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The WDVV Equations in Pure Seiberg–Witten Theory

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Abstract. We review the relationship between pure four-dimensional Seiberg–Witten theory and the periodic Toda chain. We discuss the definition of the prepotential and give two proofs that it satisfies the generalized Witten–Dijkgraaf–Verlinde–Verlinde equations. A number of steps in the definitions and proofs that is missing in the literature is supplied.

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Introduction

In two-dimensional topological conformal field theory the following remarkable system of third-order nonlinear partial differential equations for a function F of N variables emerged [48, 12]

$$F_i F_1^{-1} F_j = F_i F_1^{-1} F_j, \quad i, j = 1, \ldots, N. \tag{1}$$

Here F_i is the matrix

$$(F_i)_{jk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k}. \tag{2}$$

Moreover, it is required that F_1 is a constant and invertible matrix. Usually this system is called the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) system, for a physical review paper see for instance [11]. Although extremely difficult to solve in general, this overdetermined system of nonlinear partial differential equations admits exact solutions. For instance, within the theory of Frobenius manifolds, a substantial class of polynomial solutions has been constructed associated to Coxeter groups [44, 14].

Generalizations of this system, not requiring F_1 to be constant, have been introduced and studied in the context of four- and five-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory (see [34–36]). In 1994, Seiberg and Witten [45] solved the low energy behaviour of pure $\mathcal{N} = 2$ Super-Yang–Mills theory in terms of the prepotential $\mathcal{F}$. The essential ingredients in their construction for each simple Lie algebra $\mathfrak{g}$ of rank N are: a family of Riemann surfaces $C_{\mathfrak{g}}$, a meromorphic differential λ_{SW} and a special selection of $2N$ cycles on $C_{\mathfrak{g}}$. The prepotential is defined in terms of the period integrals of λ_{SW} as

$$a_i = \int_{A_i} \lambda_{\mathrm{SW}}, \qquad \frac{\partial \mathcal{F}}{\partial a_i} = \int_{B_i} \lambda_{\mathrm{SW}}, \tag{3}$$

where the a_i play the role of moduli parameters of the family of surfaces. These formulae define the prepotential $\mathcal{F}(a_1, \ldots, a_N)$ implicitly.

This paper consists of two main parts, the first of which is used to review the construction of the Seiberg–Witten prepotential. One of the keys to understanding the prepotential is the relation with integrable systems. In particular, the pure 4-dimensional theory is related to the periodic Toda chain [17, 38]. There is a complete lexicon translating the main objects of Seiberg–Witten theory to those of the Toda chain and vice versa. Essentially, the prepotential identifies the Liouville torus inside the Jacobian of the spectral curve. In our review of the three main ingredients of the prepotential (the curves, meromorphic differential and the cycles) we will focus on technical aspects. It is shown explicitly that the derivatives of the Seiberg–Witten differential with respect to the moduli are holomorphic, and following [22] we define a subset of $2N$ special cycles A_i, B_j. We complete this set of cycles to a full canonical set with the property that the period integrals of the Seiberg–Witten differential over the nonspecial A cycles is zero. The possibility to complete the set in this way is actually necessary to define the prepotential.

The second part of the paper concerns the generalized Witten–Dijkgraaf–Verlinde–Verlinde (or WDVV) equations. The main goal is to review the proof that the Seiberg–Witten prepotentials satisfy this highly nontrivial system of partial differential equations. The two main aspects of the proof are the construction of a family of associative and commutative algebras on the one hand, and the relation of its structure constants to the prepotential on the other. It is explained how to find the family of algebras for any simple Lie algebra [35, 19] and we discuss two methods to relate its structure constants to the prepotential. The first method [26] uses Picard–Fuchs equations related to Landau–Ginzburg theory (or the theory of isolated singularities) and its validity is therefore restricted to simply laced Lie algebras. It is not immediately clear how to proceed in nonsimply laced cases, but we complete a proof suggested in [26] for the cases of B_N and C_N. The second method [36] involves a residue formula and is valid for any simple Lie algebra.

1. The Seiberg–Witten Data

The prepotentials considered in this paper originally arose as the solution to $\mathcal{N} = 2$ supersymmetric Yang–Mills theory, also called Seiberg–Witten theory [45]. Although this physical context is essential for a full understanding of the prepotentials, it would take too much time to expose it here in full detail. For reviews on the subject, see, for example, [3, 5, 10].

On the other hand, the prepotentials can be described in the framework of an integrable system called the periodic Toda chain [17, 38]. We will assume that the reader either has the necessary physical background or has some knowledge of integrable dynamical systems. In this paper we use the Toda chain context, as the background and motivation for answering certain questions which are relevant in the construction of the prepotentials.

The mathematical definition of the prepotential for a simple Lie algebra $\mathfrak{g}$ involves three main ingredients:

- The first ingredient is a family of Riemann surfaces given in terms of a set of affine curves
$$C_{\mathfrak{g}} = \left\{(x, z) \in \mathbf{C}^2 \mid P(x, z, u_1, \ldots, u_N) = 0\right\}, \tag{4}$$
where the u_i serve as complex moduli parameters, N is the rank of $\mathfrak{g}$ and a Riemann surface in the family $C_{\mathfrak{g}}$ has genus $g \geqslant N$.
- The second ingredient is a specific meromorphic differential λ_{SW} on $C_{\mathfrak{g}}$ which is called the Seiberg–Witten differential. Its derivatives with respect to the moduli are holomorphic differentials on curves in the family $C_{\mathfrak{g}}$.
- The third ingredient is a particular choice of $2N$ independent cycles on $C_{\mathfrak{g}}$ out of a total $2g$. In terms of a canonical basis $\{A_i, B_j\}$ of the first homology group, the choice consists of N cycles of type A and N cycles of type B in such a way that the restriction of the intersection form to this subset is nondegenerate: $A_i \circ B_j \simeq \delta_{ij}$.

In the rest of this section, we will explain the construction of the family of curves, the Seiberg–Witten differential and the choice of cycles. Once these ingredients are introduced, we define the prepotential in terms of period integrals of λ_{SW} over the chosen $2N$ cycles. Since varying the moduli will influence the period integrals, the (locally defined) prepotential is a function on moduli space.

1.1. A SIMPLE EXAMPLE: TYPE A LIE ALGEBRA

The Seiberg–Witten data and the construction of the prepotential is complicated and technical for general simple Lie algebras. To get warmed up, we first give the relatively simple example of Lie algebra $\mathfrak{g} = A_N$ here separately.

1.1.1. *The Family of Curves*

A Riemann surface can be looked upon in various ways. Due to the Lie algebraic nature of our setup, we will often consider it as an algebraic curve in $\mathbf{P}^2$. On the other hand, we need the realization of the Riemann surface in terms of a complex manifold in order to study the holomorphic differentials on it. We will use the usual relation between these two realizations, see, for example, [6, 28].

An affine curve C in $\mathbf{C}^2$ is defined through a polynomial P as
$$C = \left\{(x, y) \in \mathbf{C}^2 \mid P(x, y) = 0\right\}. \tag{5}$$

The corresponding algebraic curve in $\mathbf{P}^2$ is given by adding the appropriate points at infinity. In terms of affine curves, a *family* of Riemann surfaces C is by definition

$$C = \{(x, y) \in \mathbf{C}^2 \mid P(x, y, u_1, \ldots, u_N) = 0\}, \tag{6}$$

where for generic values of the complex parameters $u_1, \ldots, u_N$ the genus of a curve in the family C is fixed to some number g. For special values however, the genus may decrease. Denoting by $\mathcal{M}$ the manifold $\mathbf{C}^N \setminus \Delta$ with the special values of the u_i removed, we can look upon the family as a fibration of Riemann surfaces over $\mathcal{M}$. The space $\mathcal{M}$ is called the moduli space of the family and the u_i are called the moduli.

Returning to the specific example under consideration, the family of Riemann surfaces C_{A_N} is given by

$$\begin{aligned} &C_{A_N} = \{(x, y) \in \mathbf{C}^2 \mid P(x, y, u_i) = y^2 - W(x, u_i)^2 + 4 = 0\}, \\ &W(x, u_i) = x^{N+1} + u_1 x^{N-1} + \cdots + u_{N-1}x + u_N. \end{aligned} \tag{7}$$

The curves in the family (7) are hyperelliptic, which makes their investigation relatively simple. Moreover, as a matter of fortunate coincidence in the type A_N case the rank N of the Lie algebra equals the genus g of the curves and these are the main reasons why it serves as the simplest example.

To get an idea of the structure of the moduli space $\mathcal{M}$, we mention that for all Lie algebras $\mathcal{M}$ is known to be a Kähler manifold with Kähler metric defined in terms of the prepotential. If we denote the prepotential, which we will introduce later, by $\mathcal{F}(a_1, \ldots, a_N)$ then the metric is given in terms of the coordinates a_i by

$$(\mathrm{d}s)^2 = \sum_{i,j} \mathrm{Im}\left(\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}\right) \mathrm{d}a_i \, \mathrm{d}\bar{a}_j. \tag{8}$$

In fact, manifolds with Kähler metric of the form (8) are known as rigid special Kähler manifolds [8]. The above relation is the reason for the name prepotential, since it serves as the basic building block for the Kähler potential.

1.1.2. *The Seiberg–Witten Differential and its Derivatives*

Moving on to the second ingredient in the construction of the prepotential $\mathcal{F}$, the Seiberg–Witten differential λ_{SW} is given by

$$\lambda_{\mathrm{SW}} = \log\left(\frac{y + W(x)}{2}\right) \mathrm{d}x. \tag{9}$$

The special property of λ_{SW} is that its derivatives with respect to the moduli are all holomorphic. We will first explain what it means to differentiate (see [33]).

We can regard the equation $P(x, y, u) = 0$ as defining implicitly the function $y(x, u_k)$. The derivative of y with respect to the moduli gives

$$\frac{\partial y}{\partial u_k} = -\frac{P_{u_k}}{P_y}, \tag{10}$$

where $P_{u_k} = \partial P/\partial u_k$. Using x as a local coordinate on the Riemann surface, we can extend this differentiation to differential forms $\omega = \phi\,\mathrm{d}x$ by

$$\frac{\partial}{\partial u_i}(\phi\,\mathrm{d}x) = \left(\frac{\partial\phi}{\partial u_i} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u_i}\right)\mathrm{d}x. \tag{11}$$

Alternatively, we can use y as a local coordinate and regard $P = 0$ as implicitly defining $x(y, u_i)$. We can calculate the derivative of $\omega = -\phi\frac{P_y}{P_x}\,\mathrm{d}y$ again and see if we get the same answer as in (11). In general this is the case only up to total differential forms $d(\frac{\phi P_{u_i}}{P_x})$ so that taking a derivative of differential forms with respect to the moduli is unique only in cohomology.

