\Omega} + \|\boldsymbol{v}_h^\perp\|_{0,\Omega} \right] \\ &\quad + |(\boldsymbol{v} - \boldsymbol{v}_h, \nabla_{\boldsymbol{\alpha}} p_h)|\,. \end{aligned} \tag{6.3}$$

As $\boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathrm{div}_{\boldsymbol{\alpha}}^0; \Omega)$ and $\nabla_{\boldsymbol{\alpha}} p_h \in \nabla_{\boldsymbol{\alpha}} Q_h^{\boldsymbol{\alpha},c}$, we know that $(\boldsymbol{v}, \nabla_{\boldsymbol{\alpha}} p_h) = 0$ and $[\![p_h]\!]_N = 0$ on $\mathcal{F}_h$. As $\boldsymbol{v}_h \in \boldsymbol{K}_h^{\boldsymbol{\alpha}}$, we know that there exist some $\boldsymbol{\lambda}_h \in M_h^{\boldsymbol{\alpha}}$ such that $(\boldsymbol{v}_h, \boldsymbol{\lambda}_h) \in \mathrm{Ker}(B_h)$.

For the last term of (6.3), we obtain that

$$
\begin{aligned}
(\boldsymbol{v}-\boldsymbol{v}_h, \nabla_{\boldsymbol{\alpha}} p_h) &= -(\boldsymbol{v}_h, \nabla_{\boldsymbol{\alpha}} p_h) \\
&= B_h(\boldsymbol{v}_h, \boldsymbol{\lambda}_h; p_h) - \int_{\mathcal{F}_h} \{\!\{\boldsymbol{v}_h\}\!\} \cdot \overline{[\![p_h]\!]}_N ds + \int_{\mathcal{F}_h} c_F \overline{[\![p_h]\!]}_N \cdot \boldsymbol{\lambda}_h ds \\
&= 0.
\end{aligned}
$$

□

According to the decomposition in Theorem 3.1, we can define the following projection

$$
P : \boldsymbol{L}^2_{\text{per}}(\Omega) \to \boldsymbol{H}_{\text{per}}(\text{div}^0_{\boldsymbol{\alpha}}; \Omega). \tag{6.4}
$$

From Corollary 3.1, we obtain that if we restrict P to $\boldsymbol{V}$, P is onto $\boldsymbol{W}$. Theorem 6.1 implies the following result.

Corollary 6.1 *For h small enough, the projection $P : \boldsymbol{V} \to \boldsymbol{W}$ satisfies*

$$
\|\boldsymbol{v}_h - P\boldsymbol{v}_h\|_{\boldsymbol{V}(h)} \le \eta_h \|\boldsymbol{v}_h\|_{\boldsymbol{V}(h)} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{K}^{\boldsymbol{\alpha}}_h,
$$

with $\eta_h \to 0$ as $h \to 0$.

Proof We rewrite Theorem 6.1: for all h small enough, there exists $\Pi_h : \boldsymbol{K}^{\boldsymbol{\alpha}}_h \to \boldsymbol{W}$, such that $\Pi_h \in \mathcal{L}(\boldsymbol{L}^2_{\text{per}}(\Omega), \boldsymbol{W})$ and

$$
\|\boldsymbol{v}_h - \Pi_h\boldsymbol{v}_h\|_{0,\Omega} \le \eta_h \|\boldsymbol{v}_h\|_{\boldsymbol{V}(h)} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{K}^{\boldsymbol{\alpha}}_h,
$$

with $\eta_h \to 0$ as $h \to 0$. As $\Pi_h\boldsymbol{v}_h \in \boldsymbol{W}$, we have

$$
\begin{aligned}
\|\boldsymbol{v}_h - P\boldsymbol{v}_h\|_{\boldsymbol{V}(h)} &= \|(I-P)\boldsymbol{v}_h\|_{\boldsymbol{V}(h)} = \|(I-P)(\boldsymbol{v}_h - \Pi_h\boldsymbol{v}_h)\|_{\boldsymbol{V}(h)} \\
&\le \|I-P\|_{\mathcal{L}(\boldsymbol{L}^2_{\text{per}}(\Omega), \boldsymbol{V}(h))} \|(\boldsymbol{v}_h - \Pi_h\boldsymbol{v}_h)\|_{0,\Omega} \\
&\le C\eta_h \|\boldsymbol{v}_h\|_{\boldsymbol{V}(h)},
\end{aligned}
$$

with $\eta_h \to 0$ as $h \to 0$. □

Next, let Λ be a countable set of mesh sizes with only accumulation point 0. We show a kind of discrete compactness property of $\boldsymbol{K}^{\boldsymbol{\alpha}}_h$, similar to Definition 7.13 in [34].

Definition 6.1 We say $\boldsymbol{K}^{\boldsymbol{\alpha}}_h$ with $h \in \Lambda$ has the discrete compactness property, if for every sequence $\{\boldsymbol{v}_h\}_{h\in\Lambda}$, such that

(i) $\boldsymbol{v}_h \in \boldsymbol{K}^{\boldsymbol{\alpha}}_h$ for each $h \in \Lambda$;

(ii) there is a constant $C > 0$ independent of $\boldsymbol{v}_h$ such that $\|\boldsymbol{v}_h\|_{\boldsymbol{V}(h)} \le C$ independent of $h \in \Lambda$,

there exist a subsequence, still denoted $\{\boldsymbol{v}_h\}_{h\in\Lambda}$, and a function $\boldsymbol{v} \in \boldsymbol{W}$ such that

$$
\|\boldsymbol{v}_h - \boldsymbol{v}\|_{0,\Omega} \to 0 \quad \text{as } h \to 0 \text{ in } \Lambda.
$$

Theorem 6.2 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \ne \boldsymbol{0}$, $\boldsymbol{K}^{\boldsymbol{\alpha}}_h$ with $h \in \Lambda$ has the discrete compactness property.*

Proof From Corollary 6.1, we know that for each $\boldsymbol{v}_h \in \boldsymbol{K}^{\boldsymbol{\alpha}}_h$, there exists a $\boldsymbol{v} = P\boldsymbol{v}_h \in \boldsymbol{H}_{\text{per}}(\text{div}^0_{\boldsymbol{\alpha}}; \Omega)$ such that

$$
\|\boldsymbol{v}_h - P\boldsymbol{v}_h\|_{\boldsymbol{V}(h)} \le \eta_h \|\boldsymbol{v}_h\|_{\boldsymbol{V}(h)},
$$

with $\eta_h \to 0$ as $h \to 0$, where P is defined in (6.4).

According to Lemma 6.2 , we decompose $\boldsymbol{v}_h$ as $\boldsymbol{v}_h = \boldsymbol{v}_h^c + \boldsymbol{v}_h^{\perp}$, where $\boldsymbol{v}_h^c \in \boldsymbol{V}_h^{\boldsymbol{\alpha},c} \subset \boldsymbol{V}$ and $\boldsymbol{v}_h^{\perp} \in \boldsymbol{V}_h^{\boldsymbol{\alpha},\perp}$. For $\boldsymbol{v}_h^c$, there is a $\boldsymbol{v}_h^0 = P\boldsymbol{v}_h^c \in \boldsymbol{W}$ by the decomposition (3.4). Then, the sequence $\{\boldsymbol{v}_h^0\}_{h\in\Lambda} \subset \boldsymbol{W}$ is bounded in the norm $\|\cdot\|_{\boldsymbol{V}(h),-}$, and also in $\|\cdot\|_{\boldsymbol{V}(h)}$. From the compactness of $\boldsymbol{W}$ endowed with $\|\cdot\|_{\boldsymbol{V}(h)}$ in $\boldsymbol{L}^2_{\text{per}}(\Omega)$ (by Theorem 3.2), there exists a subsequence $\{\boldsymbol{v}_h^0\}_{h\in\Lambda'}$ and $\boldsymbol{v} \in \boldsymbol{W}$, such that

$$\|\boldsymbol{v}_h^0 - \boldsymbol{v}\|_{0,\Omega} \to 0 \quad \text{as } h \to 0 \text{ in } \Lambda'.$$

Therefore, for $\boldsymbol{v}_h \in \boldsymbol{K}_h^{\boldsymbol{\alpha}}$,

$$\begin{aligned}\|\boldsymbol{v}_h - \boldsymbol{v}\|_{0,\Omega} &= \|\boldsymbol{v}_h - P(\boldsymbol{v}_h - \boldsymbol{v}_h^0 - \boldsymbol{v}_h^{\perp}) - \boldsymbol{v}\|_{0,\Omega}\\ &\le \|\boldsymbol{v}_h - P\boldsymbol{v}_h\|_{0,\Omega} + \|\boldsymbol{v}_h^0 - \boldsymbol{v}\|_{0,\Omega} + \|P\boldsymbol{v}_h^{\perp}\|_{0,\Omega}\\ &\to 0.\end{aligned}$$

Here we use Theorem 6.1, Corollary 6.1, the L^2-stability of P and the stability of the decomposition in Lemma 6.2 □

With the discrete compactness property of $\boldsymbol{K}_h^{\boldsymbol{\alpha}}$, we obtain that the set $\mathscr{T} = \{T_h : \boldsymbol{L}^2_{\text{per}}(\Omega) \to \boldsymbol{L}^2_{\text{per}}(\Omega),\ h \in \Lambda\}$ is a collectively compact set through a standard analysis in [34, Definition 2.47 and Theorem 7.14]. Furthermore, combining the result of pointwise convergence in Theorem 5.5 and Lemma 2.49 in [34] gives the following result.

Lemma 6.3 *Since $\mathscr{T}$ is a collectively compact set of bounded linear operators and the operators are pointwise convergent to the compact operator $T : \boldsymbol{L}^2_{\text{per}}(\Omega) \to \boldsymbol{L}^2_{\text{per}}(\Omega)$, we have*

$$\|(T_h - T)T\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)} \to 0 \ \text{ and } \ \|(T_h - T)T_h\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)} \to 0 \ \text{ as } \ h \to 0 \text{ in } \Lambda.$$

From the definition of T in (3.7) and the definition of T_h in (5.2) and (5.3), we know that T and T_h are self-adjoint operators. Hence we have

$$\begin{aligned}\|T - T_h\|^2_{\boldsymbol{L}^2_{\text{per}}(\Omega)} &= \sup_{0\neq \boldsymbol{v}\in \boldsymbol{L}^2_{\text{per}}(\Omega)} \frac{((T-T_h)\boldsymbol{v}, (T-T_h)\boldsymbol{v})}{\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2_{\text{per}}(\Omega)}} = \sup_{0\neq \boldsymbol{v}\in \boldsymbol{L}^2_{\text{per}}(\Omega)} \frac{((T-T_h)^2\boldsymbol{v}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2_{\text{per}}(\Omega)}}\\ &= \sup_{0\neq \boldsymbol{v}\in \boldsymbol{L}^2_{\text{per}}(\Omega)} \frac{(((T-T_h)T - (T-T_h)T_h)\boldsymbol{v}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2_{\text{per}}(\Omega)}}\\ &\le \|(T_h - T)T\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)} + \|(T_h - T)T_h\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)}.\end{aligned}$$

From the discussion above, we obtain the following corollary.

Corollary 6.2 *Since $\mathscr{T}$ is a collectively compact set of self-adjoint, bounded, linear operators and the operators T_h are pointwise convergent to a self-adjoint compact operator $T : \boldsymbol{L}^2_{\text{per}}(\Omega) \to \boldsymbol{L}^2_{\text{per}}(\Omega)$, we have*

$$\|T_h - T\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)} \to 0 \text{ as } h \to 0.$$

7 Spectral Theory for Mixed DG Formulation

In this section we introduce some basic notation that is used in the analysis of the discrete Maxwell eigenvalue problem. From the analysis in Sect. 3, we know that the range of T is included in $\boldsymbol{V}$. To study the eigenproblem, we restrict T to $\boldsymbol{V}$. Similarly, we restrict T_h to $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$. We denote the spectrum set and resolvent set of T by $\sigma(T)$ and $\rho(T)$, respectively. The corresponding numerical spectrum set and resolvent set of T_h are $\sigma(T_h)$ and $\rho(T_h)$. Let z be a complex number, both in $\rho(T)$ and $\rho(T_h)$.

Next, we define the spectral operators. Let $\omega^2 \in \sigma(T)$ with algebraic multiplicity m, and $\Gamma \subset \rho(T)$ be a Jordan curve. Here, we make Γ small enough so that there is no other point in $\sigma(T)$ than ω^2 in the region bounded by Γ. The spectral operator $E : \boldsymbol{V} \to \boldsymbol{V}$ of T with respect to ω^2 is defined by

$$E = \frac{1}{2\pi i} \int_\Gamma R_z(T) dz.$$

For h small enough, $\Gamma \subset \rho(T_h)$, and the spectral operator $E_h : \boldsymbol{V}_h^{\boldsymbol{\alpha}} \to \boldsymbol{V}_h^{\boldsymbol{\alpha}}$ of T_h is defined by

$$E_h = \frac{1}{2\pi i} \int_\Gamma R_z(T_h) dz.$$

Here $R_z(T) = (z - T)^{-1} : \boldsymbol{V} \to \boldsymbol{V}$ and $R_z(T_h) = (z - T_h)^{-1} : \boldsymbol{V}_h^{\boldsymbol{\alpha}} \to \boldsymbol{V}_h^{\boldsymbol{\alpha}}$ are the resolvent operators of T and T_h, respectively. Let $R(E)$ and $R(E_h)$ denote the ranges of E and E_h in $\boldsymbol{V}$ and $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$, respectively. From the definition of the contour integral, we obtain that $R(E)$ and $R(E_h)$ are the spaces of exact and numerical eigenfunctions, respectively, of the eigenvalues surrounded by Γ.

Let us define the distance between two spaces. For two closed subspaces Y and Z of $\boldsymbol{L}^2_{\text{per}}(\Omega)$, we set $\delta(Y, Z) = \sup_{y \in Y, \|y\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)}=1} \delta(y, Z)$, where $\delta(y, Z) = \inf_{z \in Z} \|y - z\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)}$, and $\hat{\delta}(Y, Z) = \max\{\delta(Y, Z), \delta(Z, Y)\}$. From the analysis in the previous sections, we have two properties of T_h and $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$.

Property 7.1 $\lim_{h \to 0} \|T - T_h\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)} = 0$.

Property 7.2 $\forall\, \boldsymbol{v} \in \boldsymbol{V}, \quad \lim_{h \to 0} \delta(\boldsymbol{v}, \boldsymbol{V}_h^{\boldsymbol{\alpha}}) = 0$.

We list the following two theorems without proofs, as they are both standard results in spectral approximation theory, see [13].

Theorem 7.1 *Assume Property 7.1 and let $\mathscr{D} \subset \mathbb{C}$ be an open set containing $\sigma(T)$. Then there exists an $h_0 > 0$ such that $\sigma(T_h) \subset \mathscr{D}$, $\forall\, h < h_0$.*

Theorem 7.2 *Assume Property 7.1 and Property 7.2. Then,*

$$\lim_{h \to 0} \hat{\delta}(R(E), R(E_h)) = 0.$$

The above two theorems guarantee the non-pollution and completeness of the spectrum and eigenspaces.

Assume that ω^2 is an eigenvalue of the operator T with multiplicity m, that is, the dimension of the space $R(E)$ is m. For sufficiently small h, let $\omega^2_{h,1}, \omega^2_{h,2}, \ldots, \omega^2_{h,m}$ be the m eigenvalues of T_h close to ω^2. We define the numerical mean eigenvalue with regard to ω^2 as

$$\hat{\omega}_h^2 = \frac{1}{m} \sum_{i=1}^{m} \omega^2_{h,j}.$$

There is a well-established theory for approximation of general compact operators. In our case, the problem consists of self-adjoint operators T and T_h, so we will use the self-adjoint version of [39, Theorem 3] for the error in eigenvalue approximation. Also, see [34, Theorem 2.52].