Now we come back to the derivatives of λ_{SW}, which we will show to be cohomologous to a set of linearly independent holomorphic differentials. Using x as a local coordinate, the derivatives of λ_{SW} are

$$\frac{\partial\lambda_{\mathrm{SW}}}{\partial u_k} = \frac{1}{y+W}\left(\frac{W}{y}+1\right)\frac{\partial W}{\partial u_k}\,\mathrm{d}x = x^{N-k}\frac{\mathrm{d}x}{y} \tag{12}$$

and it is well-known that these give a basis of the holomorphic differentials of the hyperelliptic Riemann surfaces in the family C_{A_N} (see also Appendix A).

1.1.3. *The Special Cycles*

For a generic simple Lie algebra the rank is smaller than the genus of the family of curves and a selection of $2N$ of the $2g$ cycles has to be made. For type A_N no such selection is necessary, and therefore we can immediately proceed to the definition of the prepotential.

1.1.4. *The Prepotential for Type A Lie Algebra*

We consider the period integrals of λ_{SW} over a set of canonical A cycles of the curve

$$a_i = \oint_{A_i}\lambda_{\mathrm{SW}}. \tag{13}$$

The a_i are moduli dependent and we can use their definition as a local change of variables on the moduli space. The Jacobian of this transformation is nonzero since

$$\frac{\partial a_i}{\partial u_j} = \oint_{A_i}\frac{\partial\lambda_{\mathrm{SW}}}{\partial u_j} \tag{14}$$

and a matrix built from the integrals of all holomorphic differentials over all A cycles is always nondegenerate (see, e.g., [16]). Here we have pulled differentiation with respect to moduli through the integration sign. The justification for this is that the integral does not depend on the particular cycle A_i but only on its homology class. This allows to choose a representative of this class which encircles the branch cuts widely, so that changing the position of a branch point slightly doesn't change the cycle. This then allows to differentiate with respect to the moduli under the integration sign.

One can calculate the derivatives of λ_{SW} with respect to the variables a_i by using the chain rule and we find that the $\partial\lambda_{\mathrm{SW}}/\partial a_i$ form a canonical set of holomorphic differential forms since

$$\oint_{A_j} \frac{\partial\lambda_{\mathrm{SW}}}{\partial a_i} = \frac{\partial a_j}{\partial a_i} = \delta_{ij}. \tag{15}$$

We introduce the integrals of λ_{SW} over the B cycles

$$b_j = \oint_{B_j} \lambda_{\mathrm{SW}}. \tag{16}$$

Differentiating the b_j with respect to the moduli we find

$$\frac{\partial b_j}{\partial a_i} = \oint_{B_j} \frac{\partial\lambda_{\mathrm{SW}}}{\partial a_i} = \Pi_{ij}, \tag{17}$$

where Π_{ij} is the period matrix of the Riemann surface, which according to Riemann's bilinear relations is symmetric (see, e.g., [16]). Therefore the b_j can be integrated locally and

$$b_j = \frac{\partial\mathcal{F}}{\partial a_j}. \tag{18}$$

The function $\mathcal{F}(a_1, \ldots, a_N)$ is called the prepotential.

DEFINITION 1. Associated to the type A_N Lie algebra, we define the family of curves C_{A_N} by (7) and a meromorphic differential λ_{SW} by (9). For a choice of canonical basis of cycles, the prepotential $\mathcal{F}(a_1, \ldots, a_N)$ is defined locally on the moduli space $\mathcal{M}$ by

$$\begin{aligned} a_i &= \oint_{A_i} \lambda_{\mathrm{SW}}, \\ b_j &= \oint_{B_j} \lambda_{\mathrm{SW}} = \frac{\partial\mathcal{F}}{\partial a_j}. \end{aligned} \tag{19}$$

1.1.5. *Duality*

Different choices of A and B cycles give different locally defined prepotentials. As we will see later, all of these prepotentials simultaneously satisfy the WDVV equations and in that respect the particular choice of cycles is immaterial to us.

The fact that $\mathcal{F}$ cannot be extended to a global function on the moduli space was known already to Seiberg and Witten [45] for the simplest case of A_1. Instead of $\mathcal{F}$ being a function on $\mathcal{M}$, one finds that (a_i, b_j) is a section of a flat bundle over $\mathcal{M}$ with structure group $\Gamma \subset \mathrm{Sp}(2N, \mathbf{Z}) \times \mathrm{U}(1)$. Let us elaborate on this flat bundle. Since the moduli space $\mathcal{M}$ is constructed as a submanifold of $\mathbf{C}^N$, it will in general have a nontrivial fundamental group. One can circle along the nontrivial homotopy elements and pick up a monodromy on the cycles of the Riemann surface. Typically, the homology element encircles a gap of complex codimension one in $\mathbf{C}^N$ in which one or more cycles of the Riemann surface get pinched. The monodromy is given by the Picard–Lefschetz theorem, which prescribes that the effect of a pinched cycle δ on another cycle ζ is

$$\zeta \to \zeta - (\zeta \circ \delta)\delta,$$

where $\circ$ denotes the intersection of the two. A small calculation shows that under these transformations a canonical homology basis remains canonical, in other words the monodromy operator is symplectic.

Together with the transformation on λ_{SW}, which may undergo a change in phase, this explains why the structure group of the bundle is a subgroup of $\mathrm{Sp}(2N, \mathbf{Z}) \times \mathrm{U}(1)$. The matrix of transformed variables

$$\frac{\partial \tilde{b}_j}{\partial \tilde{a}_i}$$

is therefore again symmetric and can be integrated locally to a new function $\tilde{\mathcal{F}}(\tilde{a}_1, \ldots, \tilde{a}_N)$. This leads to different functions $\mathcal{F}$ locally for each patch of $\mathcal{M}$. In the physics literature, a lot of effort is spent on determining the monodromies and the cycles for each patch. Our point of view however concerns only the WDVV equations, therefore we will put all choices of cycles (and all resulting prepotentials) in one equivalence class.

1.2. THE SEIBERG–WITTEN FAMILY OF RIEMANN SURFACES

In the original sense, dynamical integrable systems are Hamiltonian systems of particles with interactions whose equations of motion can be explicitly solved. A relatively simple approach towards solving such systems has had remarkable succes: this approach is called the isospectral deformation method or the method of Lax pairs [31]. The idea is that the equations of motion are equivalent to

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M] \tag{20}$$

in terms of a set of matrices L and M. If one can find such matrices, then as time passes by the matrix L changes by a conjugation into $U(t)LU(t)^{-1}$. The spectrum of L is therefore time independent and so are the functions $\mathrm{Tr}(L^k)$. If there are enough functionally independent traces which are in involution, then this proves the integrability of the system.

Here we consider a dynamical integrable system known as the periodic Toda chain, which involves particles with exponential nearest neighbour interaction. It was shown to be related to Seiberg–Witten theory in [17, 38], the analysis of this section will follow closely that of [38]. The periodic Toda chain is a system that can be associated to any simple Lie algebra $\mathfrak{g}$ with rank denoted by N. Associated with $\mathfrak{g}$ is a so-called affine Lie algebra $\mathfrak{g}^{(1)}$ which is the Lie algebra of Laurent polynomials in a variable z with coefficients in $\mathfrak{g}$. In terms of a root system R^1 for $\mathfrak{g}^{(1)}$, the Hamiltonian reads

$$H = \frac{1}{2}\sum_{i=1}^{N+1} p_i^2 - \sum_{\alpha\in R} \mathrm{e}^{-(\alpha,q)}, \tag{21}$$

where $q = q_0\alpha_0 + \cdots + q_N\alpha_N$ is a linear combination of the simple roots, α_0 being the affine root which is absent in the root system of $\mathfrak{g}$ (see also Figure 1). The center of mass decouples from this system, leaving only an N-dimensional phase space.

A certain Lax pair for this system (for more information see, e.g., [43]) involves the matrix L given by

$$\begin{aligned} L &= \rho(A), \\ A &= \sum_{i=1}^{N}(d_i\mathbf{h_i} + c_i\mathbf{e_i} + \mathbf{f_i}) + z\mathbf{e_0} + \frac{c_0}{z}\mathbf{f_0}. \end{aligned} \tag{22}$$

Here the $\mathbf{e_i}, \mathbf{f_i}$ are the simple root generators of $\mathfrak{g}$ corresponding to α_i and $-\alpha_i$ respectively. The $\mathbf{h_i}$ are the elements of the Cartan subalgebra and $\mathbf{e_0}$ is the affine root generator. The c_i, d_i are the so-called Flaschka coordinates on the phase space, with the property that a certain product

$$\mu = \prod_{i=0}^{N} c_i^{n_i} \tag{23}$$

is time independent, thus leaving again $2N$ phase space variables (see Figure 1 for the specific values of the n_i). Roughly speaking, the parameter μ plays the role of the energy scale in Seiberg–Witten theory. Finally, the parameter z that appears in L is called a spectral parameter and regardless of its value Equations (20) are equivalent to the equations of motion. Due to the spectral parameter one can consider the spectral equation for L

$$P(x, z) = \det[L(z) - x \cdot I] = 0 \tag{24}$$

as a family of affine curves, with the phase space variables acting as moduli. Such spectral curves are time independent since they describe the spectrum of L, and they play an essential role in solving the dynamical system.

Roughly speaking, the family of Riemann surfaces $C_{\mathfrak{g}}$ necessary for the Seiberg–Witten data is given by the spectral curve (24) for the periodic Toda chain, whose

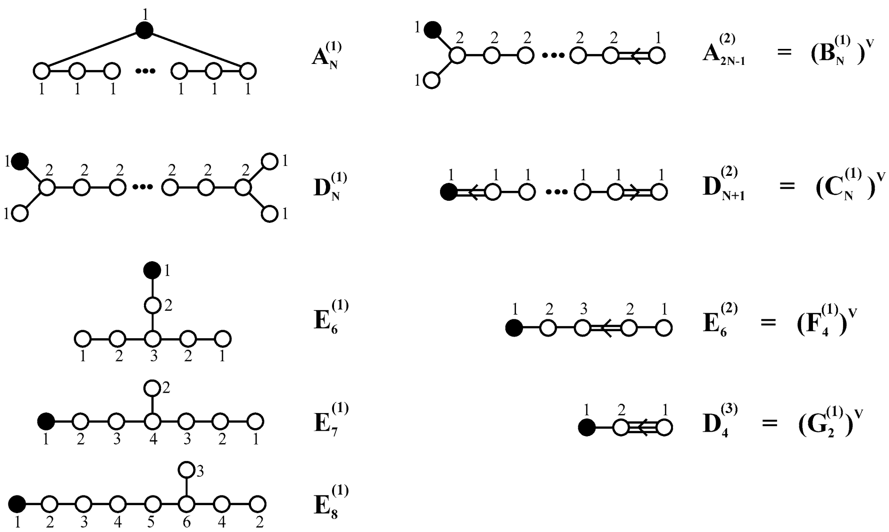

Figure 1. The left side contains the affine Dynkin diagrams for simply laced Lie algebras, the right side shows the twisted affine Dynkin diagrams for nonsimply laced Lie algebras. These are obtained by dividing out the automorphism of the Dynkin diagram of the corresponding simply laced algebra. The affine roots are coloured black and the numbers n_i which occur in the definition (23) of μ are indicated for each root.

Lax pair is defined in terms of the affine Lie algebra $\mathfrak{g}^{(1)}$ with the parameter z playing the role of the loop variable. Due to a physical requirement however, we should not consider the affine algebra $\mathfrak{g}^{(1)}$ but its dual $(\mathfrak{g}^{(1)})^\vee$ which is obtained by replacing roots with coroots. One of the consequences is that the degree of μ becomes the dual Coxeter number rather than the Coxeter number itself. For the simply laced algebras, the distinction is absent and one can continue directly. For the nonsimply laced algebras, $(\mathfrak{g}^{(1)})^\vee$ can be obtained from a simply laced algebra $\tilde{\mathfrak{g}}$ by dividing out an automorphism group π of $\tilde{\mathfrak{g}}$ [27]. In terms of the Dynkin diagram of $\tilde{\mathfrak{g}}$ the automorphism group consists either of reflections (A_{2N-1}, E_6, D_{N+1}) or rotations (D_4), see Figure 1. The spectral curve (24) is now given in terms of those roots of $\tilde{\mathfrak{g}}$ that are invariant under π. For instance, instead of the highest (long) root of $\mathfrak{g}$, one considers the highest (short) root of $\tilde{\mathfrak{g}}$ invariant under π.

DEFINITION 2. The family of Seiberg–Witten curves for four-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with gauge group $\mathfrak{g}$ is given by the spectral curves (24) associated with the periodic Toda chain for $(\mathfrak{g}^{(1)})^\vee$ and the smallest* representation ρ.

The Lax operator (23) can be assigned a natural degree by using the principal grading of the Lie algebra [27] and by assigning degrees $1, 2, h_{\mathfrak{g}}^\vee$ to d_i, c_i, z respectively, where $h_{\mathfrak{g}}^\vee$ is the dual Coxeter number of the Weyl group of $\mathfrak{g}$. This

* Later on, we will argue that the choice of representation is in fact irrelevant.

Table I. A list of the Coxeter numbers, dual Coxeter numbers and exponents of the simple Lie algebras.

Lie algebra $\mathfrak{g}$	$(\hat{g})^\vee$	$h_\mathfrak{g}$	$h_\mathfrak{g}^\vee$	Exponents
A_N	$A_N^{(1)}$	$N+1$	$N+1$	$1, 2, \ldots, N$
B_N	$A_{2N-1}^{(2)}$	$2N$	$2N-1$	$1, 3, \ldots, 2N-1$
C_N	$D_{N+1}^{(2)}$	$2N$	$N+1$	$1, 3, \ldots, 2N-1$
D_N	$D_N^{(1)}$	$2N-2$	$2N-2$	$1, 3, \ldots, 2N-3, N-1$
E_6	$E_6^{(1)}$	12	12	$1, 4, 5, 7, 8, 11$
E_7	$E_7^{(1)}$	18	18	$1, 5, 7, 9, 11, 13, 17$
E_8	$E_8^{(1)}$	30	30	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$E_6^{(2)}$	12	9	$1, 5, 7, 11$
G_2	$D_4^{(3)}$	6	4	$1, 5$

choice makes the Lax operator L homogeneous of degree 1. We denote this Lie algebraic degree of an object ϕ by $[\phi]_L$. The grading is respected by Equation (24) and since this equation is Weyl invariant the coefficients of $x^k z^l$ in $P(x, z)$ are polynomials (of a particular degree) in the Casimir invariants u_k of $\mathfrak{g}$. Since there are $N = \mathrm{rank}(\mathfrak{g})$ invariants, the spectral curve can be viewed as a family of curves depending on the N moduli u_k. Some Lie algebraic data is given in Table I.

The list of Seiberg–Witten curves is [38, 24]

$$
\begin{aligned}
&A_N \quad z + \frac{\mu}{z} + x^{N+1} + u_1 x^{N-1} + \cdots + u_N = 0,\\
&B_N \quad x\left(z + \frac{\mu}{z}\right) + x^{2N} + u_1 x^{2N-2} + u_2 x^{2N-4} + \cdots + u_N = 0,\\
&C_N \quad \left(z - \frac{\mu}{z}\right)^2 + x^2(x^{2N} + u_1 x^{2N-2} + u_2 x^{2N-4} + \cdots + u_N) = 0,\\
&D_N \quad x^2\left(z + \frac{\mu}{z}\right) + x^{2N} + u_1 x^{2N-2} + \cdots + u_{N-2}x^4 + u_N x^2 + u_{N-1}^2 = 0,\\
&E_6 \quad \frac{1}{2}x^3\left(z + \frac{\mu}{z} + u_6\right)^2 - q_1(x)\left(z + \frac{\mu}{z} + u_6\right) + q_2(x) = 0,\\
&F_4 \quad -8\left(z + \frac{\mu}{z}\right)^3 + s_1(x)\left(z + \frac{\mu}{z}\right)^2 + s_2(x)\left(z + \frac{\mu}{z}\right) + s_3(x) = 0,\\
&G_2 \quad 3\left(z - \frac{\mu}{z}\right)^2 + 2(2ux^2 - x^4)\left(z + \frac{\mu}{z}\right) - x^8 + 2ux^6 - u^2x^4 + vx^2 = 0.
\end{aligned}
\tag{25}
$$

Although the prepotential for G_2 depends only on two variables and therefore trivially satisfies the WDVV equations, we have included the Seiberg–Witten curves for G_2 in the list. The curves for E_7 and E_8 have been omitted because they are big and cumbersome. The expressions for $s_i(x), q_i(x)$ can be found in Appendix C. Note how for simply laced Lie algebras the z dependence is characterized by

$$P\left(z+\frac{\mu}{z}, x, u_1, \ldots, u_N\right) = P\left(z+\frac{\mu}{z}=0, x, u_1, \ldots, u_N + z + \frac{\mu}{z}\right). \quad (26)$$

As we will see, there is a direct relation between the A–D–E Seiberg–Witten curves for any representation on the one hand and the A–D–E Landau–Ginzburg superpotentials [46, 12] or miniversal deformations of isolated singularities [4] on the other. Equation (26) helps establish this relation, and the twisting procedure necessary to define the Seiberg–Witten curves for the nonsimply laced Lie algebras disturbes it. If it wasn't for this twisting, there would be a relation with the corresponding boundary singularities [49].

For the classical Lie algebras there exists a change of variables that gives the curves in the following standard hyperelliptic form (see also Section 1.1)

Type A_N: $y = z - \dfrac{\mu}{z}$

$$y^2 = \left(x^{N+1} + u_1 x^{N-1} + u_2 x^{N-2} + \cdots + u_N\right)^2 - 4\mu.$$

Type B_N: $y = x\left(z - \dfrac{\mu}{z}\right)$

$$y^2 = \left(x^{2N} + u_1 x^{2N-2} + u_2 x^{2N-4} + \cdots + u_N\right)^2 - 4\mu x^2.$$

Type C_N: $y = \dfrac{1}{x}\left(z^2 - \dfrac{\mu^2}{z^2}\right)$

$$y^2 = \left(x^{2N} + u_1 x^{2N-2} + \cdots + u_N\right)\left(x^2\left(x^{2N} + u_1 x^{2N-2} + \cdots + u_N\right) + 4\mu\right).$$

Type D_N: $y = x^2\left(z - \dfrac{\mu}{z}\right)$

$$y^2 = \left(x^{2N} + u_1 x^{2N-2} + \cdots + u_{N-2}x^4 + u_N x^2 + u_{N-1}^2\right)^2 - 4\mu x^4.$$

The curves for E_6, F_4 and G_2 however are *not* hyperelliptic. There is a simple test to see if a curve is hyperelliptic or not: taking tensor products $\omega_i \otimes \omega_j$ of holomorphic 1-forms, one obtains so-called quadratic holomorphic differentials (see Appendix A). The number of holomorphic 1-forms on any curve equals its genus g, but only for hyperelliptic curves their tensor products span a $(2g-1)$-dimensional subspace of the holomorphic quadratic differentials. For any other type of curves the span is bigger. For generic values of the moduli u_i all curves within one family have the same genus, and a list of these genera is given in Table II.

If Seiberg–Witten theory is to be related to the periodic Toda chain, the choice of representation (which does not appear in the definition of the Toda Hamiltonian) should be irrelevant. This issue is part of a bigger picture concerning the relation between the spectral curves and the Liouville torus promised by integrability

Table II. The genera for the Seiberg–Witten curves of ADE type.

$\mathfrak{g}$	A_N	B_N	C_N	D_N	E_6	F_4	G_2
g	N	$2N-1$	$2N$	$2N-1$	34	46	11

of the system. Adler and van Moerbeke [1] first raised this issue and suggested that the Liouville torus is a subvariety of the Jacobians of each of the spectral curves, regardless of the specific representation used. This program was subsequently worked out by a number of mathematicians such as Kanev, Mérindol, Beauville and especially Donagi [13]. As shown in [38], identifying the Liouville torus within the Jacobian of spectral curves is crucial for the understanding of Seiberg–Witten prepotentials. It provides us with a motivation for the definition of the Seiberg–Witten differential as well as the choice of cycles.

Decomposition of the Jacobians of Spectral Curves

There is a number of reasons why the Jacobians of spectral curves split into pieces. Although there is an infinite number of irreducible representations and therefore an infinite number of spectral curves, the Jacobian of each of these spectral curves contains a $2N$-dimensional Abelian subvariety called the distinghuished Prym. Following Donagi [13], we will describe here briefly how to find this Prym variety. Later we will see that the choice of Seiberg–Witten differential and the choice of cycles gives us a direct method of identifying the distinghuished Prym.