Theorem 7.3 *Let $\phi_1, \dots, \phi_m$ be any basis for $R(E)$. Then there is a constant C such that*

$$|\omega^2 - \hat{\omega}_h^2| \leq \frac{1}{m} \sum_{i,j=1}^{m} |((T - T_h)\phi_i, \phi_j)| + C\|(T - T_h)|_{R(E)}\|^2_{\boldsymbol{L}^2_{\text{per}}(\Omega)}. \tag{7.1}$$

The second term in the estimate (7.1) is straightforward to bound. Indeed,

$$\|(T - T_h)|_{R(E)}\|^2_{\boldsymbol{L}^2_{\text{per}}(\Omega)} \leq \sup_{\substack{\phi \in R(E) \\ \|\phi\|_{\boldsymbol{V}(h)}=1}} \|T\phi - T_h\phi\|^2_{\boldsymbol{V}(h)} \leq Ch^{2\min\{s,k+1\}}, \tag{7.2}$$

where we used the error estimate given in Theorem 5.5. Next, we make the first term in (7.1) more precise.

Lemma 7.1 *Under the same assumption as in Theorem 7.3 we have*

$$\begin{aligned} |((T - T_h)\phi, \psi)| \leq C\Big(&\|(T - T_h)\phi\|_{\boldsymbol{V}(h)}\|(T - T_h)\psi\|_{\boldsymbol{V}(h)} \\ &+ \|(T - T_h)\phi\|_{\boldsymbol{V}(h)}\|(T_p - T_{p,h})\psi\|_{Q(h)} \\ &+ \|(T - T_h)\psi\|_{\boldsymbol{V}(h)}\|(T_p - T_{p,h})\phi\|_{Q(h)} \\ &+ \|(T_p - T_{p,h})\phi\|_{Q(h)}\|(T_p - T_{p,h})\psi\|_{Q(h)}\Big). \end{aligned} \tag{7.3}$$

for any $\phi, \psi \in R(E)$.

Proof Let $\phi, \psi \in R(E)$. Then since $T\phi, T\psi \in \boldsymbol{H}^1(\Omega)$ and $T_p\phi, T_p\psi \in H^1(\Omega)$ by Theorem 3.3, for any $(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q$, $(T\psi, T_p\psi)$ satisfies (5.2) and (5.3) due to Lemma 5.2, that is,

$$(\psi, \boldsymbol{v}) = a_h(T\psi, \boldsymbol{v}) + b_h(\boldsymbol{v}, T_p\psi) + \overline{b_h(T\psi, q)} - c_h(T_p\psi, q).$$

Letting $(\boldsymbol{v}, q) = (T\phi, T_p\phi)$, we get

$$(\psi, T\phi) = a_h(T\psi, T\phi) + b_h(T\phi, T_p\psi) + \overline{b_h(T\psi, T_p\phi)} - c_h(T_p\psi, T_p\phi). \tag{7.4}$$

By Lemma 5.2, picking $(\boldsymbol{v}, q) = (T_h\phi, T_{p,h}\phi)$,

$$(\psi, T_h\phi) = a_h(T\psi, T_h\phi) + b_h(T_h\phi, T_p\psi) + \overline{b_h(T\psi, T_{p,h}\phi)} - c_h(T_p\psi, T_{p,h}\phi). \tag{7.5}$$

Subtracting (7.4) from (7.5), we obtain

$$\begin{aligned} (\psi, (T - T_h)\phi) =& a_h(T\psi, (T - T_h)\phi) + b_h((T - T_h)\phi, T_p\psi) \\ &+ \overline{b_h(T\psi, (T_p - T_{p,h})\phi)} - c_h(T_p\psi, (T_p - T_{p,h})\phi). \end{aligned} \tag{7.6}$$

Next choosing $(\boldsymbol{v}, q) = (T_h\psi, T_{p,h}\psi)$ in (5.2) and (5.3),

$$(T_h\psi, \phi) = a_h(T_h\psi, T_h\phi) + b_h(T_h\phi, T_{p,h}\psi) + \overline{b_h(T_h\psi, T_{p,h}\phi)} - c_h(T_{p,h}\psi, T_{p,h}\phi). \tag{7.7}$$

Also, by consistency Lemma 5.2, we have

$$(T_h\psi, \phi) = a_h(T_h\psi, T\phi) + b_h(T\phi, T_{p,h}\psi) + \overline{b_h(T_h\psi, T_p\phi)} - c_h(T_{p,h}\psi, T_p\phi). \tag{7.8}$$

Subtraction (7.7) from (7.8), we obtain

$$0 = a_h(T_h\psi, (T - T_h)\phi) + b_h((T - T_h)\phi, T_{p,h}\psi) \\ + \overline{b_h(T_h\psi, (T_p - T_{p,h})\phi)} - c_h(T_{p,h}\psi, (T_p - T_{p,h})\phi). \tag{7.9}$$

Combining (7.6) and (7.9)

$$(\psi, (T - T_h)\phi) = a_h((T - T_h)\psi, (T - T_h)\phi) + b_h((T - T_h)\phi, (T_p - T_{p,h})\psi) \\ + \overline{b_h((T - T_h)\psi, (T_p - T_{p,h})\phi)} - c_h((T_p - T_{p,h})\psi, (T_p - T_{p,h})\phi).$$

The result follows by the continuity of a_h, the Cauchy-Schwarz inequality and the definition of $Q(h)$. □

Combining (7.2) and Lemma 7.1, we obtain the following estimate for the error of the numerical mean eigenvalues.

Corollary 7.1 *Under the assumption of Theorem 7.3, we have*

$$|\omega^2 - \hat{\omega}_h^2| \le Ch^{2\min\{s,k+1\}}.$$

From the compactness of T, the convergence properties of T_h and $T_{p,h}$, and the estimate in (7.2), we obtain an estimate for the gap between the eigenfunction spaces using Theorem 1 in [39].

Theorem 7.4 *There is a constant C such that*

$$\hat{\delta}(R(E), R(E_h)) \le C\|(T - T_h)|_{R(E)}\|_{\boldsymbol{L}^2_{\text{per}}(\Omega)} \le Ch^{\min\{s,k+1\}},$$

for all small h, where $(T - T_h)|_{R(E)}$ denotes the restriction of $T - T_h$ to $R(E)$.

8 Primal DG Formulation

Alternatively, we can also compute all relevant eigenvalues using the following primal formulation: For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \boldsymbol{0}$, find $(\boldsymbol{u}_h, \omega_h^2) \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \times \mathbb{C}$ with $(\boldsymbol{u}_h, \omega_h^2) \neq (\boldsymbol{0}, 0)$ such that

$$a_h^{\text{sip}}(\boldsymbol{u}_h, \boldsymbol{v}) = \omega_h^2(\boldsymbol{u}_h, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}. \tag{8.1}$$

Here a_h^{sip} need not have the stabilization term related to the Lagrange multiplier, that is,

$$a_h(\boldsymbol{u}, \boldsymbol{v}) = a_h^{\text{sip}}(\boldsymbol{u}, \boldsymbol{v}) + \int_{\mathcal{F}_h} b_F [\![\boldsymbol{u}]\!]_N [\![\boldsymbol{v}]\!]_N ds.$$

It is easy to see that a_h^{sip} is symmetric and coercive in the semi-norm $|\cdot|_{\boldsymbol{V}(h),-}$ and continuous in the norm $\|\cdot\|_{\boldsymbol{V}(h),-}$. This DG form corresponds to the following weak formulation: Find $\boldsymbol{u} \in \boldsymbol{V}$ with $\boldsymbol{u} \neq \boldsymbol{0}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = \omega^2(\boldsymbol{u}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{8.2}$$

where $a(\cdot,\cdot)$ is defined in (3.6). Note that this method is different from the mixed formulation defined in (5.1), where we incorporate the constraint on the divergence by introducing a new variable through a Lagrange multiplier. That's not the case here. So rather we define a new weak formulation for the analysis, that is, for $\boldsymbol{f} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$ let $\boldsymbol{u} = T^{\text{sip}}\boldsymbol{f} \in \boldsymbol{V}$ with $\boldsymbol{u} \neq \mathbf{0}$ be the solution of

$$d(\boldsymbol{u}, \boldsymbol{v}) := a(\boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{8.3}$$

We refer the reader to [17, Sections 2.2.1, 2.2.2] for a discussion on the Galerkin approximation of the weak formulation (8.2). As a consequence, Theorem 3.4 holds for the eigenproblem (8.2). Corresponding to (8.3) we also define a new discrete form d_h:

$$d_h(\boldsymbol{u}_h, \boldsymbol{v}) = a_h^{\text{sip}}(\boldsymbol{u}_h, \boldsymbol{v}) + (\boldsymbol{u}_h, \boldsymbol{v}).$$

Here, provided that $\mathfrak{a}$ is large enough, d_h is coercive not only on the kernel of the bilinear form a_h^{sip}, but also in the $\|\cdot\|_{\boldsymbol{V}(h),-}$ norm, i.e.,

$$d_h(\boldsymbol{v}, \boldsymbol{v}) \geq C\|\boldsymbol{v}\|^2_{\boldsymbol{V}(h),-}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}},$$

and is also continuous, that is,

$$d_h(\boldsymbol{w}, \boldsymbol{v}) \leq C\|\boldsymbol{w}\|_{\boldsymbol{V}(h),-}\|\boldsymbol{v}\|_{\boldsymbol{V}(h),-}, \quad \forall\, \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}.$$

Here the generic constant C is independent of the mesh size. The corresponding DG operator T_h^{sip} is defined for any $\boldsymbol{f} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$ as $\boldsymbol{u}_h = T_h^{\text{sip}}\boldsymbol{f} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}$ such that

$$d_h(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}. \tag{8.4}$$

Remark 8.1 $(\boldsymbol{u}_h, \omega^2)$ is a solution of (8.1) if and only if $(\boldsymbol{u}_h, \frac{1}{1+\omega^2})$ is an eigenpair of T_h^{sip}.

The following result is a consequence of the coercivity of d_h.

Theorem 8.1 (Existence, uniqueness) *Let $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq \mathbf{0}$. For any $\boldsymbol{f} \in \boldsymbol{L}^2_{\text{per}}(\Omega)$ if $\mathfrak{a}$ is large enough, the DG problem defined by (8.4) has a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}$ satisfying*

$$\|\boldsymbol{u}_h\|_{\boldsymbol{V}(h),-} \leq C\|\boldsymbol{f}\|_{0,\Omega},$$

where $C > 0$ is independent of the mesh size and $\boldsymbol{f}$.

Proof Setting $\boldsymbol{f} = \mathbf{0}$ in (8.4) and using the coercivity of d_h in the $\|\cdot\|_{\boldsymbol{V}(h),-}$ norm gives $\boldsymbol{u}_h = \mathbf{0}$. □

We have the following a priori error estimate:

Theorem 8.2 (Convergence) *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of (8.3) and (8.4), respectively. Suppose that $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$ and $\epsilon^{-1}\nabla \times \boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$ where $s > \frac{1}{2}$. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}(h),-} \leq Ch^{\min\{s,k+1\}}\left(\|\boldsymbol{u}\|_{s,\Omega} + \|\epsilon^{-1}\nabla_\alpha \times \boldsymbol{u}\|_{s,\Omega}\right) \tag{8.5}$$

where $C > 0$ is a constant independent of the mesh size.

Proof Let $\boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}$. By the triangle inequality, the coercivity and continuity of d_h, and the consistency of the DG method

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}(h),-} \leq C \inf_{\boldsymbol{v}\in\boldsymbol{V}_h^{\boldsymbol{\alpha}}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{V}(h),-}.$$

The result follows from Lemma 4.2. □

We define the kernel of the bilinear form a_h^{sip} as

$$K_h := \left\{ \boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} : a_h^{\text{sip}}(\boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall\, \boldsymbol{w} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \right\}.$$

The orthogonal complement of K_h in $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$ is :

$$K_h^{\perp} := \left\{ \boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} : (\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{V}(h),-} = 0 \quad \forall\, \boldsymbol{w} \in K_h \right\}.$$

Here, consistent with the norm $\| \cdot \|_{\boldsymbol{V}(h),-}$, the inner product $(\cdot,\cdot)_{\boldsymbol{V}(h),-}$ is defined as $(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{V}(h),-} = (\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v}, \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{w}) + (\boldsymbol{v}, \boldsymbol{w}) + (h^{-1}[\![\boldsymbol{v}]\!]_T, [\![\boldsymbol{w}]\!]_T)$. Note that for any $\boldsymbol{w} \in K_h$, the coercivity of a_h in the semi-norm gives $0 = a_h^{\text{sip}}(\boldsymbol{w}, \boldsymbol{w}) \geq C|\boldsymbol{w}|_{\boldsymbol{V}(h),-}$. This implies that $|\boldsymbol{w}|_{\boldsymbol{V}(h),-} = 0$ and consequently $\boldsymbol{w} \in \boldsymbol{V}^0$, which means $K_h \subset \boldsymbol{V}^0$. Therefore, for any $\boldsymbol{w} \in K_h$, $\boldsymbol{v} \in \boldsymbol{V}_h^{\boldsymbol{\alpha}}$, $(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{V}(h),-} = (\boldsymbol{v}, \boldsymbol{w})$. The results of Sect. 6 (Theorem 6.1, Corollary 6.1 and Theorem 6.2, Lemma 6.3, Corollary 6.2) still hold true replacing $\boldsymbol{K}_h^{\boldsymbol{\alpha}}$ by $K_h^{\perp}$ and T, T_h by T^{sip}, T_h^{sip}. Also see [9, Proposition 7.8, Proposition 7.13]. The following discrete Friedrich's inequality can be found in [3, Lemma 7].

Theorem 8.3 (Discrete Friedrich's inequality) *There exists a constant* $C > 0$ *independent of the mesh size such that*

$$\|\boldsymbol{v}\|_{0,\Omega} \leq C|\boldsymbol{v}|_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h^{\alpha,c,0}.$$

This results in a coercivity property of a_h which can be proven similar to [9, Lemma 7.6].

Theorem 8.4 (Coercivity of a_h on $K_h^{\perp}$) *There exists a constant* $C > 0$ *independent of the mesh size such that*

$$\|\boldsymbol{v}\|_{0,\Omega} \leq C a_h(\boldsymbol{v}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in K_h^{\perp}.$$

This theorem guarantees the isolation of the discrete essential spectrum $\{1\}$ of the operator T_h^{sip} [9, Proposition 4.1].

Theorem 8.5 *Suppose that* $1 \neq \lambda_h \in \sigma(T_h^{\text{sip}})$. *Then there exists* $0 < \beta < 1$ *independent of the mesh size such that*

$$0 < \lambda_h \leq \beta.$$

Theorems 7.1 and 7.2 hold here by replacing T, T_h by T^{sip}, T_h^{sip} concluding the non-pollution and completeness of the spectrum and eigenspaces.

The results in Sect. 7 can be obtained with a slight modification. However Theorem 7.3 holds in this case for the continuous and discrete eigenvalues $\lambda = \frac{1}{1+\omega^2}$ and $\lambda_h = \frac{1}{1+\omega_h^2}$ of the operators T^{sip} and T_h^{sip}. The same result can be obtained for the eigenvalues ω and ω_h of (8.2) and (8.1) by simply observing

$$\lambda - \lambda_h = \frac{1}{1+\omega^2} - \frac{1}{1+\omega_h^2} = \frac{\omega + \omega_h}{1 + \omega^2 + \omega_h^2 + \omega^2\omega_h^2}(\omega_h - \omega).$$

This guarantees that any eigenvalue of problem (8.2) will be approximated when h is sufficiently small.