- A spectral curve $C_{\mathfrak{g},\rho}$ for an irreducible representation ρ splits into irreducible curves, with a component $C_{\mathfrak{g},\lambda}$ for each Weyl orbit of the weights of ρ. Here λ denotes the highest weight in the orbit. Still, there is an infinite number of Weyl orbits of weights.
- There is a *finite* collection of so-called parabolic curves $C_{\mathfrak{g},P}$ which are birationally equivalent to the $C_{\mathfrak{g},\lambda}$. A parabolic curve is parametrized by the parabolic subgroup W_P of the Weyl group W that stabilizes the weight λ. Each parabolic curve can be expressed in terms of a single much larger curve $\hat{C}$, called the cameral curve. This curve has a natural Weyl group action and the quotients $\hat{C}/W_P$ are the parabolic curves.
- The representation of the Weyl group on the (tangent bundle of the) Jacobian $Jac(\hat{C})$ splits into irreducible representations, thus splitting the Jacobian into Prym subvarieties. Consequently the Jacobian of a parabolic curve also splits. Among the factors of the Jacobian there is for any parabolic curve a piece coming from the reflection representation of the Weyl group. This representation has multiplicity one in the decomposition of the Weyl group action into irreducibles and therefore the Jacobian of all parabolic curves contains a unique subvariety called the distinghuished Prym. Donagi and Ksir [30] then

go on to calculate the dimension of the distinghuished Prym and find that it is twice the rank of $\mathfrak{g}$, which is precisely the dimension of the phase space of the periodic Toda chain.

Following [38], we describe in the next subsection a direct way of identifying the distinghuished Prym in terms of the Seiberg–Witten differential and a choice of cycles.

1.3. MORE ABOUT THE SEIBERG–WITTEN CURVES

It is convenient to view the spectral curve as a branched cover of the z sphere. For generic values of the moduli and z, the Lax operator $L = \rho(A)$ is the representation of a regular semisimple element A of the Lie algebra. This implies that a Cartan subalgebra of $\mathfrak{g}$ can be defined by means of the centralizer of L. Since all Cartan subalgebras are conjugate, L is conjugate to an element $\mathbf{v}(z) \cdot \mathbf{h} = \sum_{i=1}^{N} v_i(z) h_i$ in the standard Cartan subalgebra. The eigenvalues x of $L(z)$ are therefore given by $x = \mathbf{v}(z) \cdot \omega_k$ where the ω_k denote the weights of the representation. The spectral curve can now be denoted by

$$\prod_{k=1}^{\dim \rho} (x - \mathbf{v}(z) \cdot \omega_k) = 0. \tag{27}$$

If the weight space of one of the weights ω is more than one-dimensional, we remove all but one factor $x - \mathbf{v}(z) \cdot \omega$. Since the weights form a Weyl invariant subset of the root space, the spectral curve splits according to their Weyl orbits. Representations with only one Weyl orbit of weights are called miniscule. If the representation is not miniscule, we focus on the piece containing the highest weight λ. The resulting curve is called a parabolic cover C_P of $\mathbf{P}^1$ (on which z lives). The parabolic subgroup W_P of the Weyl group is identified as the stabilizer group of the highest weight λ.

We will now discuss the pieces of plumbing that connect the different sheets of the foliation, starting with the finite values of x. For generic values of z, we know that $L(z)$ is a regular semisimple element of $\mathfrak{g}$ conjugate to $\mathbf{v}(z) \cdot h$. By using the action of the Weyl group, we can take $\mathrm{Im}(\mathbf{v}) \cdot h$ to be in the fundamental Weyl chamber. Naturally $\mathbf{v}(z)$ is in general not a rational function, so Equation (27) can't be used effectively to study the Seiberg–Witten curve as an algebraic variety. The equation can help however with the identification of the branch points. Branch points of the curve occur for those z for which $\partial P/\partial x = 0$, in other words if two eigenvalues of $\rho(L)$ come together. This happens for example when $\mathbf{v}(z) \cdot \mathbf{h}$ hits a wall of the fundamental Weyl chamber, i.e. when $\mathbf{v}(z) \cdot \alpha_k = 0$ for some simple root α_k. If this is the case, then the weight ω_i and its reflection $\omega_j = \sigma_{\alpha_k} \omega_i$ give the same eigenvalue since

$$\mathbf{v} \cdot \omega_j = \mathbf{v} \cdot \sigma_{\alpha_k} \omega_i = \sigma_{\alpha_k} \mathbf{v} \cdot \omega_i = \mathbf{v} \cdot \omega_i. \tag{28}$$

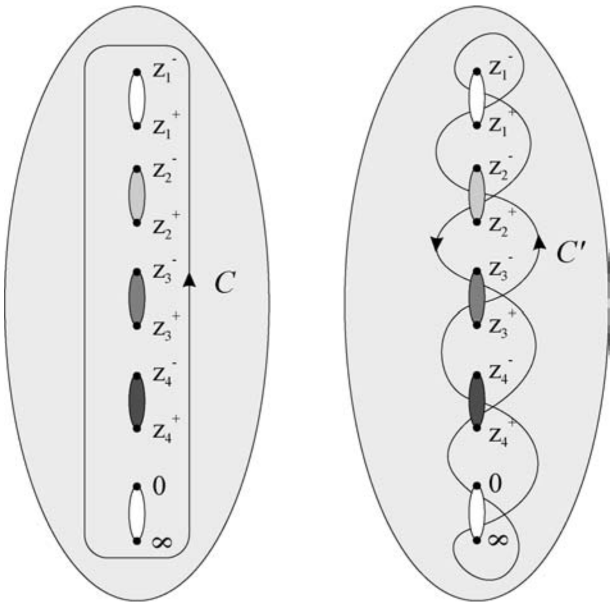

Figure 2. The z-sphere is given twice for a rank 4 simple Lie algebra together with the branch points: $z = 0, \infty$ and the $z_i^{\pm}$. The curve C in the left picture is trivial and is therefore closed when lifted to the Riemann surface. Since all branch cuts are of square root type the same is true for C' in the right picture.

From the expression (23) for the Lax operator one finds [38] that the curves exhibit⋆ a symmetry $z \to \mu/z$ where μ was defined in (23). Therefore the branch points come in pairs to form square root branch cuts. There can also be other branch points or even singular points for which $\mathbf{v}(z) \cdot \mathbf{h}$ does not hit a wall of the fundamental Weyl chamber, and these points are called accidental. There is a criterion in [13] for absense of accidental singularities, which is almost never satisfied. A pedestrian's test would be to calculate the genus directly from the curve and comparing it with the genus of the parabolic cover using the above description of the plumbing. If they disagree, there is an accidental singularity and the birational map between the spectral curve $C_{\mathfrak{g},\lambda}$ and the parabolic curve $C_{\mathfrak{g},P}$ is not an isomorphism. Usually the spectral curve contains more singularities than the parabolic curve, and one has to desingularize the spectral curve.

The preceding recipe tells us how to connect the sheets of the cover for finite values of z. For $z = 0$ and $z = \infty$ there is also a good description of what happens in terms of the root system of $\mathfrak{g}$. On the $\mathbf{P}^1$ base on which z takes its values we have given branch points $z_i^{\pm}$ corresponding to each simple root α_i of $\mathfrak{g}$, whose various lifts to the sheets of the foliation make up the branch cuts for finite values of x. Of course any lift of a closed curve C on the z sphere encircling all the branch

⋆ There exist other Lax operators from which the symmetry is more apparently visible [43]. One can of course also check the symmetry directly for the curves 25.

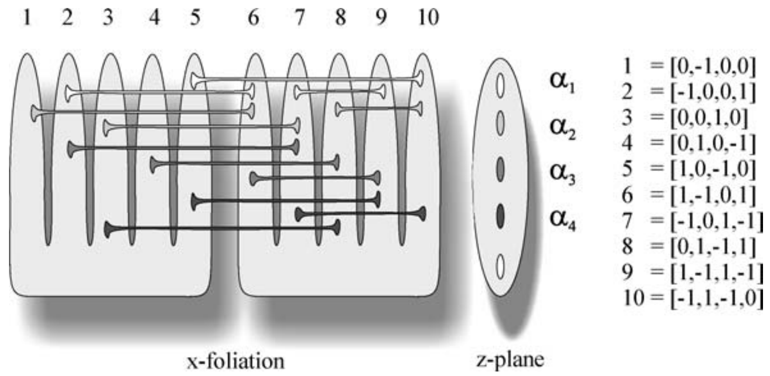

Figure 3. The Riemann surface for A_4 in the antisymmetric 10-dimensional representation. The genus of the curve is $g = 11$ and we have labeled the weights by their coefficients in terms of the fundamental weights. Picture taken from [38].

points must come back to the sheet it started on since we can deform C to a trivial curve on the z sphere. Because the branch cuts are of square root type, any lift of the closed curve C' in Figure 2 must also come back to the same sheet. Adding C and C' we see that any lift of a closed curve encircling all the z_i^- and $z = 0$ must also come back to the same sheet, so that encircling only $z = 0$ has the same effect as encircling all the branch points z_i^-. Therefore, starting on the sheet S_ω with weight ω and then making a circle around $z = 0$, one ends up on the sheet with weight $s\omega$ where s is the Coxeter element of the Weyl group of $\mathfrak{g}$. So the branch cut between $z = 0$ and $z = \infty$ connects all the sheets whose weights are in one orbit of the cyclic group $\mathbf{Z}_{h_\mathfrak{g}^\vee}$ generated by s. In Figure 3 we have given the example of Lie algebra A_4 in the 10-dimensional representation [38]. The weights are given for each sheet, and two sheets are connected above the α_i cut if and only if their weights are exchanged under σ_{α_i}. The Coxeter element s splits the weights into two groups of 5, which specifies how the sheets are connected at infinity. The genus of the curve is thus $g = 11$, which is the same answer as one gets from a direct calculation using the equation for the spectral curve given in (119). This shows that there are no accidental points.

As another example, we consider again A_4 but now in the 24-dimensional adjoint representation. This representation is not miniscule, because the weights split into two disjoint Weyl orbits: the roots of A_4 each of which has multiplicity 1, and the zero vector which has multiplicity 4. Consequently the Riemann surface splits into two parts, and we concentrate on the part containing the highest weight. The genus of the curve is $g = 25$, see Figure 4. Again accidental points are absent since a direct calculation of the genus using the spectral curve (119) gives the same result.

As a final example, consider Figure 5 where the curve for E_6 is depicted. The 27 weights of the smallest representation are labeled by the coefficients in the expansion in terms of fundamental weights [23], so $[1, 0, 0, 0, 0, 0]$ stands for the

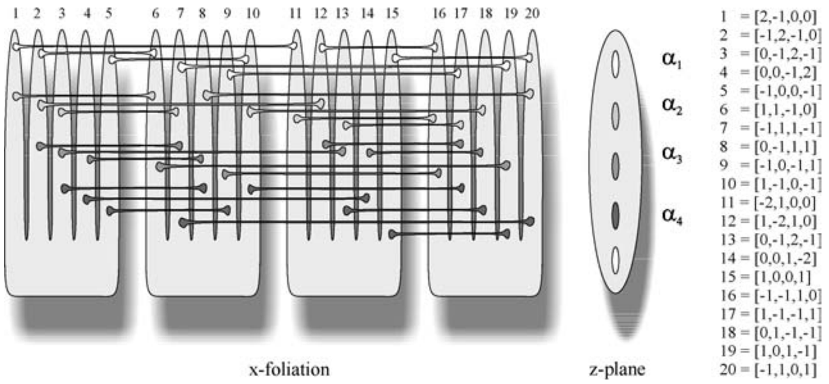

Figure 4. The Riemann surface for A_4 in the 24-dimensional adjoint representation. Since the spectral curve splits into two parts, we have concentrated on the part containing the highest weight. The genus of the curve is $g = 25$. As usual we have labeled the weights by their coefficients with respect to the fundamental weights [23].

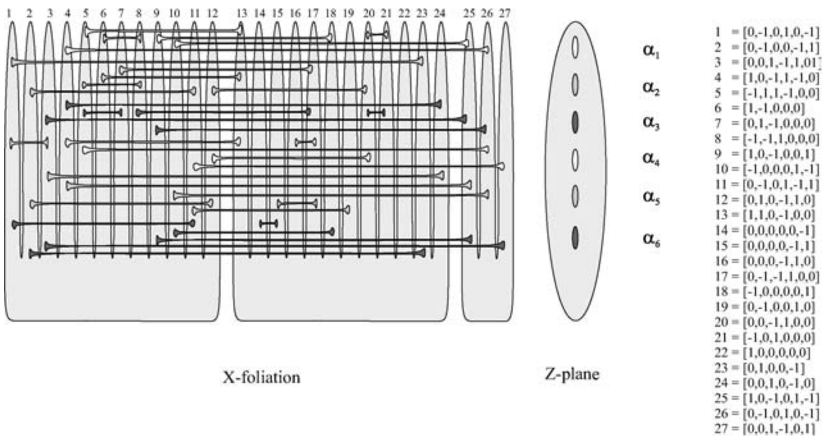

Figure 5. The Riemann surface for E_6 in the 27-dimensional representation. In the z-plane the branch cuts are depicted according to the six simple roots of E_6 (in standard notation) and the cut from $z = 0$ to ∞ is omitted. Above each root there are six pieces of plumbing connecting the three Coxeter orbits. The genus of the curve is $g = 34$.

fundamental dominant weight λ_1 which is also the highest weight for this representation. Each weight has multiplicity one, the 27 sheets are connected at $z = 0, \infty$ by the Coxeter element and the orbits have dimension 12, 12 and 3. Above each simple root there are 6 square root branch cuts, giving the Riemann surface genus 34 which is the same as the value found in Table II. This shows that there are no accidental points. For a more elaborate description of the E_6 curve see [32].

1.4. THE SEIBERG–WITTEN DIFFERENTIAL AND ITS DERIVATIVES

The second ingredient of the Seiberg–Witten data is a special meromorphic differential λ_{SW}.

DEFINITION 3. The Seiberg–Witten differential λ_{SW} is given by

$$\lambda_{\mathrm{SW}} = \log(z)\,\mathrm{d}x = \mathrm{d}(x\log(z)) - x\frac{\mathrm{d}z}{z} \simeq -x\frac{\mathrm{d}z}{z}, \tag{29}$$

where $\simeq$ denotes equality modulo total differentials.

Since we will mainly be interested in the period integrals of λ_{SW}, only its cohomology class is important. In the specific case of Lie algebra A_N, the differential form (9) reduces to (29) since

$$\begin{aligned}&\log(y+W)\,\mathrm{d}x - \log(2)\,\mathrm{d}x\\ &\quad= \log\left(z - \frac{\mu}{z} + z + \frac{\mu}{z}\right)\mathrm{d}x - \log(2)\,\mathrm{d}x = \log(z)\,\mathrm{d}x.\end{aligned}$$

In terms of the Toda system, λ_{SW} plays the role of the action differential $p\,\mathrm{d}q$ [38]. The main special property of λ_{SW} that we are interested in is that its derivatives with respect to the moduli parameters u_k give holomorphic differentials.

1.4.1. *Holomorphic Differentials*

In this section it is shown that the derivatives of λ_{SW} are holomorphic (see Appendix A for the construction of holomorphic differential forms on a general Riemann surface). Let a Riemann surface be given by an affine equation

$$P(x,z) = 0. \tag{30}$$

In particular, we are interested in the affine curves obtained from the Seiberg–Witten family (25). In order to make those curves affine, we multiply them with a monomial z^k of minimal degree necessary to make P polynomial. Viewing the curve as defining implicitly $x(z)$, the branch points are given by $P_x = 0$ and $P_z \neq 0$. Consider the differential form

$$\omega = \frac{\phi(x,z)\,\mathrm{d}z}{P_x} = -\frac{\phi(x,z)\,\mathrm{d}x}{P_z}. \tag{31}$$

Denoting the degree* of P by $[P] = d$, one finds that for ϕ a polynomial of degree smaller or equal to $d-3$, the differential form ω is nonsingular for all points except the singularities. In particular, ω is nonsingular in the branch points and due to the condition on the degree of ϕ also at infinity. If there are no singular points, a basis of holomorphic forms can be constructed from the ω as above, and their

* Note the difference between the notation $[\cdot]$ of degrees of polynomials in terms of their variables and the Lie algebraic degree $[\cdot]_L$.

number is $\frac{1}{2}(d-1)(d-2)$ which is in accordance with the degree-genus formula for nonsingular curves (see, e.g., [28]).

We will first check that the derivatives of λ_{SW} are holomorphic outside the singular points. Denote the Seiberg–Witten curves by

$$P(x,z,u) = \sum_{i=0}^{r} \left(z^2+\mu\right)^i z^{r-i} q_i(x,u) \tag{32}$$

and the degree of P is given by

$$[P] = [q_0] + r. \tag{33}$$

The derivatives of λ_{SW} with respect to the moduli are given by

$$\frac{\partial \lambda_{SW}}{\partial u_k} = -\left(\frac{\partial}{\partial u_k} x\right)\frac{\mathrm{d}z}{z} = \frac{P_{u_k}}{z}\frac{\mathrm{d}z}{P_x}. \tag{34}$$

It can be checked explicitly for every Seiberg–Witten curve in (25) that q_r is moduli independent. Hence P_{u_k}/z is a polynomial and taking into account that it is homogeneous in terms of the Lie algebraic grading, in which z has $h_{\mathfrak{g}}^{\vee}$ times the degree of x, we find that its polynomial degree is

$$\left[\frac{P_{u_k}}{z}\right] \leqslant [q_0] + r - 1 - [u_k]_L = d - 1 - [u_k]_L \tag{35}$$

and since the u_k are the Casimir invariants of the Lie algebra, their Lie algebraic degree is greater or equal to 2. Therefore the derivatives of λ_{SW} are holomorphic for nonsingular curves.

The restrictions that follow from the singularities are straightforward. In the affine coordinate patch (not at infinity) one can write $x(z)$ as a convergent power series if $P_x \neq 0$ using the implicit function theorem. For singular points, using the method of Puiseux expansions one can write $x(z)$ as a *fractional* power series instead, with a number of different series for each individual singularity [28]. The form ω should be nonsingular when each of these fractional power series is substituted into it. The singular points at infinity are treated in the same way after a change of variables on $\mathbf{P}^2$ to the relevant coordinate patch.

For the classical Lie algebras we have given the curves in standard hyperelliptic form in (27) from which it is easy to see that the derivatives of λ_{SW} are holomorphic. For E_6, F_4 and G_2 explicit computations were done using the computer algebra package Maple, which show that the derivatives of λ_{SW} are nonsingular not only in the branch points of the curve and at infinity but even in its singular points. We therefore arrive at the following proposition

PROPOSITION 4. *The derivatives of* λ_{SW} *with respect to the moduli are holomorphic for all simple Lie algebras.*

As an example, we consider the curve of G_2 of genus 11, given in (25). As explained in Appendix A, the 11 holomorphic forms are $\frac{\phi_k(x,z)\,\mathrm{d}z}{P_x}$ with ϕ_k given by:

$$\{\phi_k\} = \left\{x^6 z, x^5 z, x^4 z, x^3 z, x^2 z^2, x^2 z, x^2, xz, x, xz^2, z^2 - \mu\right\}. \tag{36}$$

On the other hand, the derivatives of λ_{SW} are given by

$$\begin{aligned} \frac{\partial \lambda_{\mathrm{SW}}}{\partial u} &= \frac{P_u\,\mathrm{d}z}{zP_x} = \left(2x^6 z - 2ux^4 z + 2x^2 z^2 + 2x^2\right)\frac{\mathrm{d}z}{P_x}, \\ \frac{\partial \lambda_{\mathrm{SW}}}{\partial v} &= \frac{P_v\,\mathrm{d}z}{zP_x} = x^2 z \frac{\mathrm{d}z}{P_x} \end{aligned} \tag{37}$$

and can be written as linear combinations of the holomorphic forms. As an aside, we note that the subspace spanned by the tensor products of holomorphic forms is 40-dimensional. This shows explicitly that the G_2 curve is not hyperelliptic,* since in that case the subspace should have dimension $2g - 1 = 21$.

1.5. THE SUBSET OF CYCLES

The third and final ingredient of the Seiberg–Witten data is a special subset of $2N$ independent cycles. For A_N in the fundamental representation one can take all cycles and no selection is necessary. The Seiberg–Witten curves of the other classical Lie algebras in the fundamental representation possess an involution which makes it easy to identify the special cycles. For the remaining cases there exists a more general method [38, 22] based on the action of the Weyl group on the curves. Here we treat only the simply laced Lie algebras, referring the reader to [22] for the nonsimply laced ones.

1.5.1. *The Special Cycles for the B, C, D Lie Algebras*

We regard the curves in their hyperelliptic form (27). Each of them has the involution $\sigma(x) = -x$. This helps us to identify the special cycles immediately: consider the curves as defining implicitly $y(x)$, and draw the branch cuts in the x-plane in such a way that the cuts come in pairs $K_i^{\pm}$ related by σ. We denote the counterclockwise contour around $K_i^{\pm}$ on the first sheet by $C_i^{\pm}$. The special A cycles are then defined by

$$A_i = C_i^{+} - C_i^{-}. \tag{38}$$

The special B cycles are the obvious ones going from K_i^{-} to K_i^{+} on the first sheet and back again on the second.

* Recently however, there have been suggestions [2] for an alternative hyperelliptic G_2 curve.

1.5.2. *Cycles for Simply Laced Lie Algebras*

Here we will discuss the more general method of identifying the special cycles, based on the action of the Weyl group on the family of curves as discussed in Section 1.2. This method is independent of the particular representation used to define the Seiberg–Witten curves and it solves the Adler–van Moerbeke problem of identifying the Liouville torus inside the Jacobian of the Toda spectral curve for any representation.