9 Numerical Experiments

In this section, we present numerical examples to verify the convergence rate of the eigenvalues computed with the mixed DG method and its application to computing the band structure of photonic crystals, both in two and three dimensions. Specially, in two-dimensional cases, we extend vector fields $\boldsymbol{u}(x, y) = (u_1(x, y), u_2(x, y))$ and unit vector $\boldsymbol{n} = (n_1, n_2)$ into three dimensions by $\boldsymbol{u}(x, y, z) = (u_1(x, y, 0), u_2(x, y, 0), 0)$ and $\boldsymbol{n} = (n_1, n_2, 0)$. Then we deduce that

$$\nabla \times \boldsymbol{u} = \left(0, 0, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right),$$
$$\nabla \times (\nabla \times \boldsymbol{u}) = \left(\frac{\partial}{\partial y}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right), -\frac{\partial}{\partial x}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right), 0\right),$$
$$\text{and} \quad \boldsymbol{n} \times \boldsymbol{u} = (0, 0, n_1 u_2 - n_2 u_1).$$

We use the mixed interior penalty DG method, given by (5.1), and set the stabilization parameters as: $\mathfrak{a} = 100l^2$, $\mathfrak{b} = 0.01$ and $\mathfrak{c} = 10$, where l is the degree of the polynomial basis functions. During the assembly of the global matrices, we use high order Gaussian quadrature to compute the local matrices for every element in the mesh. The DG discretization results in the following matrices for the eigenvalue problem (5.1):

$$a_h(\boldsymbol{v}_j, \boldsymbol{v}_i) \rightarrow \mathcal{A}_{ij},$$
$$b_h(\boldsymbol{v}_i, q_j) \rightarrow \mathcal{B}_{ij},$$
$$c_h(q_j, q_i) \rightarrow \mathcal{C}_{ij},$$
$$(\boldsymbol{v}_j, \boldsymbol{v}_i) \rightarrow \mathcal{M}_{ij}.$$

For the photonic band structure, we need to compute the eigenvalues in the first n bands by solving the following generalized eigenvalue problem:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & \mathcal{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_h \\ \boldsymbol{p}_h \end{pmatrix} = \omega_h^2 \begin{pmatrix} \mathcal{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_h \\ \boldsymbol{p}_h \end{pmatrix}, \tag{9.1}$$

where we need the smallest eigenvalues to compute the band structure of photonic crystals. The eigenvalues are computed using the MATLAB iterative eigenvalue solver $eigs$ for large sparse matrices. In the computations, we change the above problem to a problem of finding the largest eigenvalues as follows:

$$\begin{pmatrix} \mathcal{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_h \\ \boldsymbol{p}_h \end{pmatrix} = \tilde{\omega}_h^2 \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & \mathcal{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_h \\ \boldsymbol{p}_h \end{pmatrix}. \tag{9.2}$$

After solving the eigenvalue problem (9.2), we then obtain the solution of (9.1) by using $\omega_h^2 = \frac{1}{\tilde{\omega}_h^2}$. This is considerably more efficient than computing eigenvalues close to zero using the iterative eigenvalue solver $eigs$.

Alternatively, we could also compute all relevant eigenvalues using the primal bilinear form:

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) = \omega_h^2(\boldsymbol{u}_h, \boldsymbol{v}), \tag{9.3}$$

whose corresponding matrices are $\mathcal{A}\boldsymbol{u}_h = \omega_h^2 \mathcal{M}\boldsymbol{u}_h$, but in that case we have to deal with a large number of zero eigenvalues, which make the computations much less efficient.

In the next subsection we first consider the two-dimensional homogeneous case ($\epsilon = 1$, $\boldsymbol{\alpha} = (\pi, 0)$, $l = 2$, number of triangles $N_e = 256$) as an example to illustrate the advantage

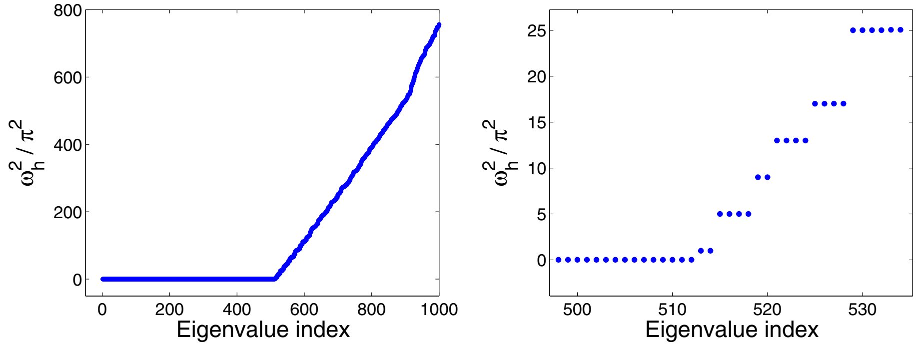

Fig. 1 First 1000 eigenvalues computed using the bilinear form $a_h(\boldsymbol{u}_h, \boldsymbol{v}) = \omega_h^2(\boldsymbol{u}_h, \boldsymbol{v})$ $(left)$. Zoom near first non-zero eigenvalues $(right)$

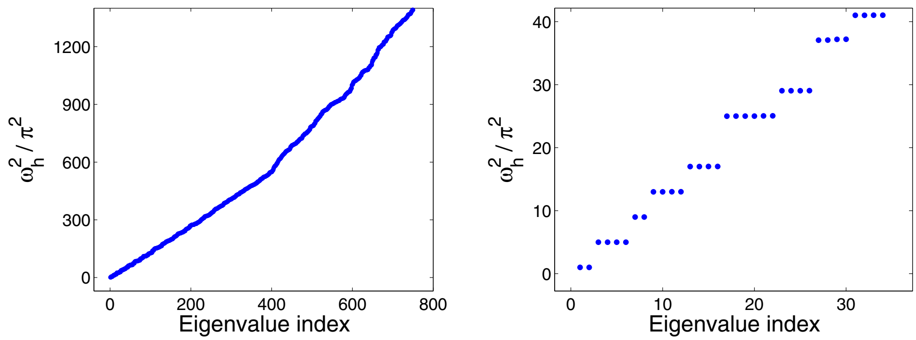

Fig. 2 First 750 eigenvalues computed using the mixed DG form (5.1) $(left)$. Zoom of the eigenvalues near zero $(right)$

of the mixed form (5.1) compared with the primal bilinear form (9.3). The number of degrees of freedom in the bilinear form $a_h(\boldsymbol{u}_h, \boldsymbol{v})$ is 2048. Figure 1 shows the first 1000 smallest eigenvalues of (9.3), among which there are 512 zero eigenvalues, 25 % of the total number of degrees of freedom. These zero eigenvalues are of no use, and also considerably slow down the iterative computation of the smallest non-zero eigenvalues using $eigs$. This results in a significant computational cost compared to the mixed DG method, which does not have zero eigenvalues, as shown in the theoretical analysis, and also allows the approach outlined in (9.2). If we use the mixed form, we just need to compute the first set of eigenvalues of (9.1), which is much cheaper, although the number of degrees of freedom is larger than for (9.3), with 3584 compared with 2048. Figure 2 shows that the zero eigenvalues are not present for the mixed formulation.

9.1 Convergence of the Discrete Eigenvalues

In this subsection, we compute the convergence of the eigenvalues calculated with the mixed DG form (5.1) on unit domains $[0, 1]^d$ $(d = 2, 3)$, with a homogeneous material with coefficient $\epsilon = 1$. In this case, the exact eigenvalues of the analytical problem (2.3) can easily be computed. Let $\boldsymbol{u} = \boldsymbol{C_I} e^{i\boldsymbol{I}\cdot\boldsymbol{x}}$ and $p = c_{\boldsymbol{I}} e^{i\boldsymbol{I}\cdot\boldsymbol{x}}$, where $\boldsymbol{I} \in \mathcal{J}$, $\boldsymbol{C_I} \in \mathbb{C}^d$ and $c_{\boldsymbol{I}} \in \mathbb{C}$.

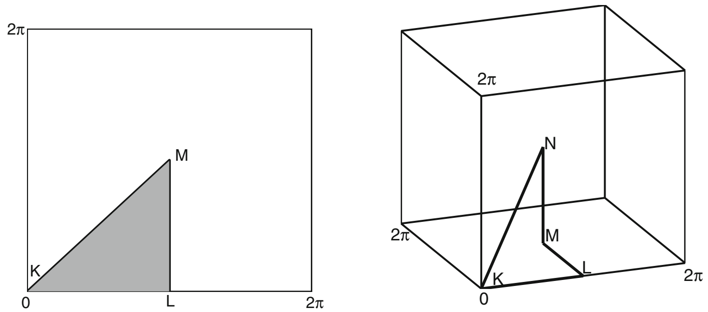

Fig. 3 Reciprocal lattice of $[0, 1]^2$ (*left*), where the shadowed region is the irreducible zone. Reciprocal lattice of $[0, 1]^3$ (*right*), where K, L, M and N are the vertices of the irreducible zone

Then (2.3) becomes

$$i(\boldsymbol{\alpha} + \boldsymbol{I}) \times i(\boldsymbol{\alpha} + \boldsymbol{I}) \times C_{\boldsymbol{I}} + i(\boldsymbol{\alpha} + \boldsymbol{I})c_{\boldsymbol{I}} = \omega^2 C_{\boldsymbol{I}}, \tag{9.4}$$

$$i(\boldsymbol{\alpha} + \boldsymbol{I}) \cdot C_{\boldsymbol{I}} = 0. \tag{9.5}$$

From (9.5), we obtain that $C_{\boldsymbol{I}}$ is orthogonal to $(\boldsymbol{\alpha} + \boldsymbol{I})$, then $c_{\boldsymbol{I}} = 0$ in (9.4), and the exact eigenvalues are equal to

$$\omega^2 = |\boldsymbol{\alpha} + \boldsymbol{I}|^2, \quad \boldsymbol{I} \in \mathcal{J}. \tag{9.6}$$

Fixing $\boldsymbol{\alpha}$ and letting $\boldsymbol{I}$ take all the values in $\mathcal{J}$, we obtain all the exact eigenvalues for the homogeneous material from (9.6).

In the two-dimensional case, we take $[0, 1]^2$ as computational domain, whose corresponding reciprocal lattice is shown in Fig. 3. Let $\boldsymbol{\alpha}$ move along the edge of the irreducible Brillouin zone in Fig. 3. We compute the photonic bands by using 262 triangles and third order Nédélec elements. Fig. 4 shows a comparison between the numerical and exact eigenvalues, from which we can observe that there are no spurious eigenvalues in the computations.

Choosing $\boldsymbol{\alpha} = (\pi, 0)$, we investigate the convergence rate of the eigenvalues computed with the mixed DG form (5.1). From (9.6), it is easy to see that the first two smallest exact eigenvalues are π^2 and $5\pi^2$ with, respectively, multiplicity 2 and 4. In these computations, we use a triangular mesh, and refine it by dividing each triangle into four new triangles. Tables 1, 2 and 3 show the errors in the computed eigenvalues and their convergence rates using the first, second and third order Nédélec finite elements of the first family from [36]. The convergence rates are twice the order of the polynomial basis functions, which verify the result of Corollary 7.1.

Next, we present the same test case in three dimensions. The difference is that we now use a cubic mesh. The computational domain is $[0, 1]^3$, and its reciprocal lattice is shown in Fig. 3. Figure 5 shows the band structure computed using third order Nédélec elements. For the computation of the convergence rates of the eigenvalues, we choose $\boldsymbol{\alpha} = (\pi, 0, 0)$, and from (9.6), we know that the first two smallest exact values are π^2 and $5\pi^2$ with, respectively, multiplicity 4 and 16. Tables 4, 5 and 6 show the errors in the computed eigenvalues and their convergence rates, from which we see that the convergence rate is also twice the order of the polynomial basis functions.

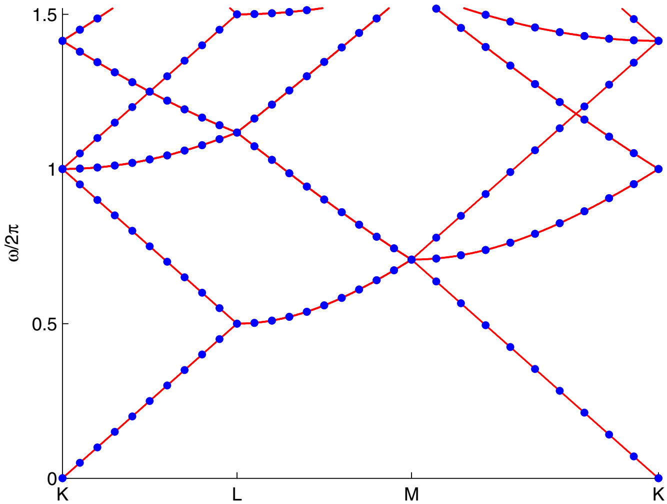

Fig. 4 Exact band structure for 2D homogeneous material (*lines*), and eigenvalues computed with the mixed DG discretization (5.1) using third order Nédélec polynomials on a mesh with 262 triangular elements (*dots*)

Table 1 Eigenvalue convergence rates for a 2D homogeneous material with $\epsilon = 1$ using zero order Nédélec elements with the mixed DG formulation (5.1)

Mesh	$N_e = 16$		$N_e = 64$		$N_e = 256$		$N_e = 1024$	
ω^2	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order
π^2	4.07e−02	–	1.86e−02	1.13	5.13e−03	1.86	1.31e−03	1.97
π^2	3.82e−01	–	9.10e−02	2.07	2.25e−02	2.02	5.61e−03	2.00
$5\pi^2$	7.85e+00	–	1.83e+00	2.10	4.64e−01	1.98	1.17e−01	1.99
$5\pi^2$	4.91e+00	–	1.42e+00	1.79	3.74e−01	1.92	9.48e−02	1.98
$5\pi^2$	4.79e+00	–	3.51e−01	3.77	4.99e−02	2.82	1.03e−02	2.27
$5\pi^2$	9.32e−01	–	1.02e−01	3.19	4.19e−02	1.29	1.15e−02	1.87

Table 2 Eigenvalue convergence rates for a 2D homogeneous material with $\epsilon = 1$ using first order Nédélec elements with the mixed DG formulation (5.1)

Mesh	$N_e = 16$		$N_e = 64$		$N_e = 256$		$N_e = 1024$	
ω^2	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order
π^2	2.24e−03	–	2.04e−04	3.46	1.35e−05	3.91	8.37e−07	4.01
π^2	1.68e−02	–	1.03e−03	4.03	6.42e−05	4.01	4.00e−06	4.01
$5\pi^2$	4.95e−01	–	1.26e−02	5.30	6.58e−04	4.26	$4.04e - 05$	4.03
$5\pi^2$	2.94e−01	–	1.11e−02	4.73	6.36e−04	4.12	3.82e−05	4.06
$5\pi^2$	3.94e−01	–	5.09e−02	2.95	3.45e−03	3.88	2.19e−04	3.98
$5\pi^2$	1.28e+00	–	7.47e−02	4.09	4.74e−03	3.98	2.97e−04	4.00

Table 3 Eigenvalue convergence rates for a 2D homogeneous material with $\epsilon = 1$ using second order Nédélec elements with the mixed DG formulation (5.1)

Mesh	$N_e = 16$		$N_e = 64$		$N_e = 256$		$N_e = 1024$	
ω^2	$\left\|\omega^2 - \omega_h^2\right\|$	Order	$\left\|\omega^2 - \omega_h^2\right\|$	Order	$\left\|\omega^2 - \omega_h^2\right\|$	Order	$\left\|\omega^2 - \omega_h^2\right\|$	Order
π^2	4.95e−05	–	1.03e−06	5.58	1.71e−08	5.91	3.35e−010	5.68
π^2	2.42e−04	–	3.75e−06	6.01	5.85e−08	6.00	8.33e−010	6.13
$5\pi^2$	1.95e−02	–	7.63e−05	8.00	2.30e−06	5.05	4.06e−08	5.82
$5\pi^2$	3.67e−02	–	4.73e−04	6.28	7.57e−06	5.97	1.20e−07	5.98
$5\pi^2$	3.95e−02	–	9.86e−04	5.33	1.66e−05	5.89	2.64e−07	5.97
$5\pi^2$	9.54e−02	–	1.38e−03	6.11	2.18e−05	5.98	3.43e−07	5.99

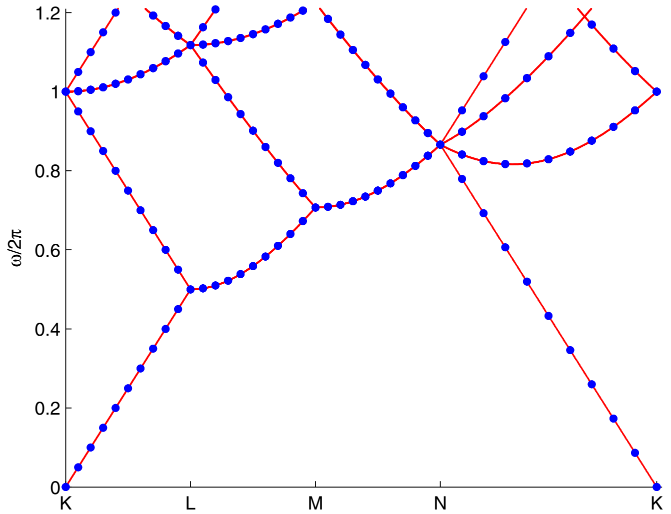

Fig. 5 Exact band structure for 3D homogeneous material (*lines*), and eigenvalues computed with the mixed DG discretization (5.1) using third order Nédélec polynomials on a mesh with 64 cubic elements (*dots*)

9.2 Band Structures of Photonic Crystals

In this section, we apply the mixed DG formulation to compute the band structure of several photonic crystals. Here, we present four examples, two two-dimensional cases and two three-dimensional cases.

In two dimensions, we consider three common structures, each made of air $\epsilon = 1$ and a dielectric material with $\epsilon = 8.9$, as presented in [3]. Figure 6 gives a sketch of the first structure, which is a periodic lattice $[0, 1]^2$ with a dielectric disk ($r = 0.165$) in the middle of each cell. We discretize the computational domain with a mesh comprised of 490 triangular cells, and use second order Nédélec polynomials. Figure 7 shows the transverse magnetic eigenmodes structure where $(\omega^2, \boldsymbol{u})$ is computed using the mixed DG form (5.1). From these results, we can observe a band gap for low magnetic frequencies between the 1^{st} and 2^{nd} band. The second structure is a bar frame ($d = 0.2$) shown in Fig. 8. It can be reduced to a lattice with unit cell $[0, 1]^2$. We discretize the domain with a mesh comprised of 262 triangular

Table 4 Eigenvalue convergence rates for a 3D homogeneous material with $\epsilon = 1$ using zero order Nédélec elements computed using the mixed DG formulation (5.1)

Mesh	$N_e = 27$		$N_e = 125$		$N_e = 343$		$N_e = 1000$	
ω^2	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order
π^2	9.30e−01	–	3.29e−01	2.04	1.67e−01	2.02	8.14e−02	2.01
π^2	9.30e−01	–	3.29e−01	2.04	1.67e−01	2.02	8.14e−02	2.01
π^2	9.30e−01	–	3.29e−01	2.04	1.67e−01	2.02	8.14e−02	2.01
π^2	9.30e−01	–	3.29e−01	2.04	1.67e−01	2.02	8.14e−02	2.01
$5\pi^2$	1.54e+001	–	5.74e+000	1.94	2.88e+000	2.05	1.40e+000	2.03
$5\pi^2$	1.54e+001	–	5.74e+000	1.94	2.88e+000	2.05	1.40e+000	2.03
$5\pi^2$	1.54e+001	–	5.74e+000	1.94	2.88e+000	2.05	1.40e+000	2.03
$5\pi^2$	1.54e+001	–	5.74e+000	1.94	2.88e+000	2.05	1.40e+000	2.03

Table 5 Eigenvalue convergence rates for a 3D homogeneous material with $\epsilon = 1$ using first order Nédélec elements computed using the mixed DG formulation (5.1)

Mesh	$N_e = 27$		$N_e = 125$		$N_e = 343$		$N_e = 512$	
ω^2	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order
π^2	1.56e−02	–	2.09e−03	3.93	5.49e−04	3.97	3.23e−04	3.98
π^2	1.56e−02	–	2.09e−03	3.93	5.49e−04	3.97	3.23e−04	3.98
π^2	1.56e−02	–	2.09e−03	3.93	5.49e−04	3.97	3.23e−04	3.98
π^2	1.56e−02	–	2.09e−03	3.93	5.49e−04	3.97	3.23e−04	3.98
$5\pi^2$	8.80e−01	–	1.28e−01	3.77	3.47e−02	3.89	2.05e−02	3.93
$5\pi^2$	8.80e−01	–	1.28e−01	3.77	3.47e−02	3.89	2.05e−02	3.93
$5\pi^2$	8.80e−01	–	1.28e−01	3.77	3.47e−02	3.89	2.05e−02	3.93
$5\pi^2$	8.80e−01	–	1.28e−01	3.77	3.47e−02	3.89	2.05e−02	3.93

Table 6 Eigenvalue convergence rates for a 3D homogeneous material with $\epsilon = 1$ using second order Nédélec elements computed using the mixed DG formulation (5.1)

Mesh	$N_e = 27$		$N_e = 64$		$N_e = 125$		$N_e = 216$	
ω^2	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order	$\lvert\omega^2 - \omega_h^2\rvert$	Order
π^2	1.24e−04	–	2.25e−05	5.94	5.94e−06	5.96	2.00e−06	5.98
π^2	1.24e−04	–	2.25e−05	5.94	5.94e−06	5.96	2.00e−06	5.98
π^2	1.24e−04	–	2.25e−05	5.94	5.94e−06	5.96	2.00e−06	5.98
π^2	1.24e−04	–	2.25e−05	5.94	5.94e−06	5.96	2.00e−06	5.98
$5\pi^2$	2.84e-02	–	5.41e−03	5.76	1.46e−03	5.86	4.99e−04	5.91
$5\pi^2$	2.84e−02	–	5.41e−03	5.76	1.46e−03	5.86	4.99e−04	5.91
$5\pi^2$	2.84e−02	–	5.41e−03	5.76	1.46e−03	5.86	4.99e−04	5.91
$5\pi^2$	2.84e−02	–	5.41e−03	5.76	1.46e−03	5.86	4.99e−04	5.91

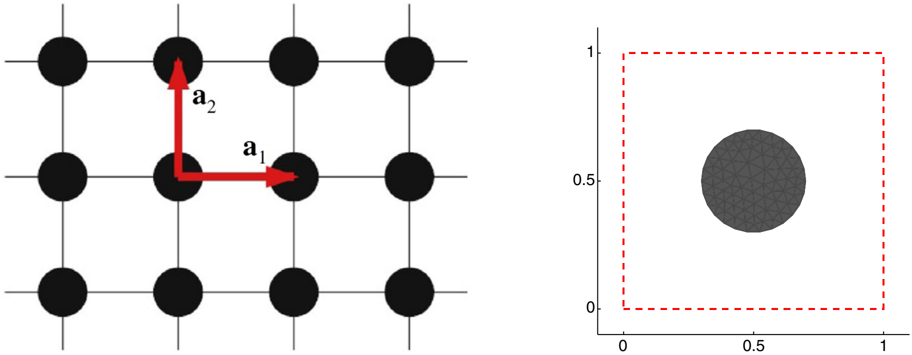

Fig. 6 2D photonic crystals consisting of a periodic lattice containing a dielectric disc (r = 0.2) with $\epsilon = 8.9$ (*left*). Zoom of individual cell (*right*)

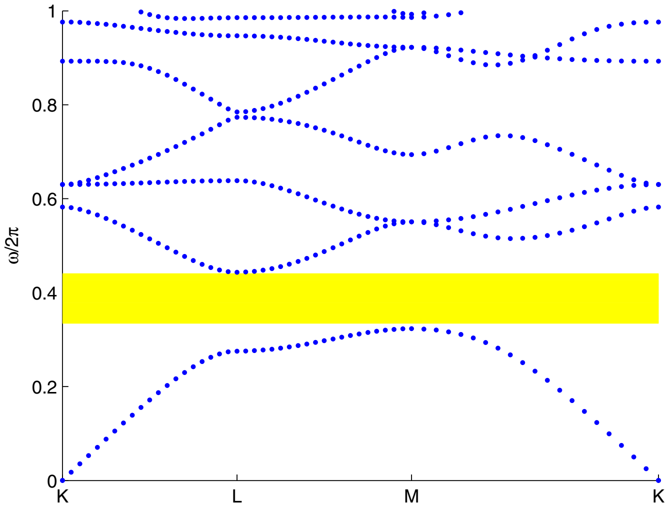

Fig. 7 Band structure of transverse magnetic modes for the 2D geometry shown in Fig. 6. A band gap exists between the first and second band around $\omega/2\pi = 0.4$

cells, and use second order Nédélec polynomials. Figure 9 shows that a thin magnetic band gap occurs between the first and second band. Figure 10 shows another structure with many circular holes ($r = 0.48$), whose individual periodic cell is a rhombus with acute angle $\theta = 60°$. In this case, the dielectric constant of the material is $\epsilon = 13$. We discretize the domain with 1384 cells, and also use second order Nédélec polynomials. Figure 11 shows that there are several magnetic band gaps among low bands.

We consider also two three-dimensional test cases, presented in [14] and [44]. These two photonic crystals both contain silicon $\epsilon = 13$ and air $\epsilon = 1$, and can be reduced to a lattice with unit cell $[0, 1]^3$, which means that they share the same reciprocal lattice, as shown in Fig. 3. The first structure is called "scaffold" with a frame thickness $d = 0.25$, as shown in Fig. 12. We use a cubic mesh to discretize the photonic crystals and use second order Nédélec polynomials as basis functions. The numerical results in Fig. 13 predict that there is a small band gap around $\omega/2\pi c = 0.4$. The frame thickness of the other photonic crystal is

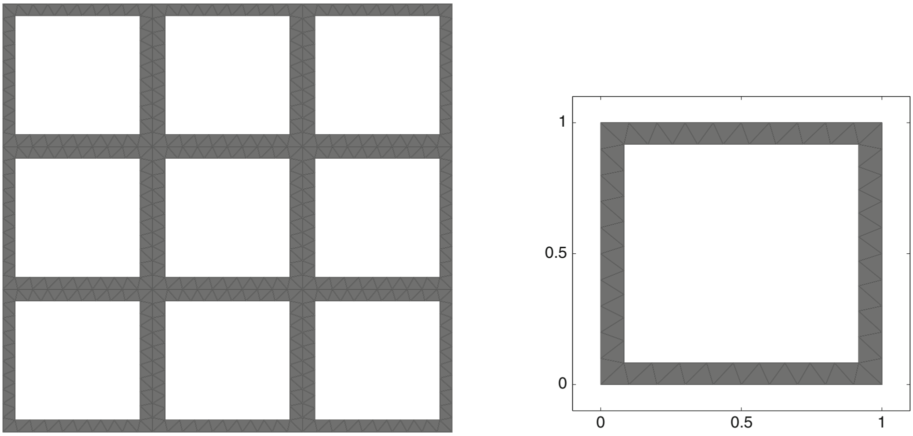

Fig. 8 2D photonic crystals containing *square holes* ($left$). Zoom of individual cell ($right$)

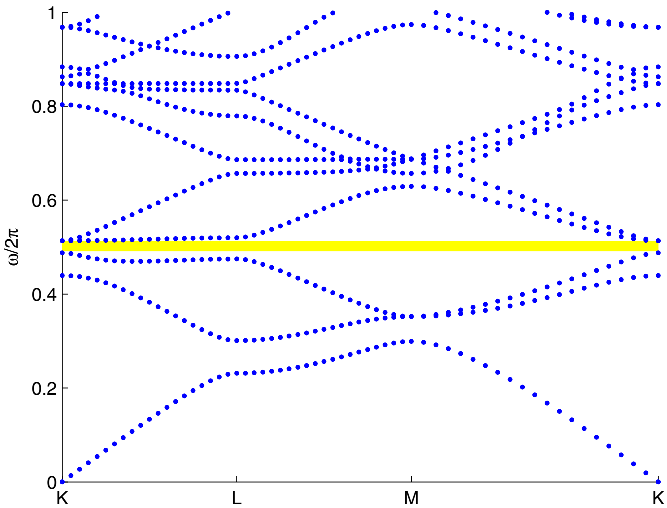

Fig. 9 Band structure of transverse magnetic modes for 2D geometry shown in Fig. 8. A small band gap exists between the third and fourth band near $\omega/2\pi = 0.5$

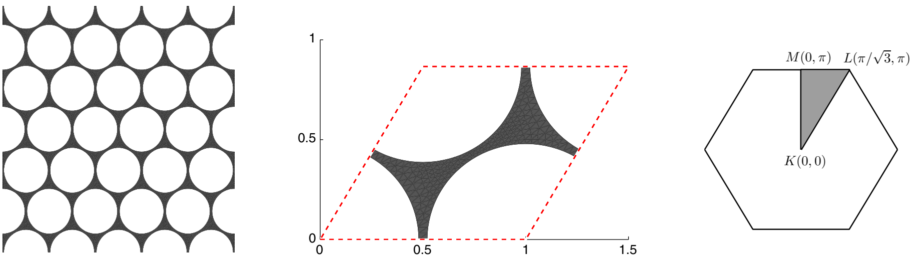

Fig. 10 2D photonic crystals containing circular holes ($left$). Zoom of individual cell ($middle$). The corresponding reciprocal lattice and irreducible zone ($right$)

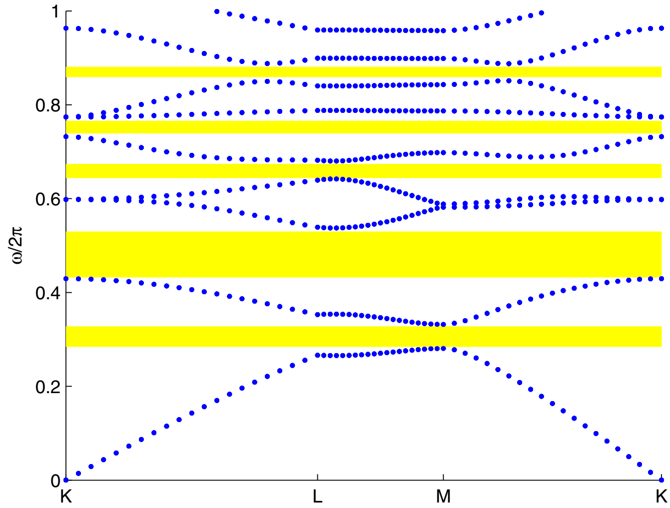

Fig. 11 Band structure of transverse magnetic modes for the 2D geometry shown in Fig. 10

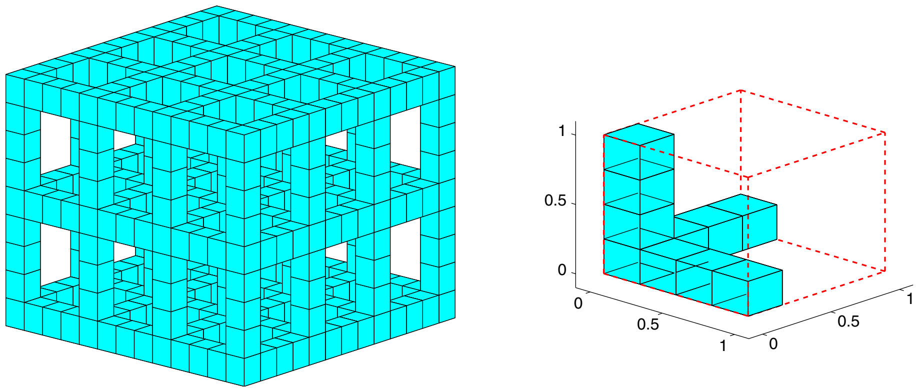

Fig. 12 3D photonic crystals consisting of a periodic lattice with regular *bars* ($\epsilon = 13,\ d = 0.25$) and air ($\epsilon = 1$) ($left$). Zoom of individual cell ($right$)

also 0.25, but the distribution of the bars is different from that in Fig. 14. The results shown in Fig. 15 predict a wide band gap for the photonic crystals shown in Fig. 14.