First we note that any lift A_i^ω of a counterclockwise closed contour C_i around only the α_i cut on the z sphere to the sheet S_ω labeled by the weight ω is a closed curve on that sheet. If $\alpha_i \cdot \omega = 0$ then A_i^ω is trivial, otherwise it's not. Since the branch cuts come in pairs, the cycle A_i^ω is homologous to $-A_i^{\sigma_{\alpha_i}\omega}$. By multiplying the contribution of each cycle by $\omega \cdot \alpha_i$ the contributions from the two different sheets add up since $\sigma_{\alpha_i}\omega \cdot \alpha_i = -\omega \cdot \alpha_i$. It is convenient to introduce the combinations

$$\hat{A}_i^\omega = \frac{1}{2}\big(A_i^\omega - A_i^{\sigma_{\alpha_i}\omega}\big). \tag{39}$$

These are the building blocks of the A cycles.

DEFINITION 5. The special A cycles are given by

$$A_i = N_{i,\rho}\sum_\omega (\omega \cdot \alpha_i) A_i^\omega = N_{i,\rho}\sum_\omega (\omega \cdot \alpha_i)\hat{A}_i^\omega, \tag{40}$$

where A_i^ω is the lift of C_i to the sheet characterized by the weight ω. The absolute value of $\omega \cdot \alpha_i$ determines how many times to wind around the cut and its sign determines in what direction to wind: a positive value means anti-clockwise and negative means clockwise. The normalisation factor $N_{i,\rho}$ is given by

$$N_{i,\rho} = \frac{1}{\sum_\omega |(\omega \cdot \alpha_i)|^2}. \tag{41}$$

There is an action of the Weyl group on the cycles, by letting it act on the weights of the representation. The image of a cycle A_i under a reflection σ_{α_j} is

$$\sigma_{\alpha_j} A_i = N_{i,\rho}\sum_\omega (\sigma_{\alpha_j}\omega \cdot \sigma_{\alpha_j}\alpha_i)\hat{A}_i^{\sigma_{\alpha_j}\omega} = N_{i,\rho}\sum_\omega (\omega \cdot \sigma_{\alpha_j}\alpha_i)\hat{A}_i^\omega \tag{42}$$

and therefore the cycles transform in the reflection representation of the Weyl group.

On the other hand, we need a set of B cycles. To define the cycle B_i, we draw a number of lifts B_i^ω to the sheet S_ω of the open curve D_i going from $z = 0$ to $z = z_i^-$ on the z sphere. The number and direction of the lifts is again determined by $\omega \cdot \alpha_i$: for example, $\omega \cdot \alpha_i = 1$ means one strand going up from $z = 0$ to $z = z_i^-$ on S_ω, while $\omega \cdot \alpha_i = -2$ means two strands going down from $z = z_i^-$ to $z = 0$ (see

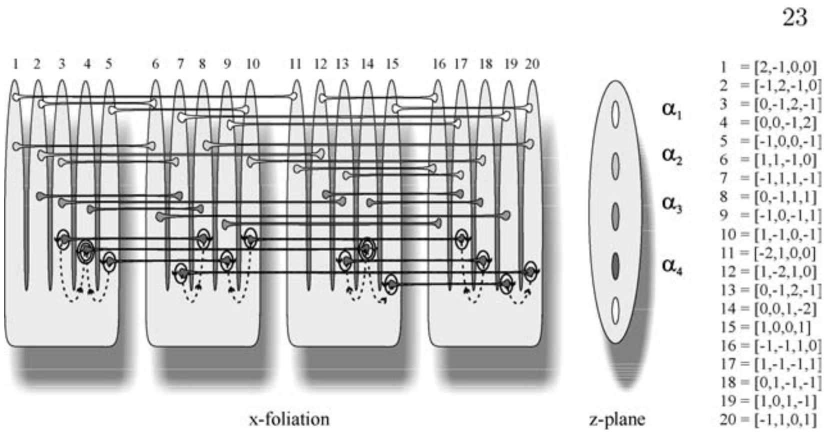

Figure 6. The Riemann surface for A_4 in the 24-dimensional adjoint representation, including the cycles above the fourth simple root. The fourth root and fourth weight are equal and their norm is two, thus causing two cycles of type A to encircle that branch cut and two strands to go up to the branch cut to form a special B cycle. The special A cycle is therefore obtained by adding all type A cycles in the picture and the special B cycle by adding the B type cycles, denoted by dotted lines.

Figure 6). Then for each Coxeter orbit $\mathcal{O}_k$ of sheets, we connect the strands through the cuts between $z = 0$ and $z = \infty$. To prove that this gives a closed curve B_i, we note that the number of strands going down to $z = 0$ on $\mathcal{O}_k$ equals the number of strands going up, since $\sum_{\omega\in\mathcal{O}_k}(\omega \cdot \alpha_i) = 0 \cdot \alpha_i = 0$. Therefore we can connect the strands on every Coxeter orbit, which shows that B_i is indeed closed. Again, it is convenient to introduce the linear combination

$$\hat{B}_i^{\omega} = \frac{1}{2}\big(B_i^{\omega} - B_i^{\sigma_{\alpha_i}\omega}\big). \tag{43}$$

We are now ready to define the special B cycles, see also Figure 6.

DEFINITION 6. The special B cycles are given by

$$B_i = N_{i,\rho}\sum_{\omega} \omega \cdot \alpha_i B_i^{\omega} = N_{i,\rho}\sum_{\omega} \omega \cdot \alpha_i \hat{B}_i^{\omega}, \tag{44}$$

where B_i^{ω} is the lift of the open curve from $z = 0$ to z_i^- to the sheet S_ω. The number $\omega \cdot \alpha_i$ decides on the direction and number of strands. The curve is then closed up through the cuts between $z = 0$ and $z = \infty$.

The normalisation factor $N_{i,\rho}$ is chosen in such a way that the period integrals of λ_{SW} are representation independent: on S_ω we have $\lambda_{\mathrm{SW}} = -\mathbf{v}(z) \cdot \omega\frac{\mathrm{d}z}{z}$ due to (29) and therefore

$$\oint_{A_i} \lambda_{\mathrm{SW}} \;=\; N_{i,\rho}\sum_{\omega} \omega \cdot \alpha_i \oint_{\hat{A}_i^{\omega}} \lambda_{\mathrm{SW}} = N_{i,\rho}\sum_{\omega\cdot\alpha_i>0} \omega \cdot \alpha_i \oint_{(A_i^{\omega}-A_i^{\sigma_{\alpha_i}\omega})} \lambda_{\mathrm{SW}}$$

$$
\begin{aligned}
&= N_{i,\rho} \sum_{\omega\cdot\alpha_i>0} \omega\cdot\alpha_i \oint_{C_i} (-\mathbf{v}(z)\cdot\omega + \mathbf{v}(z)\cdot\sigma_{\alpha_i}\omega)\frac{\mathrm{d}z}{z} \\
&= -N_{i,\rho} \sum_{\omega\cdot\alpha_i>0} \frac{(\omega\cdot\alpha_i)^2}{\alpha_i\cdot\alpha_i} \oint_{C_i} 2\mathbf{v}(z)\cdot\alpha_i\frac{\mathrm{d}z}{z} \\
&= -N_{i,\rho} \sum_{\omega} \frac{(\omega\cdot\alpha_i)^2}{\alpha_i\cdot\alpha_i} \oint_{C_i} \mathbf{v}(z)\cdot\alpha_i\frac{\mathrm{d}z}{z} \\
&= \frac{-1}{\alpha_i\cdot\alpha_i} \oint_{C_i} \mathbf{v}(z)\cdot\alpha_i\frac{\mathrm{d}z}{z} \qquad (45)
\end{aligned}
$$

which is indeed representation independent. A similar reasoning shows that the period integrals of λ_{SW} over the B cycles are independent of ρ. This is also true for the nonsimply laced Lie algebras [22].

To show that the A and B cycles just defined have the proper intersection numbers, we proceed as follows. It is clear that $A_i \circ A_j = B_i \circ B_j = 0$ and $A_i \circ B_j = -B_j \circ A_i = \gamma_i\delta_{ij}$ for some number γ_i. To determine the γ_i, we count the intersection on each sheet S_ω. Up to the normalisation, the number of strands from the B cycle that cross the closed curve from the A cycle is $|\omega\cdot\alpha_i|$ and there are also $|\omega\cdot\alpha_i|$ copies of the A cycles. Since the contribution to the intersection is always positive we find that the contribution from the sheet S_ω is $|(\omega\cdot\alpha_i)|^2$. Summing the contributions for all sheets and taking into account the normalisation we find

$$
A_i \circ B_j = (N_{i,\rho})^2 \sum_{\omega} |\omega\cdot\alpha_i|^2\delta_{ij} = \frac{1}{\sum_\omega(\omega\cdot\alpha_i)^2}\delta_{ij}. \qquad (46)
$$

Now consider the bilinear form $\sum_\omega(\omega\cdot x)(\omega\cdot y)$ on the linear space where the roots take their values. This bilinear form is invariant under the Weyl group and therefore we find that it equals a multiple of the Euclidean inner product on the root space. So in the end we find that the intersection matrix

$$
A_i \circ B_j \sim \frac{1}{\alpha_i\cdot\alpha_i}\delta_{ij} \sim \delta_{ij}. \qquad (47)
$$

1.6. DEFINITION OF THE PREPOTENTIAL

The Seiberg–Witten data has been introduced, consisting of the family of curves $C_{\mathfrak{g},\rho}$ (Definition 2), the Seiberg–Witten differential λ_{SW} (Definition 3) and a canonical subset of $2N$ cycles A_i and B_j with the usual intersection numbers (Definitions 5 and 6). We will need the following lemma

LEMMA 7. *There exists an additional set of cycles $A_{N+1},\ldots,A_g$ with the appropriate intersection numbers with the special cycles, and with the property that the period integrals of λ_{SW} around them are zero. In particular, this lemma implies that the special cycles are a subset of a canonical homology basis.*

Proof. For the classical Lie algebras, the additional A cycles are given by the C-invariant combinations

$$\tilde{A}_i = C_i^+ + C_-^-, \tag{48}$$

see also (38). Since the period integrals are independent of x, we find that C acts as the identity on them. On the other hand, the involution C sends $\lambda_{\rm SW}$ to $-\lambda_{\rm SW}$ and therefore we conclude

$$C\biggl(\oint_{\tilde{A}_i} \lambda_{\rm SW}\biggr) = -\oint_{\tilde{A}_i} \lambda_{\rm SW} = 0. \tag{49}$$

For the simply laced Lie algebras, there is a special cycle A_i for each root α_i. After our construction of additional cycles, the number of A cycles equals the number of branch cuts for finite values of x. These are too many cycles since the genus is the number of branch cuts minus the number of cuts necessary to connect the different Coxeter orbits of weights. Selecting a subset with g elements (including the special cycles) gives the set of A cycles promised by the lemma.