10 Conclusions

The computation of photonic band structures describing the behavior of light in photonic crystals requires the accurate solution of Maxwell eigenvalue problems. The numerical solution of these eigenvalue problems is done in this article using discontinuous Galerkin methods, since they are well suited to obtain higher order accuracy on unstructured meshes and can efficiently deal with discontinuous material interfaces and singularities. We considered both a mixed DG formulation with modified Nédélec elements and a primal DG formulation, which does not explicitly enforce the divergence constraint. We analyzed the convergence properties

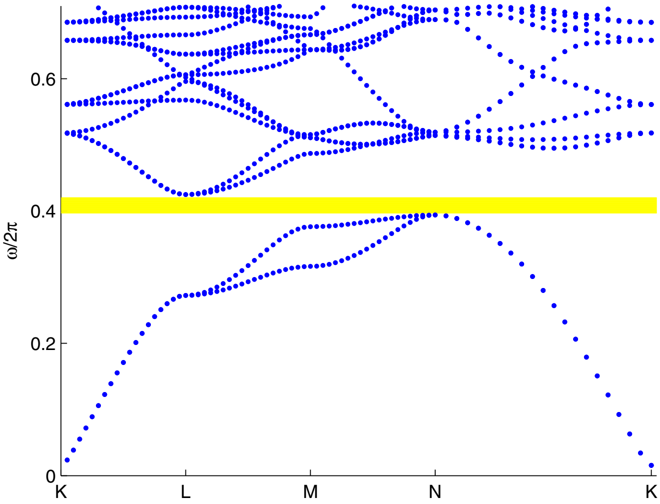

Fig. 13 Band structure for 3D geometry shown in Fig. 12. A band gap exists between the second and third band near $\omega/2\pi = 0.4$

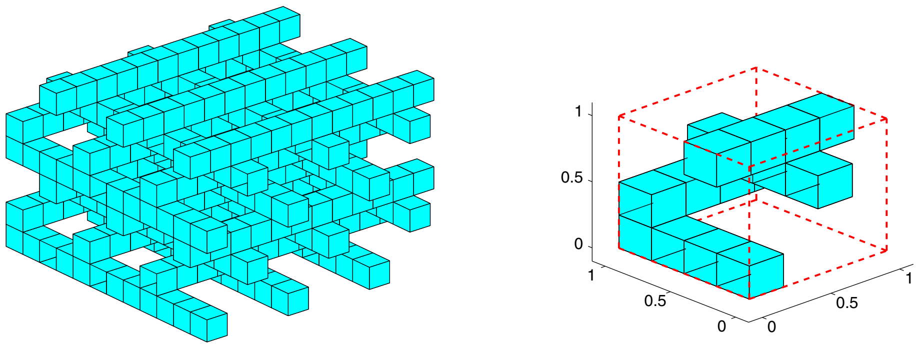

Fig. 14 3D photonic crystals consisting of a periodic lattice with regular bars ($\epsilon = 13,\ d = 0.25$) and air ($\epsilon = 1$) ($left$). Zoom of individual cell ($right$)

of both DG formulations in detail. For a sufficiently fine mesh both DG formulations do not have spurious eigenvalues and the numerical solution of the eigenvalue problem converges to the exact solution. We proved that the convergence rate of the numerical eigenvalues is twice the minimum of the order of the polynomial basis functions and the regularity of the solution of the Maxwell equations, which is verified by the numerical results for a homogeneous material. To prove the non-pollution of the numerical spectrum of the Maxwell operator, we proved a discrete compactness property for the corresponding DG space. For practical computations, the mixed DG formulation, which explicitly enforces the divergence constraint, is considerably more efficient, even though the number of degrees of freedom of the mixed DG formulation is significantly larger than for the primal DG formulation. The mixed DG formulation is applied to compute the band structures of several 2D and 3D photonic crystals. These results demonstrate its potential to compute band structures in complex photonic crystals.

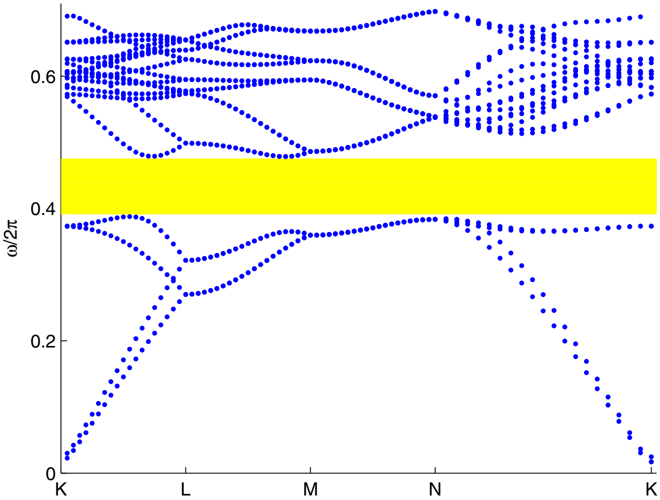

Fig. 15 Band structure for 3D geometry shown in Fig. 14. A band gap exists between the second and third band near $\omega/2\pi = 0.45$

Appendix

Appendix A: Continuity and Semi-Ellipticity

Lemma 10.1 *For all* $\boldsymbol{v} \in \boldsymbol{V}(h)$ *and* $q \in Q(h)$*,*

$$\|\epsilon^{-\frac{1}{2}}\mathcal{L}(\boldsymbol{v})\|_{0,\Omega} \le C\|e^{-\frac{1}{2}}h^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\|_{0,\mathcal{F}_h},$$

$$\|\mathcal{M}(q)\|_{0,\Omega} \le C\|h^{-\frac{1}{2}}[\![q]\!]_N\|_{0,\mathcal{F}_h}.$$

with a constant $C > 0$ *that is independent of the mesh size and the coefficient* ϵ*.*

Proof

$$\begin{aligned}\|\epsilon^{-\frac{1}{2}}\mathcal{L}(\boldsymbol{v})\|_{0,\Omega} &= \sup_{\boldsymbol{w}\in\boldsymbol{V}_h^{\boldsymbol{\alpha}}} \frac{\int_\Omega \epsilon^{-\frac{1}{2}}\mathcal{L}(\boldsymbol{v})\cdot\overline{\boldsymbol{w}}d\boldsymbol{x}}{\|\boldsymbol{w}\|_{0,\Omega}} \\ &= \sup_{\boldsymbol{w}\in\boldsymbol{V}_h^{\boldsymbol{\alpha}}} \frac{\int_{\mathcal{F}_h} \epsilon^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\cdot\overline{\{\!\{\boldsymbol{w}\}\!\}}ds}{\|\boldsymbol{w}\|_{0,\Omega}} \\ &\le \sup_{\boldsymbol{w}\in\boldsymbol{V}_h^{\boldsymbol{\alpha}}} \frac{\|e^{-\frac{1}{2}}h^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\|_{0,\mathcal{F}_h}\|h^{\frac{1}{2}}\{\!\{\boldsymbol{w}\}\!\}\|_{0,\mathcal{F}_h}}{\|\boldsymbol{w}\|_{0,\Omega}} \\ &\le C\|e^{-\frac{1}{2}}h^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\|_{0,\mathcal{F}_h}.\end{aligned}$$

Here we use the inverse inequality Lemma 11 in [41]:

$$\|h^{\frac{1}{2}}\{\!\{\boldsymbol{w}\}\!\}\|_{0,\mathcal{F}_h}^2 \le C\sum_{K\in\mathcal{T}_h} h_K\|\boldsymbol{w}\|_{0,\partial K}^2 \le C\sum_{K\in\mathcal{T}_h}\|\boldsymbol{w}\|_{0,K}^2 = C\|\boldsymbol{w}\|_{0,\Omega}^2.$$

The proof of the other estimate is similar. □

Theorem 10.1 *There exist constants $a_1 > 0$ and $a_2 > 0$, independent of the mesh size and the coefficient ϵ, such that*

$$|A_h(\boldsymbol{u},\boldsymbol{\lambda};\boldsymbol{v},\boldsymbol{\eta})| \le a_1\|(\boldsymbol{u},\boldsymbol{\lambda})\|_{U(h)}\|(\boldsymbol{v},\boldsymbol{\eta})\|_{U(h)} \qquad \forall\ (\boldsymbol{u},\boldsymbol{\lambda}),(\boldsymbol{v},\boldsymbol{\eta}) \in \boldsymbol{U}(h),$$
$$|B_h(\boldsymbol{v},\boldsymbol{\eta};q)| \le a_2\|(\boldsymbol{v},\boldsymbol{\eta})\|_{U(h)}\|q\|_{Q(h)} \qquad \forall\ (\boldsymbol{v},\boldsymbol{\eta}) \in \boldsymbol{U}(h),\ \forall\, q \in Q(h).$$

Proof

$$\begin{aligned}
|A_h(\boldsymbol{u},\boldsymbol{\lambda};\boldsymbol{v},\boldsymbol{\eta})| \le{}& \|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{u}\|_{0,\Omega}\|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{v}\|_{0,\Omega}\\
&+C(\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{u}]\!]_T\|_{0,\mathcal{F}_h}\|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{v}\|_{0,\Omega}\\
&+\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{v}]\!]_T\|_{0,\Omega}\|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{u}\|_{0,\Omega})\\
&+\mathfrak{a}\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{u}]\!]_T\|_{0,\mathcal{F}_h}\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{v}]\!]_T\|_{0,\mathcal{F}_h}\\
&+\mathfrak{b}\|h^{\frac12}[\![\boldsymbol{u}]\!]_N\|_{0,\mathcal{F}_h}\|h^{\frac12}[\![\boldsymbol{v}]\!]_N\|_{0,\mathcal{F}_h}\\
&+\mathfrak{c}\|h^{-\frac12}\boldsymbol{\lambda}\|_{0,\mathcal{F}_h}\|h^{-\frac12}\boldsymbol{\eta}\|_{0,\mathcal{F}_h}\\
\le{}& a_1\|(\boldsymbol{u},\boldsymbol{\lambda})\|_{U(h)}\|(\boldsymbol{v},\boldsymbol{\eta})\|_{U(h)}.
\end{aligned}$$

$$\begin{aligned}
|B_h(\boldsymbol{v},\boldsymbol{\eta};q)| \le{}& \|\boldsymbol{v}\|_{0,\Omega}\|\nabla_{\boldsymbol{\alpha},h}q\|_{0,\Omega}+C\|\boldsymbol{v}\|_{0,\Omega}\|h^{-\frac12}[\![q]\!]_N\|_{0,\mathcal{F}_h}\\
&+\mathfrak{c}\|h^{-\frac12}\boldsymbol{\eta}\|_{0,\mathcal{F}_h}\|h^{-\frac12}[\![q]\!]_N\|_{0,\mathcal{F}_h}\\
\le{}& a_2\|(\boldsymbol{v},\boldsymbol{\eta})\|_{U(h)}\|q\|_{Q(h)}.
\end{aligned}$$

□

Lemma 10.2 *For $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \ne \boldsymbol{0}$, given that $\mathfrak{a} > 0$ is large enough, $\mathfrak{b} > 0$ and $\mathfrak{c} > 0$, there exists a $C > 0$ independent of h, such that*

$$A_h(\boldsymbol{u},\boldsymbol{\lambda};\boldsymbol{u},\boldsymbol{\lambda}) \ge C|(\boldsymbol{u},\boldsymbol{\lambda})|^2_{U(h)} \qquad \forall(\boldsymbol{u},\boldsymbol{\lambda}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}}. \tag{10.1}$$

Proof

$$\begin{aligned}
A_h(\boldsymbol{u},\boldsymbol{\lambda};\boldsymbol{u},\boldsymbol{\lambda}) \ge{}& \|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{u}\|^2_{0,\Omega}-2C\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{u}]\!]_T\|_{0,\mathcal{F}_h}\|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{u}\|_{0,\Omega}\\
&+\mathfrak{a}\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{u}]\!]_T\|^2_{0,\mathcal{F}_h}+\mathfrak{b}\|h^{\frac12}[\![\boldsymbol{u}]\!]_N\|^2_{0,\mathcal{F}_h}+\mathfrak{c}\|h^{-\frac12}\boldsymbol{\lambda}\|^2_{0,\mathcal{F}_h}\\
={}& \frac12\|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{u}\|^2_{0,\Omega}+\frac12\left(\|\epsilon^{-\frac12}\nabla_{\boldsymbol{\alpha},h}\times\boldsymbol{u}\|_{0,\Omega}-2C\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{u}]\!]_T\|_{0,\mathcal{F}_h}\right)^2\\
&+(\mathfrak{a}-2C^2)\|e^{-\frac12}h^{-\frac12}[\![\boldsymbol{u}]\!]_T\|^2_{0,\mathcal{F}_h}+\mathfrak{b}\|h^{\frac12}[\![\boldsymbol{u}]\!]_N\|^2_{0,\mathcal{F}_h}+\mathfrak{c}\|h^{-\frac12}\boldsymbol{\lambda}\|^2_{0,\mathcal{F}_h}\\
\ge{}& C|(\boldsymbol{u},\boldsymbol{\lambda})|^2_{U(h)}.
\end{aligned}$$

where we used the estimate in Lemma 10.1. □

Appendix B: Inf-Sup Condition

For the proof of Lemma 5.5, we first need the following auxiliary result.

Lemma 10.3 *Given N real numbers $\{\alpha_1,\ldots,\alpha_N\}$ let $\beta = \frac{1}{N}\sum_{j=1}^N \alpha_j$. Then,*

$$\sum_{j=1}^{N}\left|\alpha_j-\beta\right|^2 \le C\sum_{j=1}^{N-1}\left|\alpha_{j+1}-\alpha_j\right|^2, \tag{10.2}$$

where $C > 0$ depends only on N.