Take a simple root α_i. There are just as many branch cuts above α_i as there are weights ω with $\alpha_i \cdot \omega > 0$. Corresponding to α_i, take a weight ω_j so that $\omega_j \cdot \alpha_i > 0$. We introduce the subset Ω_{ij} of the set of weights Ω by

$$\Omega_{ij} = \bigl\{\omega' \in \Omega \mid (\omega' \cdot \alpha_i) > 0,\ \omega' \neq \omega_j\bigr\}. \tag{50}$$

For every $\omega_k \in \Omega_{ij}$ we define the cycle

$$A_i(\omega_k) = A_i^{\omega_j} - \frac{\alpha_i \cdot \omega_j}{\alpha_i \cdot \omega_k} A_i^{\omega_k}, \tag{51}$$

where $A_i^{\omega_j}$ is defined in Section 1.5. Together with the special cycle A_i this gives a number of cycles for each simple root α_i equal to the number of branch cuts for α_i. Obviously the intersection of all type A cycles among each other is zero.

We calculate the intersection numbers of the new $A_i(\omega_k)$ cycles with the special cycles B_l and find

$$\begin{aligned} A_i(\omega_k) \circ B_l &= N_{l,\rho}\biggl(A_i^{\omega_j} - \frac{\alpha_i \cdot \omega_j}{\alpha_i \cdot \omega_k} A_i^{\omega_k}\biggr) \circ \sum_{\omega} \alpha_i \cdot \omega B_l^{\omega} \\ &= N_{j,\rho}\biggl(\alpha_i \cdot \omega_j - \frac{\alpha_i \cdot \omega_j}{\alpha_i \cdot \omega_k} \alpha_i \cdot \omega_k\biggr)\delta_{il} = 0. \end{aligned} \tag{52}$$

Moreover, the period integrals of $\lambda_{\rm SW}$ over the cycles $A_i(\omega_k)$ are zero:

$$\begin{aligned} \oint_{A_i(\omega_k)} \lambda_{\rm SW} &= \oint_{\hat{A}_i^{\omega_j}} \lambda_{\rm SW} - \frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \omega_k} \oint_{\hat{A}_i^{\omega_k}} \lambda_{\rm SW} \\ &= -\frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \alpha_i} \oint_{C_i} \alpha_i \cdot \mathbf{v}(z) \frac{\mathrm{d}z}{z} + \frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \omega_k} \frac{\alpha_i \cdot \omega_k}{\alpha_i \cdot \alpha_i} \oint_{C_i} \alpha_i \cdot \mathbf{v}(z) \frac{\mathrm{d}z}{z} \\ &= 0. \end{aligned} \tag{53}$$

Repeating this construction of cycles for each simple root, we find that the number of A cycles now equals the number of branch cuts for finite values of x. As mentioned before, these are too many since some cycles are needed to connect the different Coxeter orbits of weights. We can always make a selection such that the cycles that are left out connect the Coxeter orbits. Thus we end up with a set of g canonical A cycles promised by the lemma. For the nonsimply laced Lie algebras the construction of cycles is similar. □

As an example, we consider A_4 in the 24-dimensional representation, see Figure 6. The special A cycle above the fourth simple root is given there. Denoting by $A_4^{\omega_3}$ the closed cycle on the sheet labeled by ω_3, the extra type A cycles become

$$\begin{aligned}
A_4(\omega_4) &= A_4^{\omega_3} - \frac{1}{2}A_4^{\omega_4},\\
A_4(\omega_5) &= A_4^{\omega_3} - A_4^{\omega_5},\\
A_4(\omega_7) &= A_4^{\omega_3} - A_4^{\omega_7},\\
A_4(\omega_{10}) &= A_4^{\omega_3} - A_4^{\omega_{10}},\\
A_4(\omega_{13}) &= A_4^{\omega_3} - A_4^{\omega_{13}},\\
A_4(\omega_{15}) &= A_4^{\omega_3} - A_4^{\omega_{15}}.
\end{aligned}$$

Including the special cycle drawn in Figure 6 this gives us 7 type A cycles above the branch cut labeled by α_4.

Using Lemma 7 in combination with Proposition 4, one can define the prepotential. First consider the new variables on the moduli space

$$a_i = \oint_{A_i} \lambda_{\mathrm{SW}}. \tag{54}$$

To prove that the change of variables from u_i to a_i is nonsingular, we note that the integrals of the holomorphic differentials $\partial\lambda_{\mathrm{SW}}/\partial u_i$ around the cycles A_{N+1}, $\dots$, A_g are zero. Since the N by g matrix

$$\oint_{A_j} \frac{\partial\lambda_{\mathrm{SW}}}{\partial u_i} \tag{55}$$

must have rank N, we conclude that the determinant of the Jacobi matrix for the change of variables from u_i to a_i is nonzero. This is similar to the situation for Lie algebra A_N, which we discussed in Section 1.1. We proceed to define the b_j by

$$b_j = \oint_{B_j} \lambda_{\mathrm{SW}} \tag{56}$$

and their moduli derivatives

$$\frac{\partial b_j}{\partial a_i} = \Pi_{ij}. \tag{57}$$

Since the special A cycles are a subset of a canonical homology basis and since the holomorphic forms $\partial\lambda_{\mathrm{SW}}/\partial a_i$ are canonical with respect to this basis

$$\oint_{A_i} \frac{\partial\lambda_{\mathrm{SW}}}{\partial a_j} = \delta_{ij}, \quad 1 \leqslant i \leqslant g,\ 1 \leqslant j \leqslant N \tag{58}$$

we find that Π_{ij} is an N by N submatrix of the g by g period matrix, and therefore symmetric. Due to this symmetry we can locally integrate the b_j and find the prepotential $\mathcal{F}$.

DEFINITION 8. The prepotential $\mathcal{F}(a_1, \ldots, a_N)$ is defined locally on the moduli space by

$$b_j = \frac{\partial\mathcal{F}}{\partial a_j}. \tag{59}$$

In Section 1.5 it was shown that a_i and b_j are representation independent, which shows that although we have chosen the smallest representation to define the family of curves we could in fact have used any irreducible representation and the prepotential is independent of this choice. For any irreducible representation, we have thus identified through the subset of N special cycles and the Seiberg–Witten differential a subvariety of the Jacobian. The cycles that make up this Abelian subvariety transform in the reflection representation of the Weyl group and its period matrix is given by Π_{ij}. This Abelian subvariety is the distinghuished Prym variety discussed earlier.

2. The WDVV Equations

The WDVV equations (1) are not suitable for the Seiberg–Witten context, since there is no special variable a_1 giving rise to a constant matrix F_1 of third-order derivatives. However, if a function satisfies (1) one can replace the inverse of F_1 by the inverse of any linear combination $K = \sum_m \alpha_m F_m$ of matrices of third-order derivatives, and the resulting equation also holds. This leads to the following definition

DEFINITION 9. The generalized WDVV equations are satisfied by a function $F(a_1, \ldots, a_N)$ provided

$$F_i K^{-1} F_j = F_i K^{-1} F_j, \quad i, j = 1, \ldots, N \tag{60}$$

for an arbitrary invertible linear combination $K = \sum_m \alpha_m F_m$.

Since the observation in [34] that the perturbative parts of the pure four-dimensional Seiberg–Witten prepotentials satisfy the generalized WDVV system for classical gauge groups, an extensive literature on the subject has formed. For a partial list, see [34–36, 39, 19, 21, 40, 47, 41, 37, 42]. Seiberg–Witten theory can be varied in several directions: the dimension of space-time can be altered, the matter content of the theory can be changed and the gauge group can be chosen. In many cases the corresponding prepotentials or their perturbative limits were shown to satisfy the generalized WDVV equations, although for example in the four-dimensional case of matter in the adjoint representation they do not [36]. In this paper, we restrict ourselves to the case of the pure four-dimensional theory with arbitrary gauge group [36, 26, 19].

Roughly speaking, the (generalized) WDVV equations express the conditions for the third-order derivatives of a function F to form the structure constants of an associative, commutative algebra with unit. More precisely, we introduce objects C_{ij}^k through the relation

$$F_{ijl} = \sum_k C_{ij}^k K_{kl}, \tag{61}$$

where K is the linear combination referred to in the definition of the generalized WDVV equations.

PROPOSITION 10. *The objects C_{ij}^k form the structure constants of an associative, commutative algebra with unit.*

Proof. One can rewrite the WDVV equations (60) as the commutation relations $[C_i, C_j] = 0$ for the matrices $(C_i)_j^k = C_{ij}^k$. This proves associativity of the algebra. Moreover, clearly $C_{ij}^k = C_{ji}^k$ so the algebra is commutative. Existence of a unit is shown through the observation that $\sum_m \alpha_m C_m = I$, the identity matrix. □

For this reason the WDVV equations are also called associativity equations in the literature. In this section we prove that the pure four-dimensional Seiberg–Witten prepotentials satisfy the WDVV system by first identifying the associative commutative algebra as an algebra of holomorphic differential forms on the Seiberg–Witten curve, and subsequently proving the relation (61) between the prepotential and the structure constants of the algebra. There are two methods in the literature for proving this relation, one based on Picard–Fuchs equations for period integrals [26] and the other based on a residue formula [36]. We explain and relate the two.

We note once more that in the choice of special cycles we have the freedom to make a discrete symplectic transformation, thus changing the definition of the prepotential. The upcoming proof is independent of this freedom, which therefore shows that the symplectic transformations form a symmetry group of the WDVV equations. This fact was also proven more directly in [9].

2.1. THE FAMILY OF ASSOCIATIVE ALGEBRAS

For each simple Lie algebra, we will construct a family of polynomial algebras over an ideal. Since they are polynomial, they are automatically commutative and associative. Furthermore, the choice of a unit element will eventually determine the precise linear combination K appearing in the WDVV equations (60). See Appendix B for more information about quotient rings of polynomial rings in several variables.

We denote the algebraic curves (25) by

$$P\left(x, z + \frac{\mu}{z}, u_i\right) = 0 \tag{62}$$

and consider them as the double cover of a torus

$$\begin{aligned} &P(x, w, u_i) = 0, \\ &z + \frac{\mu}{z} = w. \end{aligned} \tag{63}$$

The function P is now a polynomial in the two variables x, w. We introduce the ideal $I = \langle P, P_x \rangle$ in $\mathbf{C}[x, w]$. We will check that the $P_{a_i} = \partial P / \partial a_i$ span a subalgebra of $\mathbf{C}[x, w]/I$.