Proof For any j, the Cauchy-Schwarz inequality gives

$$\left|\alpha_j-\beta\right|^2=\frac{1}{N^2}\left|\sum_{i=1}^{N}(\alpha_j-\alpha_i)\right|^2\le\frac{N-1}{N^2}\sum_{i=1}^{N}\left|\alpha_j-\alpha_i\right|^2.$$

Summing over j, we obtain

$$\begin{aligned}\sum_{j=1}^{N}\left|\alpha_j-\beta\right|^2&\le\frac{2(N-1)^2}{N^2}\sum_{j=1}^{N}\sum_{i=1}^{j-1}\sum_{k=i}^{j-1}|\alpha_{k+1}-\alpha_k|^2\\&\le\frac{2(N-1)^2}{N^2}\sum_{j=1}^{N}\sum_{i=1}^{N-1}\sum_{k=1}^{N-1}|\alpha_{k+1}-\alpha_k|^2\\&\le\frac{2(N-1)^4}{N^2}\sum_{j=1}^{N-1}\left|\alpha_{j+1}-\alpha_j\right|^2\\&=C(N)\sum_{j=1}^{N-1}\left|\alpha_{j+1}-\alpha_j\right|^2.\end{aligned}$$

□

Proof of Lemma 5.5 Given $q_h\in Q_h^{\boldsymbol{\alpha}}$, we construct a function $\chi\in Q_h^{\boldsymbol{\alpha},c}$ as follows: At every node of the mesh $\mathcal{T}_h$ corresponding to a Lagrangian type degree of freedom for $Q_h^{\boldsymbol{\alpha},c}$, the value of χ is set to the average of the values of q_h at that node.

For each $K\in\mathcal{T}_h$, let $\mathcal{N}_K=\{\boldsymbol{x}_K^{(j)}, j=1,\ldots,m\}$ be the Lagrange nodes (points) of K and $\{\phi_K^{(j)}, j=1,\ldots,m\}$ the corresponding (local) basis functions satisfying $\phi_K^{(j)}(\boldsymbol{x}_K^{(i)})=\delta_{ij}$. Set $\mathcal{N}=\cup_{K\in\mathcal{T}_h}\mathcal{N}_K$. We view $\mathcal{N}$ as the union of two classes:

$$\begin{aligned}\mathcal{N}_i&=\{\nu\in\mathcal{N}:\nu\text{ is interior to an element}\},\\\mathcal{N}_f&=\{\nu\in\mathcal{N}:\nu\in\partial K,\ \text{for some }K\in\mathcal{T}_h\},\end{aligned}$$

We note that $\mathcal{N}_f$ can be divided into two sets $\mathcal{N}_f^i$ and $\mathcal{N}_f^b$: $\mathcal{N}_f^i$ is the set of nodes on interior faces, while $\mathcal{N}_f^b$ is the set of nodes on the boundary of a face $f\subset\partial\Omega$. As Ω and $\mathcal{T}_h$ are both periodic, for every $\nu^1\in\mathcal{N}_f^b$, there exist a unique ν^2 also in $\mathcal{N}_f^b$ being the corresponding periodic point of ν^1. From the definition $Q_h^{\boldsymbol{\alpha},c}=Q_h^{\boldsymbol{\alpha}}\cap H^1_{\text{per}}(\Omega)$, $Q_h^{\boldsymbol{\alpha},c}$ is a periodic conforming finite element space. To satisfy the periodicity of $Q_h^{\boldsymbol{\alpha},c}$, we can let ν^1 and ν^2 share the same degree of freedom. Then we regard ν^1 and ν^2 as the 'same' node in our computational domain. Furthermore, the nodes in $\mathcal{N}_f^b$ can therefore be considered as nodes in $\mathcal{N}_f^i$. In the following discussion, we consider the nodes in $\mathcal{N}_f^b$ and $\mathcal{N}_f^i$ therefore in the same way.

For each $\nu\in\mathcal{N}$, let $\omega_\nu=\{K\in\mathcal{T}_h|\nu\in K\}$ and denote its cardinality by $|\omega_\nu|$. If $\nu\in\mathcal{N}_i$, then $|\omega_\nu|=1$, and if $\nu\in\mathcal{N}_f$, $|\omega_\nu|\ge1$. Then the basis function $\phi^{(\nu)}$ in $Q_h^{\boldsymbol{\alpha},c}$ at the node $\nu\in\mathcal{N}$ can be constructed as

$$\text{supp}\,\phi^{(\nu)}=\bigcup_{K\in\omega_\nu},\quad\phi^{(\nu)}|_K=\phi_K^{(j)},\quad\boldsymbol{x}_K^{(j)}=\nu.$$

Now, given $q_h\in Q_h^{\boldsymbol{\alpha}}$, written as $q_h=\sum_{K\in\mathcal{T}_h}\sum_{j=1}^{m}\alpha_K^{(j)}\phi_K^{(j)}$, we define the function $\chi\in Q_h^{\boldsymbol{\alpha},c}$ by

$$\chi=\sum_{\nu\in\mathcal{N}}\beta^{(\nu)}\phi^{(\nu)},$$

where

$$\beta^{(\nu)} = \frac{1}{|\omega_\nu|} \sum_{\boldsymbol{x}_K^{(j)}=\nu} \alpha_K^{(j)} \qquad \text{for } \nu \in \mathcal{N}.$$

Now set $\beta_K^{(j)} = \beta^{(\nu)}$ whenever $\boldsymbol{x}_K^{(j)} = \nu$. A simple scaling argument shows that $\|\nabla_{\boldsymbol{\alpha}} \phi_K^{(j)}\|_K^2 \le c h_K^{d-2}$. Hence

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \|\nabla_{\boldsymbol{\alpha}}(q_h - \chi)\|_{0,K}^2 &\le C\, m \sum_{K\in\mathcal{T}_h} h_K^{d-2} \sum_{j=1}^{m} \left|\alpha_K^{(j)} - \beta_K^{(j)}\right|^2 \\
&\le C \sum_{\nu\in\mathcal{N}} h_\nu^{d-2} \sum_{\boldsymbol{x}_K^{(j)}=\nu,\ \boldsymbol{x}_K^{(j)}\in\mathcal{N}_K,\ K\in\mathcal{T}_h} \left|\alpha_K^{(j)} - \beta^{(\nu)}\right|^2 \\
&= C \sum_{\nu\in\mathcal{N}_f} h_\nu^{d-2} \sum_{\boldsymbol{x}_K^{(j)}=\nu,\ \boldsymbol{x}_K^{(j)}\in\mathcal{N}_K,\ K\in\mathcal{T}_h} \left|\alpha_K^{(j)} - \beta^{(\nu)}\right|^2,
\end{aligned} \tag{10.3}$$

where in the last step, we remove the nodes in $\mathcal{N}_i$ as they have no contribution by the definition of $\beta^{(\nu)}$.

We now temporarily focus on the case $d = 2$. For $\nu \in \mathcal{N}_f$ we enumerate the elements of ω_ν as $\{K_1, \ldots, K_{|\omega_\nu|}\}$ so that any consecutive pair K_i, K_{i+1} in that list shares an edge. Then from Lemma 10.3, with some constant C depending only on $|\omega_\nu|$, we have

$$\sum_{\boldsymbol{x}_K^{(j)}=\nu} \left|\alpha_K^{(j)} - \beta^{(\nu)}\right|^2 \le C \sum_{i=1}^{|\omega_\nu|-1} \left|\alpha_{K_i}^{j_i} - \alpha_{K_{i+1}}^{j_{i+1}}\right|^2. \tag{10.4}$$

For $d = 3$, it may not be possible to enumerate ω_ν in such a way. However, by allowing some repetitions of its elements, we can write $\omega_\nu = \{K_{l_1}, \ldots, K_{l_{n(\nu)}}\}$ for some $n(\nu)$, so that in this case also K_{l_i} and $K_{l_{i+1}}$ share a face or an edge. Having done so, by applying Lemma 10.3 to the list obtained by removing all repetitions of elements of ω_ν, we obtain

$$\sum_{\boldsymbol{x}_K^{(j)}=\nu} \left|\alpha_K^{(j)} - \beta^{(\nu)}\right|^2 \le C \sum_{i=1}^{n(\nu)-1} \left|\alpha_{K_i}^{j_i} - \alpha_{K_{i+1}}^{j_{i+1}}\right|^2. \tag{10.5}$$

Using (10.4) for $d = 2$, or (10.5) if $d = 3$, from (10.3) we have

$$\sum_{K\in\mathcal{T}_h} \|\nabla_{\boldsymbol{\alpha}}(q_h - \chi)\|_{0,K}^2 \le C \sum_{f\in\mathcal{F}_h} \sum_{\nu\in f} h_\nu^{d-2} \left|\alpha_{K+}^{j_\nu^+} - \alpha_{K-}^{j_\nu^-}\right|^2, \tag{10.6}$$

with $\boldsymbol{x}_{K_+}^{j_\nu^+} = \boldsymbol{x}_{K_-}^{j_\nu^-} = \nu$. Note that $\alpha_{K_+}^{j_\nu^+} - \alpha_{K_-}^{j_\nu^-}$ is the jump in the values of q_h at ν across f. Also, since the mesh $\mathcal{T}_h$ is locally quasi-uniform, it follows that

$$\begin{aligned}
\sum_{\nu\in f} h_\nu^{d-2} \left|\alpha_{K_+}^{j_\nu^+} - \alpha_{K_-}^{j_\nu^-}\right|^2 &\le C h_f^{d-2} \|[\![q_h]\!]_N\|_{L^\infty(f)}^2 \\
&\le C h_f^{-1} \|[\![q_h]\!]_N\|_{0,f}^2,
\end{aligned} \tag{10.7}$$

where the constant C depends on the number of nodes in f. The required result now follows from (10.6)–(10.7). □

Proof of Theorem 5.2 Fix $0 \neq q \in Q_h^{\boldsymbol{\alpha}}$, and use the $Q_h^{\boldsymbol{\alpha}}$-decomposition as $q = q_0 + q_1$ with $q_0 \in Q_h^{\boldsymbol{\alpha},c}$ and $q_1 \in Q_h^{\boldsymbol{\alpha},\perp}$. Choose $\boldsymbol{v}_0 = -\nabla_{\boldsymbol{\alpha}} q_0 \in \boldsymbol{V}_h^{\boldsymbol{\alpha}} \cap \boldsymbol{H}_{\text{per}}(\text{curl}_{\boldsymbol{\alpha}}^0; \Omega)$, then we have

$$B_h(\boldsymbol{v}_0, \boldsymbol{0}; q_0) = \|\nabla_{\boldsymbol{\alpha}} q_0\|_{0,\Omega}^2 = \|q_0\|_{Q(h)}^2. \tag{10.8}$$

$$\begin{aligned} \|(\boldsymbol{v}_0, \boldsymbol{0})\|_{U(h)}^2 &= \|h^{\frac{1}{2}} [\![\boldsymbol{v}_0]\!]_N\|_{0,\mathcal{F}_h}^2 + \|\boldsymbol{v}_0\|_{0,\Omega}^2 \\ &\leq C \sum_{K \in T_h} h_K \|\nabla_{\boldsymbol{\alpha}} q_0\|_{0,\partial K}^2 + \|\nabla_{\boldsymbol{\alpha}} q_0\|_{0,\Omega}^2 \\ &\leq C \|\nabla_{\boldsymbol{\alpha}} q_0\|_{0,\Omega}^2 = C\|q_0\|_{Q(h)}^2, \end{aligned} \tag{10.9}$$

where we use $\|\nabla_{\boldsymbol{\alpha}} \phi\|_{0,\partial K} \leq C h_K^{-\frac{1}{2}} \|\nabla_{\boldsymbol{\alpha}} \phi\|_{0,K}$, for any $\phi = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{x}} \tilde{\phi}$ and $\tilde{\phi} \in \mathcal{S}_l(K)$ with $C > 0$, which we obtain from the trace inequality $\|\nabla \tilde{\phi}\|_{0,\partial K} \leq C h_K^{-\frac{1}{2}} \|\nabla \tilde{\phi}\|_{0,K}$, with $C > 0$. Let $\boldsymbol{\nu}_1 = -[\![q_1]\!]_N$. Using Lemma 5.6, we obtain

$$\begin{aligned} B_h(\boldsymbol{0}, \boldsymbol{\nu}_1; q_1) &= \mathfrak{c} \int_{\mathcal{F}_h} h^{-1} [\![q_1]\!]_N{}^2 ds \geq \mathfrak{c} C_1^2 \|q_1\|_{Q(h)}^2, \\ \|(\boldsymbol{0}, \boldsymbol{\nu}_1)\|_{U(h)} &\leq \|q_1\|_{Q(h)}. \end{aligned} \tag{10.10}$$

Let $(\boldsymbol{v}, \boldsymbol{\nu}) = (\boldsymbol{v}_0, \boldsymbol{0}) + \delta(\boldsymbol{0}, \boldsymbol{\nu}_1)$ with $\delta > 0$. Since $q_0 \in Q_h^{\boldsymbol{\alpha},c}$, $[\![q_0]\!]_N = \boldsymbol{0}$ on $\mathcal{F}_h$ and $B_h(\boldsymbol{0}, \boldsymbol{\nu}_1; q_0) = c \int_{\mathcal{F}_h} h^{-1} [\![q_0]\!]_N \cdot \bar{\boldsymbol{\nu}}_1 ds = 0$, we have

$$\begin{aligned} B_h(\boldsymbol{v}, \boldsymbol{\nu}; q) &= B_h(\boldsymbol{v}_0, \boldsymbol{0}; q_0) + B_h(\boldsymbol{v}_0, \boldsymbol{0}; q_1) + \delta B_h(\boldsymbol{0}, \boldsymbol{\nu}_1; q_1) \\ &\geq \|q_0\|_{Q(h)}^2 \quad + \delta \mathfrak{c} C_1^2 \|q_1\|_{Q(h)}^2 - |B_h(\boldsymbol{v}_0, 0, q_1)|. \end{aligned}$$

Using Theorem 10.1 and (10.9), we obtain

$$\begin{aligned} |B_h(\boldsymbol{v}_0, \boldsymbol{0}; q_1)| &\leq C \|(\boldsymbol{v}_0, \boldsymbol{0})\|_{U(h)} \|q_1\|_{Q(h)} \\ &\leq C\zeta \|q_0\|_{Q(h)}^2 + \frac{C}{\zeta} \|q_1\|_{Q(h)}^2, \end{aligned}$$

with any $\zeta > 0$. Choosing suitable δ and ζ, we have

$$B_h(\boldsymbol{v}, \boldsymbol{\nu}; q) \geq (1 - C\zeta)\|q_0\|_{Q(h)}^2 + (\delta \mathfrak{c} C_1^2 - \frac{C}{\zeta})\|q_1\|_{Q(h)}^2 \geq k_1 \|q\|_{Q(h)}^2, \tag{10.11}$$

with $k_1 > 0$. From (10.9) and (10.10), we have

$$\|(\boldsymbol{v}, \boldsymbol{\nu})\|_{U(h)} = \|(\boldsymbol{v}_0, \boldsymbol{0})\|_{U(h)} + \delta \|(\boldsymbol{0}, \boldsymbol{\nu}_1)\|_{U(h)} \leq k_2 \|q\|_{Q(h)}.$$

Then the result follows with $k = k_1/k_2$. □

Appendix C: Ellipticity on the Kernel

Lemma 10.4

$$\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega) = \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{H}_{\text{per}}^1(\Omega).$$

Proof Let $\boldsymbol{v} \in \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega)$. By [16, Theorem 3.1], there exists $\boldsymbol{w} \in \boldsymbol{H}^1_{\text{per}}(\Omega)$ and $\phi \in H^1_{\text{per}}(\Omega)$ such that

$$\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v} = \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{w} + \nabla_{\boldsymbol{\alpha}} \phi, \quad \nabla_{\boldsymbol{\alpha}} \cdot \boldsymbol{w} = 0.$$

By Lemma 3.1, since $\nabla_{\boldsymbol{\alpha}} \cdot \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v} = 0$, we obtain $\phi = 0$. Therefore

$$\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{v} = \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{w} \in \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{H}^1_{\text{per}}(\Omega).$$

implying $\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega) \subset \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{H}^1_{\text{per}}(\Omega)$. The other inclusion is obvious as $\boldsymbol{H}^1_{\text{per}}(\Omega) \subset \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega)$. □

Proof of Lemma 5.7 Lemma 10.4 implies that $\nabla_{\boldsymbol{\alpha}} \times$ maps $\boldsymbol{H}^1_{\text{per}}(\Omega)$ onto $\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega)$. Let K denote the orthogonal complement of the kernel of $\nabla_{\boldsymbol{\alpha}} \times$ in $\boldsymbol{H}^1_{\text{per}}(\Omega)$. Then, the restriction $\nabla_{\boldsymbol{\alpha}} \times |_K$ of $\nabla_{\boldsymbol{\alpha}} \times$ to K also maps $\boldsymbol{H}^1_{\text{per}}(\Omega)$ onto $\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega)$. In addition to being onto, $\nabla_{\boldsymbol{\alpha}} \times |_K$ is continuous, one-to-one and has a continuous inverse due to [16, Theorem 3.1]. The operator $R = (\nabla_{\boldsymbol{\alpha}} \times |_K)^{-1} \nabla_{\boldsymbol{\alpha}} \times$ satisfies the conclusion of the lemma. □

Lemma 10.5 *For $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, we have the following estimate for the auxiliary problem (10.12):*

$$\begin{aligned} \|\epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z\|_{0,\Omega} + \|z\|_{0,\Omega} + \|\nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z\|_{0,\Omega} \\ + \|\nabla_{\boldsymbol{\alpha}} \psi\|_{0,\Omega} + \|\psi\|_{0,\Omega} \leq C_m \|\boldsymbol{u}\|_{0,\Omega}. \end{aligned}$$

Proof Taking the periodic boundary conditions into consideration and integrating by parts, we have

$$\begin{aligned} \|\boldsymbol{u}\|^2_{0,\Omega} &= (\boldsymbol{u}, \boldsymbol{u}) \\ &= (\nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z - \nabla_{\boldsymbol{\alpha}} \psi, \nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z - \nabla_{\boldsymbol{\alpha}} \psi) \\ &= (\nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z, \nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z) + (\nabla_{\boldsymbol{\alpha}} \psi, \nabla_{\boldsymbol{\alpha}} \psi) \\ &- 2Re(\nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z, \nabla_{\boldsymbol{\alpha}} \psi) \\ &= (\nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z, \nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z) + (\nabla_{\boldsymbol{\alpha}} \psi, \nabla_{\boldsymbol{\alpha}} \psi) \\ &- 2Re(\epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z, \nabla_{\boldsymbol{\alpha}} \times (\nabla_{\boldsymbol{\alpha}} \psi)) \\ &= \|\nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z\|^2_{0,\Omega} + \|\nabla_{\boldsymbol{\alpha}} \psi\|^2_{0,\Omega}, \end{aligned}$$

where $\nabla_{\boldsymbol{\alpha}} \times (\nabla_{\boldsymbol{\alpha}} \psi) = \mathbf{0}$. Combining with the estimate given in Theorem 3.3 gives the result. □

Proof of Theorem 5.3 From the seminorm ellipticity in Lemma 10.2 , it is sufficient to show that there exist $C > 0$, such that

$$\|\boldsymbol{u}\|_{0,\Omega} \leq C |(\boldsymbol{u}, \boldsymbol{v})|_{U(h)} \quad \forall (\boldsymbol{u}, \boldsymbol{v}) \in \text{Ker}(B_h).$$

Now fix $(\boldsymbol{u}, \boldsymbol{v}) \in \text{Ker}(B_h)$, and let $(z, \psi) \in \boldsymbol{V} \times Q$ satisfying

$$\begin{aligned} \nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z - \nabla_{\boldsymbol{\alpha}} \psi &= \boldsymbol{u}, \\ \nabla_{\boldsymbol{\alpha}} \cdot z &= 0, \end{aligned} \tag{10.12}$$

with periodic boundary conditions. Thereby,

$$\begin{aligned} \|\epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z\|_{0,\Omega} + \|z\|_{0,\Omega} + \|\nabla_{\boldsymbol{\alpha}} \times \epsilon^{-1} \nabla_{\boldsymbol{\alpha}} \times z\|_{0,\Omega} \\ + \|\nabla_{\boldsymbol{\alpha}} \psi\|_{0,\Omega} + \|\psi\|_{0,\Omega} \leq C_m \|\boldsymbol{u}\|_{0,\Omega}, \end{aligned} \tag{10.13}$$

where the detailed derivation of (10.13) is given in Lemma 10.5. Set $\boldsymbol{w} = \epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{z}$, clearly $\boldsymbol{w} \in \boldsymbol{H}_{\text{per}}(\text{curl}; \Omega)$. Then, from Theorem 5.7 and the inequality (10.13) there exists $\boldsymbol{w}_0 \in \boldsymbol{H}^1_{\text{per}}(\Omega)$ such that

$$\begin{aligned} \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{w}_0 =& \nabla_{\boldsymbol{\alpha}} \times \boldsymbol{w}, \\ \|\boldsymbol{w}_0\|_{1,\Omega} \le C\|\boldsymbol{w}\|_{\boldsymbol{H}_{\text{per}}(\text{curl};\Omega)} \le& C_m\|\boldsymbol{u}\|_{0,\Omega}. \end{aligned} \tag{10.14}$$

Multiplying the first equation of (10.12) by $\boldsymbol{u}$ and integrating by parts, we obtain

$$\begin{aligned} \|\boldsymbol{u}\|^2_{0,\Omega} =& \int_\Omega \boldsymbol{w}_0 \cdot \overline{\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u}} d\boldsymbol{x} - \int_{\mathcal{F}_h} \boldsymbol{w}_0 \cdot \overline{[\![\boldsymbol{u}]\!]_T} ds + \int_\Omega \psi \overline{\nabla_{\boldsymbol{\alpha},h} \cdot \boldsymbol{u}} d\boldsymbol{x} \\ &- \int_{\mathcal{F}_h} \{\!\{\psi\}\!\} \overline{[\![\boldsymbol{u}]\!]_N} ds. \end{aligned}$$

Since $(\boldsymbol{u}, \boldsymbol{v}) \in \text{Ker}(B_h)$, we choose ψ_h as the L^2-projection of ψ in $Q_h^{\boldsymbol{\alpha}}$, then we have $B_h(\boldsymbol{u}, \boldsymbol{v}; \psi_h) = 0$. Using the fact that $\psi \in Q$ in the auxiliary problem (10.12) and $[\![\psi]\!]_N = 0$ on $\mathcal{F}_h$,

$$\begin{aligned} \|\boldsymbol{u}\|^2_{0,\Omega} =& \int_\Omega \boldsymbol{w}_0 \cdot \overline{\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u}} d\boldsymbol{x} - \int_{\mathcal{F}_h} \boldsymbol{w}_0 \cdot \overline{[\![\boldsymbol{u}]\!]_T} ds + \int_\Omega (\psi - \psi_h) \overline{\nabla_{\boldsymbol{\alpha},h} \cdot \boldsymbol{u}} d\boldsymbol{x} \\ &- \int_{\mathcal{F}_h} \{\!\{\psi - \psi_h\}\!\} \overline{[\![\boldsymbol{u}]\!]_N} ds - \int_{\mathcal{F}_h} \mathfrak{c} h^{-1} \boldsymbol{v} \cdot \overline{[\![\psi - \psi_h]\!]_N} ds. \end{aligned}$$

Using (10.14), we have

$$\left| \int_\Omega \boldsymbol{w}_0 \cdot \overline{\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u}} d\boldsymbol{x} \right| \le \|\boldsymbol{w}_0\|_{1,\Omega} \|\nabla_{\boldsymbol{\alpha},h} \times \boldsymbol{u}\|_{0,\Omega} \le C_m \|\boldsymbol{u}\|_{0,\Omega} |\boldsymbol{u}|_{V(h)}.$$

Using trace inequalities and (10.14), we have

$$\begin{aligned} \left| \int_{\mathcal{F}_h} \boldsymbol{w}_0 \cdot \overline{[\![\boldsymbol{u}]\!]_T} ds \right| \le& C \left(\sum_{K \in T_h} h_K \epsilon_K \|\boldsymbol{w}_0\|^2_{0,\partial K} \right)^{\frac{1}{2}} \|e^{-\frac{1}{2}} h^{-\frac{1}{2}} [\![\boldsymbol{u}]\!]_T\|_{0,\mathcal{F}_h} \\ \le& C\|\boldsymbol{w}_0\|_{1,\Omega} \|e^{-\frac{1}{2}} h^{-\frac{1}{2}} [\![\boldsymbol{u}]\!]_T\|_{0,\mathcal{F}_h} \le C\|\boldsymbol{u}\|_{0,\Omega} |\boldsymbol{u}|_{V(h)}. \end{aligned}$$

Since ψ_h is the L^2-projection of ψ, the third term is zero. Using (10.13), we obtain the following estimate for the last two terms:

$$\begin{aligned} \left| \int_{\mathcal{F}_h} \{\!\{\psi - \psi_h\}\!\} \overline{[\![\boldsymbol{u}]\!]_N} ds \right| \le& C \left(\sum_{K \in T_h} h_k^{-1} \|\psi - \psi_h\|^2_{0,\partial K} \right)^{\frac{1}{2}} \left(\sum_{K \in T_h} \|h^{\frac{1}{2}} [\![\boldsymbol{u}]\!]_N\|^2_{0,\partial K} \right)^{\frac{1}{2}} \\ \le& C \left(\sum_{K \in T_h} h_k^{-1} \|\psi - \psi_h\|^2_{0,\partial K} \right)^{\frac{1}{2}} |\boldsymbol{u}|_{V(h)} \\ \le& C \left(\sum_{K \in T_h} \|\nabla_{\boldsymbol{\alpha},h} \psi\|^2_{0,K} + \|\psi\|^2_{0,K} \right)^{\frac{1}{2}} |\boldsymbol{u}|_{V(h)} \\ \le& C\|\boldsymbol{u}\|_{0,\Omega} |\boldsymbol{u}|_{V(h)}. \end{aligned}$$

$$\left|\int_{\mathcal{F}_h} h^{-1}\boldsymbol{v}\cdot\overline{[\![\psi-\psi_h]\!]_N}ds\right| \le C\left(\sum_{K\in T_h} h_K^{-1}\|\psi-\psi_h\|_{0,\partial K}^2\right)^{\frac{1}{2}}\left(\int_{\mathcal{F}_h} h^{-1}|\boldsymbol{v}|^2 ds\right)^{\frac{1}{2}}$$
$$\le C\|\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{v}\|_{M_h^{\boldsymbol{\alpha}}}.$$

From the results above, we have $\|\boldsymbol{u}\|_{0,\Omega} \le C|(\boldsymbol{u},\boldsymbol{v})|_{U(h)}$. □

Appendix D: The Convergence of the Operator

Proof of Theorem 5.4 Let $(\boldsymbol{u}, p)$ be the analytical solution of (3.7), and $(\boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h)$ be the numerical solution of (5.4). By the triangle inequality and the definition of $\|(\cdot,\cdot)\|_{\boldsymbol{U}(h)}$, we have

$$\|(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)} \le \|(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)} + \|(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)}, \tag{10.15}$$

for any $(\boldsymbol{v},\boldsymbol{\eta}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}}$. First, we take $(\boldsymbol{v},\boldsymbol{\eta}) \in \mathrm{Ker}(B_h)$. Since $(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h) \in \mathrm{Ker}(B_h)$, employing the ellipticity property of Theorem 5.3 and the definition of R_h^1, we have

$$\begin{aligned}
b\|(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)}^2 &\le A_h(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h;\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\\
&= A_h(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\eta};\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\\
&\quad + A_h(\boldsymbol{u}-\boldsymbol{u}_h,-\boldsymbol{\lambda}_h;\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\\
&= A_h(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\eta};\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\\
&\quad - B_h(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h;p-p_h)\\
&\quad + R_h^1(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h;\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\\
&= A_h(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\eta};\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\\
&\quad - B_h(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h;p-q) + R_h^1(\boldsymbol{u},p;\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\\
&\le a_1\|(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)}\|(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)}\\
&\quad + a_2\|(\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)}\|p-q\|_{Q(h)} + R_h^1(\boldsymbol{u},p;\boldsymbol{v}-\boldsymbol{u}_h,\boldsymbol{\eta}-\boldsymbol{\lambda}_h),
\end{aligned} \tag{10.16}$$

for any $q \in Q_h^{\boldsymbol{\alpha}}$. Combining (10.15) and (10.16), we have

$$\begin{aligned}
\|(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)} &\le \left(1+\frac{a_1}{b}\right)\inf_{(\boldsymbol{v},\boldsymbol{\eta})\in\mathrm{Ker}(B_h)}\|(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)}\\
&\quad + \frac{a_2}{b}\inf_{q\in Q_h^{\boldsymbol{\alpha}}}\|p-q\|_{Q(h)} + \frac{1}{b}\mathcal{R}_h^1(\boldsymbol{u},p).
\end{aligned} \tag{10.17}$$

Next, we prove that

$$\inf_{(\boldsymbol{v},\boldsymbol{\eta})\in\mathrm{Ker}(B_h)}\|(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)} \le \left(1+\frac{a_2}{k}\right)\inf_{(\boldsymbol{v},\boldsymbol{\eta})\in\boldsymbol{U}_h^{\boldsymbol{\alpha}}}\|(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)} + \frac{1}{k}\mathcal{R}_h^2(\boldsymbol{u}). \tag{10.18}$$

Let $(\boldsymbol{v},\boldsymbol{\eta}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}}$, and consider the following problem: find $(\boldsymbol{w},\nu) \in \boldsymbol{U}(h)$ such that

$$B_h(\boldsymbol{w},\nu;q) = B_h(\boldsymbol{u}-\boldsymbol{v},-\boldsymbol{\eta};q) - R_h^2(\boldsymbol{u};q) \qquad \forall q \in Q_h^{\boldsymbol{\alpha}}. \tag{10.19}$$

Problem (10.19) admits a solution in $\boldsymbol{U}(h)$ that is unique up to elements in $\mathrm{Ker}(B_h)$. The discrete inf-sup condition of Theorem 5.2 guarantees the existence of a solution $(\boldsymbol{w},\nu) \in \boldsymbol{U}(h)$ satisfying

$$
\begin{aligned}
\|(\boldsymbol{w},\nu)\|_{\boldsymbol{U}(h)} &\le \frac{1}{k}\left(\sup_{q\in Q_h^{\boldsymbol{\alpha}}}\frac{B_h(\boldsymbol{u}-\boldsymbol{v},-\boldsymbol{\eta};q)}{\|q\|_{Q(h)}}+\sup_{q\in Q_h^{\boldsymbol{\alpha}}}\frac{R_h^2(\boldsymbol{u};q)}{\|q\|_{Q(h)}}\right)\\
&\le \frac{a_2}{k}\|(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)}+\frac{1}{k}\mathcal{R}_h^2(\boldsymbol{u}),
\end{aligned}
\tag{10.20}
$$

where we have used the continuity of $B_h(\cdot,\cdot;\cdot)$, the definition of the norm $\|(\cdot,\cdot)\|_{\boldsymbol{U}(h)}$, and the definition of $\mathcal{R}_h^2(\cdot)$. From (10.19), $B_h(\boldsymbol{w}+\boldsymbol{v},\nu+\boldsymbol{\eta};q)=0$, for any $q\in Q_h^{\boldsymbol{\alpha}}$, so that $(\boldsymbol{w}+\boldsymbol{v},\nu+\boldsymbol{\eta})\in \mathrm{Ker}(B_h)$. Therefore, since

$$
\|(\boldsymbol{u}-(\boldsymbol{v}+\boldsymbol{w}),\boldsymbol{\eta}+\nu)\|_{\boldsymbol{U}(h)}\le\|(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)}+\|(\boldsymbol{w},\nu)\|_{\boldsymbol{U}(h)},
$$

for any $(\boldsymbol{v},\boldsymbol{\eta})\in\boldsymbol{U}(h)$, taking into account (10.20), we obtain (10.18). This, together with (10.17), yields

$$
\begin{aligned}
\|(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)} \le C\Bigg(&\inf_{(\boldsymbol{v},\boldsymbol{\eta})\in\boldsymbol{U}_h^{\boldsymbol{\alpha}}}\|(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)}\\
&+\inf_{q\in Q_h^{\boldsymbol{\alpha}}}\|p-q\|_{Q(h)}+\mathcal{R}_h^1(\boldsymbol{u},p)+\mathcal{R}_h^2(\boldsymbol{u})\Bigg),
\end{aligned}
$$

where the constant C depends on a_1, a_2 and k_1. Choosing $\boldsymbol{\eta}=\boldsymbol{0}$ gives the error bound for $(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\lambda}_h)$.

We now turn to the bound for $p-p_h$. Again by the triangle inequality, we have

$$
\|p-p_h\|_{Q(h)}\le\|p-q\|_{Q(h)}+\|q-p_h\|_{Q(h)},
\tag{10.21}
$$

for any $q\in Q_h^{\boldsymbol{\alpha}}$. Since

$$
A_h(\boldsymbol{u}-\boldsymbol{u}_h,-\boldsymbol{\lambda}_h;\boldsymbol{v},\boldsymbol{\eta})+B_h(\boldsymbol{v},\boldsymbol{\eta};p-q)+B_h(\boldsymbol{v},\boldsymbol{\eta};q-p_h)=R_h^1(\boldsymbol{u},p;\boldsymbol{v},\boldsymbol{\eta}),
$$

for any $(\boldsymbol{v},\boldsymbol{\eta})\in\boldsymbol{U}(h)$, the discrete inf-sup condition of $B_h(\cdot,\cdot;\cdot)$ gives

$$
\begin{aligned}
\|q-p_h\|_{Q(h)} &\le \frac{1}{k}\sup_{(\boldsymbol{0},\boldsymbol{0})\neq(\boldsymbol{v},\boldsymbol{\eta})\in\boldsymbol{U}_h^{\boldsymbol{\alpha}}}\frac{B_h(\boldsymbol{v},\boldsymbol{\eta};q-p_h)}{\|(\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)}}\\
&=\frac{1}{k}\sup_{(\boldsymbol{0},\boldsymbol{0})\neq(\boldsymbol{v},\boldsymbol{\eta})\in\boldsymbol{U}_h^{\boldsymbol{\alpha}}}\frac{-A_h(\boldsymbol{u}-\boldsymbol{u}_h,-\boldsymbol{\lambda}_h;\boldsymbol{v},\boldsymbol{\eta})-B_h(\boldsymbol{v},\boldsymbol{\eta};p-q)+R_h^1(\boldsymbol{u},p;\boldsymbol{v},\boldsymbol{\eta})}{\|(\boldsymbol{v},\boldsymbol{\eta})\|_{\boldsymbol{U}(h)}}\\
&\le\frac{a_1}{k}\|(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\lambda}_h)\|_{\boldsymbol{U}(h)}+\frac{a_2}{k}\|p-q\|_{Q(h)}+\frac{1}{k}\mathcal{R}_h^1(\boldsymbol{u},p).
\end{aligned}
$$

This, together with (10.21), gives a bound for $p-p_h$. □

Proof of Lemma 5.8 Let $(\boldsymbol{u}, p)$ be the analytical solution of (3.7). Let $\Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha}}}$ be the L^2-projection onto $\boldsymbol{V}_h^{\boldsymbol{\alpha}}$. For $(\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{U}_h^{\boldsymbol{\alpha}}$,

$$\begin{aligned}
\left|R_h^1(\boldsymbol{u}, p; \boldsymbol{v}, \boldsymbol{\eta})\right| &= |A_h(\boldsymbol{u}, \boldsymbol{0}; \boldsymbol{v}, \boldsymbol{\eta}) + B_h(\boldsymbol{v}, \boldsymbol{\eta}; p) - a_h(\boldsymbol{u}, \boldsymbol{v}) - b_h(\boldsymbol{v}, p)| \\
&= \left| -\int_\Omega \overline{\mathcal{L}(\boldsymbol{v})} \cdot (\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}) d\boldsymbol{x} + \int_{\mathcal{F}_h} \overline{[\![\boldsymbol{v}]\!]_T} \cdot (\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}) d\boldsymbol{x} \right| \\
&= \left| -\int_\Omega \overline{\mathcal{L}(\boldsymbol{v})} \cdot \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha}}}(\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}) d\boldsymbol{x} + \int_{\mathcal{F}_h} \overline{[\![\boldsymbol{v}]\!]_T} \cdot (\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}) ds \right| \\
&= \left| \int_{\mathcal{F}_h} \{\!\{\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha}}}(\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u})\}\!\} \cdot \overline{[\![\boldsymbol{v}]\!]_T} ds \right| \\
&\le C \left(\sum_{K \in \mathcal{T}_h} h_K \|\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha}}}(\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u})\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \|(\boldsymbol{v}, \boldsymbol{\eta})\|_{\boldsymbol{U}(h)} \\
&\le C h^{\min\{s,k+1\}} \|\epsilon^{-1}\nabla_{\boldsymbol{\alpha}} \times \boldsymbol{u}\|_{s,\Omega} \|(\boldsymbol{v}, \boldsymbol{\eta})\|_{\boldsymbol{U}(h)}.
\end{aligned}$$

Similarly, for $q \in Q_h^{\boldsymbol{\alpha}}$,

$$\begin{aligned}
\left|R_h^2(\boldsymbol{u}; q)\right| &= \left| \overline{B_h(\boldsymbol{u}, \boldsymbol{0}; q)} - \overline{b_h(\boldsymbol{u}, q)} + c_h(p, q) \right| \\
&\le \left| \int_\Omega \overline{\boldsymbol{u}} \cdot \mathcal{M}(q) d\boldsymbol{x} - \int_{\mathcal{F}_h} \overline{\{\!\{\boldsymbol{u}\}\!\}} \cdot [\![q]\!]_N ds \right| \\
&= \left| \int_\Omega \overline{\Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha}}} \boldsymbol{u}} \cdot \mathcal{M}(q) d\boldsymbol{x} - \int_{\mathcal{F}_h} \overline{\{\!\{\boldsymbol{u}\}\!\}} \cdot [\![q]\!]_N ds \right| \\
&\le \left| \int_{\mathcal{F}_h} \overline{\{\!\{\boldsymbol{u} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha}}} \boldsymbol{u}\}\!\}} \cdot [\![q]\!]_N ds \right| \\
&\le C \left(\sum_{K \in \mathcal{T}_h} h_K \|\boldsymbol{u} - \Pi_{\boldsymbol{V}_h^{\boldsymbol{\alpha}}} \boldsymbol{u}\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \|q\|_{Q(h)} \\
&\le C h^{\min\{s,k+1\}} \|\boldsymbol{u}\|_{s,\Omega} \|q\|_{Q(h)}.
\end{aligned}$$

□

References

1. Boffi, D.: Fortin operator and discrete compactness for edge elements. Numer. Math. **87**(2), 229–246 (2000)
2. Boffi, D., Brezzi, F., Fortin, M.: Mixed Finite Element Methods and Applications. Springer Series in Computational Mathematics. Springer, New York (2013)
3. Boffi, D., Conforti, M., Gastaldi, L.: Modified edge finite elements for photonic crystals. Numer. Math. **105**(2), 249–266 (2006)
4. Boffi, D., Fernandes, P., Gastaldi, L., Perugia, I.: Computational models of electromagnetic resonators: analysis of edge element approximation. SIAM J. Numer. Anal. **36**(4), 1264–1290 (1999)
5. Bossavit, A.: A rationale for edge-elements in 3-D fields computations. Magn. IEEE Trans. **24**, 74–79 (1988)
6. Bossavit, A.: Solving Maxwell equations in a closed cavity, and the question of 'spurious modes'. Magn. IEEE Trans. **26**(2), 702–705 (1990)
7. Brenner, S.C., Li, F., Sung, L.Y.: A locally divergence-free interior penalty method for two-dimensional curl-curl problems. SIAM J. Numer. Anal. **46**(3), 1190–1211 (2008)

8. Buffa, A., Houston, P., Perugia, I.: Discontinuous Galerkin computation of the Maxwell eigenvalues on simplicial meshes. J. Comput. Appl. Math. **204**(2), 317–333 (2007). Special Issue: The Seventh International Conference on Mathematical and Numerical Aspects of Waves (WAVES05)
9. Buffa, A., Perugia, I.: Discontinuous Galerkin approximation of the Maxwell eigenproblem. SIAM J. Numer. Anal. **44**(5), 2198–2226 (2006)
10. Busch, K., Knig, M., Niegemann, J.: Discontinuous Galerkin methods in nanophotonics. Laser Photon. Rev. **5**(6), 773–809 (2011)
11. Caorsi, S., Fernandes, P., Raffetto, M.: On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems. SIAM J. Numer. Anal. **38**(2), 580–607 (2001)
12. Cockburn, B., Li, F., Shu, C.W.: Locally divergence-free discontinuous Galerkin methods for the Maxwell equations. J. Comput. Phys. **194**(2), 588–610 (2004)
13. Descloux, J., Nassif, N., Rappaz, J.: On spectral approximation. Part 1. The problem of convergence. RAIRO Anal. Numer. **12**(2), 97–112 (1978)
14. Dobson, D.C.: An efficient method for band structure calculations in 2D photonic crystals. J. Comput. Phys. **149**(2), 363–376 (1999)
15. Dobson, D.C., Gopalakrishnan, J., Pasciak, J.E.: An efficient method for band structure calculations in 3D photonic crystals. J. Comput. Phys. **161**(2), 668–679 (2000)
16. Dobson, D.C., Pasciak, J.E.: Analysis of an algorithm for computing electromagnetic Bloch modes using Nédélec spaces. Comput. Methods Appl. Math. **1**(2), 138–153 (2001)
17. Dörfler, W., Lechleiter, A., Plum, M., Schneider, G., Wieners, C.: Photonic Crystals: Mathematical Analysis and Numerical Approximation. Springer, NewYork (2011)
18. Fernandes, P., Gilardi, G.: Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. Math. Models Methods Appl. Sci. **7**(07), 957–991 (1997)
19. Grote, M.J., Schneebeli, A., Schötzau, D.: Interior penalty discontinuous Galerkin method for Maxwell's equations: energy norm error estimates. J. Comput. Appl. Math. **204**(2), 375–386 (2007). Special Issue: The Seventh International Conference on Mathematical and Numerical Aspects of Waves (WAVES05)
20. Grote, M.J., Schneebeli, A., Schötzau, D.: Interior penalty discontinuous Galerkin method for Maxwell's equations: optimal L^2-norm error estimates. IMA J. Numer. Anal. **28**(3), 440–468 (2008)
21. Hesthaven, J.S., Warburton, T.: High-order nodal discontinuous Galerkin methods for the Maxwell eigenvalue problem. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **362**(1816), 493–524 (2004)
22. Hiptmair, R.: Finite elements in computational electromagnetism. Acta Numer. **11**, 237–339 (2002)
23. Houston, P., Perugia, I., Schneebeli, A., Schötzau, D.: Interior penalty method for the indefinite time-harmonic Maxwell equations. Numer. Math. **100**(3), 485–518 (2005)
24. Houston, P., Perugia, I., Schneebeli, A., Schötzau, D.: Mixed discontinuous Galerkin approximation of the Maxwell operator: the indefinite case. ESAIM Math. Model. Numer. Anal. **39**(4), 727–753 (2010)
25. Houston, P., Perugia, I., Schötzau, D.: hp-DGFEM for Maxwell's equations. In: Brezzi, F., Buffa, A., Corsaro, S., Murli, Aeditors (eds.) Numerical Mathematics and Advanced Applications, pp. 785–794. Springer, Milan (2003)
26. Houston, P., Perugia, I., Schotzau, D.: Mixed discontinuous Galerkin approximation of the Maxwell operator. SIAM J. Numer. Anal. **42**(1), 434–459 (2004)
27. Houston, P., Perugia, I., Schötzau, D.: Energy norm a posteriori error estimation for mixed discontinuous Galerkin approximations of the Maxwell operator. Comput. Methods Appl. Mech. Eng. **194**(25), 499–510 (2005). Selected papers from the 11th Conference on The Mathematics of Finite Elements and Applications
28. Houston, P., Perugia, I., Schötzau, D.: Mixed discontinuous Galerkin approximation of the Maxwell operator: non-stabilized formulation. J. Sc. Comput. **22–23**(1–3), 315–346 (2005)
29. Joannopoulos, J.D., Johnson, S.G., Winn, J.N., Meade, R.D.: Photonic Crystals: Molding the Flow of Light. Princeton University Press, Princeton (2011)
30. Kantorovich, L.: Quantum Theory of the Solid State: An Introduction, vol. 136. Springer, NewYork (2004)
31. Karakashian, O.A., Pascal, F.: A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems. SIAM J. Numer. Anal. **41**(6), 2374–2399 (2003)
32. Kikuchi, F.: Mixed and penalty formulations for finite element analysis of an eigenvalue problem in electromagnetism. Comput. Methods Appl. Mech. Eng. **64**(1), 509–521 (1987)
33. Kikuchi, F.: On a discrete compactness property for the Nédélec finite elements. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36**(3), 479–490 (1989)
34. Monk, P.: Finite Element Methods for Maxwell's Equations. Oxford University Press, Oxford (2003)
35. Monk, P., Demkowicz, L.: Discrete compactness and the approximation of Maxwell's equations in $\mathbb{R}^3$. Math. Comput. **70**(234), 507–523 (2001)
36. Nédélec, J.C.: Mixed finite elements in $\mathbb{R}^3$. Numer. Math. **35**(3), 315–341 (1980)

37. Nelson, D., Jacob, T.S.: Linear Operators. Part 1 General Theory. Interscience publishers, NewYork (1958)
38. Nguyen, N.C., Peraire, J., Cockburn, B.: Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell's equations. J. Comput. Phys. **230**(19), 7151–7175 (2011)
39. Osborn, J.E.: Spectral approximation for compact operators. Math. Comput. **29**(131), 712–725 (1975)
40. Perugia, I., Schötzau, D.: The *hp*-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations. Math. Comput. **72**(243), 1179–1214 (2003)
41. Perugia, I., Schötzau, D., Monk, P.: Stabilized interior penalty methods for the time-harmonic Maxwell equations. Comput. Methods Appl. Mech. Eng. **191**(41), 4675–4697 (2002)
42. Sármány, D., Izsák, F., van der Vegt, J.J.W.: Optimal penalty parameters for symmetric discontinuous Galerkin discretisations of the time-harmonic Maxwell equations. J. Sci. Comput. **44**(3), 219–254 (2010)
43. Soukoulis, C.M.: Photonic Band Gap Materials. Kluwer, Dordrecht (1996)
44. Sözüer, H.S., Haus, J.W.: Photonic bands: simple-cubic lattice. JOSA B **10**(2), 296–302 (1993)
45. Sun, D., Manges, J., Yuan, X., Cendes, Z.: Spurious modes in finite-element methods. Antennas Propag. Mag. IEEE **37**(5), 12–24 (1995)
46. Taflove, A., Hagness S.C.: *Computational electrodynamics: the finite-difference time-domain method*. Artech House, Inc., Boston, MA, second edition, 2000. With 1 CD-ROM (Windows)
47. Warburton, T., Embree, M.: The role of the penalty in the local discontinuous Galerkin method for Maxwell's eigenvalue problem. Comput. Methods Appl. Mech. Eng. **195**(2528), 3205–3223 (2006)