DEFINITION 11. For any simple Lie algebra $\mathfrak{g}$ whose family of Seiberg–Witten curves is given by $P(x, w) = 0$, the family of algebras $\mathcal{A}$ is defined by taking a subalgebra of $\mathbf{C}[x, w]/I$ where I is the ideal generated by P and P_x. These subalgebras are the ones generated by the P_{a_i} and are automatically associative and commutative as subalgebras of a polynomial algebra and they have a unit.

Since the Seiberg–Witten family of curves is formulated in terms of the u_i as moduli, we will give often give the algebras in terms of the $P_{u_j} = \sum_j \frac{\partial a_i}{\partial u_j} P_{a_i}$ which span the same subalgebra as the P_{a_i}. In terms of the u_j the structure constants are defined through

$$P_{u_i} P_{u_j} = \sum_{k,q} C_{ij}^k(\alpha, u) P_{u_k} \alpha_q P_{u_q} \quad \mod I, \tag{64}$$

where $\sum_q \alpha_q P_{u_q}$ serves as the unit element of the algebra. The dependence of the structure constants on the unit element and the coordinates u_j is emphasized in (64). Making the change of coordinates to the a_i we find that the structure constants transform as a (2, 1) tensor into

$$C_{ij}^k(\beta, a) = \sum_{l,m,n} \frac{\partial u_l}{\partial a_i} \frac{\partial u_m}{\partial a_j} C_{lm}^n(\alpha, u) \frac{\partial a_k}{\partial u_n} \tag{65}$$

and the new algebra unit is

$$\sum_p \beta_p P_{a_p} = \sum_{p,q} \alpha_q \frac{\partial a_p}{\partial u_q} P_{a_p}. \tag{66}$$

2.1.1. *Three Realizations of the Same Algebra*

It will be useful to have three realizations of the same algebra: to prove that it exists, we will use the polynomial multiplication

$$P_{u_i} P_{u_j} = \sum_{k,q} C_{ij}^k P_{u_k} \alpha_q P_{u_q} \quad \mod P,\ P_x. \tag{67}$$

To make the connection with flat coordinates and Landau–Ginzburg theory in Section 2.2, we will use an algebra of rational functions whose multiplication reads

$$w_{u_i} w_{u_j} = \sum_{k,q} C_{ij}^k w_{u_k} \alpha_q w_{u_q} + Q_{ij} w_x, \tag{68}$$

where $w_{u_i} = -P_{u_i}/P_w$ and w plays the role of a one-variable Landau–Ginzburg superpotential. Finally, to show in Subsection 2.5 that the algebraic function $w(x, u_i)$ is a superpotential for any choice of the representation, we will regard the algebra as an algebra of holomorphic forms [36]

$$\frac{\partial \lambda_{\mathrm{SW}}}{\partial u_i} \otimes \frac{\partial \lambda_{\mathrm{SW}}}{\partial u_j} = \sum_{k,q} C_{ij}^k \frac{\partial \lambda_{\mathrm{SW}}}{\partial u_k} \otimes \alpha_q \frac{\partial \lambda_{\mathrm{SW}}}{\partial u_q} + \frac{\bar{Q}_{ij}}{P_x} \frac{\mathrm{d}z}{z} \otimes \frac{\mathrm{d}z}{z}. \tag{69}$$

The elements of the left and right-hand sides of this equation are elements of Ω^2, the space of holomorphic quadratic differentials. See Appendix A for more details.

In the upcoming paragraphs, we will prove the existence of the algebras using the polynomial algebra (67). For the classical Lie algebras the generators of the algebra depend only on x (not on w), so that the ideal has one generator. For the exceptional Lie algebras, all depends on w as well as x and we have to use the technique of Groebner bases.

Type A_N

The family of Riemann surfaces (25) in this case is given by

$$P_{A_N}(x, w) = w + W(x, u_i) = 0, \tag{70}$$

where W is the A_N Landau–Ginzburg superpotential. The ideal $I \subset \mathbf{C}[x, w]$ is given by $I = \langle w + W, W_x \rangle$. Since $P_{u_i} = W_{u_i}$ depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$ where J is the ideal generated by W_x. The algebra therefore simplifies to

$$W_{u_i} W_{u_j} = \sum_{k,q=1}^{N} C_{ij}^k(\alpha, u) W_{u_k} \alpha_q W_{u_q} \quad \mod W_x \tag{71}$$

which is just the well-known type A Landau–Ginzburg algebra (or local algebra of the type A singularity). The algebra exists because W_x is a degree N polynomial in x generating the ideal J in $\mathbf{C}[x]$ and the $W_{u_i} = x^{N-i}$ form a basis of $\mathbf{C}[x]/J$. Since it is a polynomial algebra, it is automatically associative and commutative.

As an example, we give the structure constants $C_{ij}^k(\alpha_q = \delta_{q,4}, u)$ of A_4.

$$(C_3)_j^k = \begin{pmatrix} 0 & -\frac{3}{5}u_1 & -\frac{2}{5}u_2 & -\frac{1}{5}u_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{72}$$

and since $P_{u_i} = x^{N-i}$ we find that $C_k = C_3^{4-k}$ for $k = 1, \ldots 4$.

Type B_N

The family of Riemann surfaces in this case is given by

$$\begin{aligned} P_{B_N}(x, w) &= xw + W^{BC} = xw + x^{2N} + u_1 x^{2N-2} + u_2 x^{2N-4} + \cdots + u_N \\ &= 0, \end{aligned}$$

where W^{BC} is the type BC Landau–Ginzburg superpotential. The ideal I is given by $I = \langle xw + W^{BC}, w + W_x^{BC} \rangle$. Since $P_{u_i} = W_{u_i}^{BC}$ again depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$. To see what J should be, we calculate a Groebner basis of I in terms of a lexicographical order in which $w > x$ and we find that the only element in the basis not depending on w is $W^{BC} - xW_x^{BC}$. To see that this is an element of I we note that

$$W^{BC} - xW_x^{BC} = \left(xw + W^{BC}\right) - x\left(w + W_x^{BC}\right). \tag{73}$$

The quotient ring $\mathbf{C}[x]/J$ consists of polynomials up to degree $2N$ and since $W^{BC} - xW_x^{BC}$ contains only even degree terms the P_{u_k} span a subalgebra consisting of polynomials of even degree in $\mathbf{C}[x]/J$.

As an example, we give the structure constants $C_{ij}^k(\alpha_q = \delta_{q,3}, u)$ of the algebra for B_3 and note that this is not the Landau–Ginzburg algebra of type BC [49]. This is no coincidence: due to the twisting procedure from $\mathfrak{g}^{(1)}$ to $(\mathfrak{g}^{(1)})^\vee$ in the definition of the Seiberg–Witten family of curves, the relationship between the Seiberg–Witten algebra and the Landau–Ginzburg algebra is lost for the nonsimply laced Lie algebras. For the simply laced ones the two algebras are in fact the same, as we will see.

$$(C_1)_j^k = \begin{pmatrix} -\frac{1}{5}u_2 + \frac{9}{25}u_1^2 & \frac{1}{5}u_3 + \frac{3}{25}u_1 u_2 & -\frac{3}{25}u_1 u_3 \\ -\frac{3}{5}u_1 & -\frac{1}{5}u_2 & \frac{1}{5}u_3 \\ 1 & 0 & 0 \end{pmatrix}$$

and again $C_{k+1} = (C_k)^2$ for $k = 1, 2$.

Type C_N

The family of Riemann surfaces in this case is given by

$$P_{C_N}(x, w) = w^2 - 4\mu + x^2 W^{BC} = 0. \tag{74}$$

The ideal is given by $I = \langle w^2 - 4\mu + x^2 W^{BC}, 2xW^{BC} + x^2 W_x^{BC}\rangle$. Since P_{u_i} depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$ with J generated by $2xW^{BC} + x^2 W_x^{BC}$. The quotient ring $\mathbf{C}[x]/J$ consists of polynomials up to degree $2N + 1$ and since $2xW^{BC} + x^2 W_x^{BC}$ contains only odd degree terms the polynomials of even degree span a subalgebra in $\mathbf{C}[x]/J$. The dimension of this subalgebra however is $N + 1$, and we only have N polynomials P_{u_i}. The P_{u_i} which have degree in x greater or equal to 2 span yet a smaller subalgebra, because the lowest degree in x occurring in the ideal generator is degree 3.

Type D_N

The family of Riemann surfaces in this case is given by

$$x^2 w + x^{2N} + u_1 x^{2N-2} + \cdots + u_{N-2} x^4 + u_N x^2 + u_{N-1}^2 = 0. \tag{75}$$

The ideal I is given by $I = \langle x^2 w + W^D, 2xw + W_x^D\rangle$. Since $P_{u_i} = W_{u_i}^D$ depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$. To see what J should be, we calculate a Groebner basis of I in terms of a lexicographical order in which $w > x$ and we find that the only element in the basis not depending on w is $2W^D - xW_x^D$. This is the generator of J, and to see that this is an element of I we note that

$$2W^D - xW_x^D = 2\big(x^2 w + W^D\big) - x\big(2xw + W_x^D\big). \tag{76}$$

The quotient ring $\mathbf{C}[x]/J$ consists of polynomials up to degree $2N$ and since $2W^D - xW_x^D$ contains only even degree terms the P_{u_k} span a subalgebra consisting of polynomials of even degree in $\mathbf{C}[x]/J$. Note that this is precisely the Landau–Ginzburg algebra for type D_N.

Type E_6

Until now, the polynomial $P(x, w, u_i)$ did not contain terms mixing w with the moduli u_i. This allowed us to consider polynomial algebras in one variable. Any ideal is then generated by just one polynomial and calculations are done by dividing by this polynomial. For E_6 this is no longer the case. Since mixing does occur, we are forced to use the two-variable ring $\mathbf{C}[x, w]$ in which it is no longer guaranteed that an ideal is generated by one polynomial. Nevertheless one can construct a finite Groebner basis for the ideal in such a way that calculations in the quotient ring can be done by using a division algorithm to divide out the elements of the basis, see Appendix B.

An additional help in explicit computations is the grading that is present. As mentioned before, the principal grading of the affine Lie algebra causes the Riemann surfaces and Seiberg–Witten differential to be graded as well, and in turn the algebra that we are constructing is graded. Since the dependence on the Casimirs u_i is always polynomial, we can predict the dependence of the structure constants $C_{ij}^k(u)$ on the Casimirs. The only thing we have to calculate explicitly are the coefficients of the various terms, which are just numbers. For example, if we take $\alpha_q = \delta_{q,6}$ then the algebra becomes

$$P_{u_i} P_{u_j} = \sum_k C_{ij}^k(u) P_{u_k} P_{u_6} \mod I. \tag{77}$$

The degree of P is 27, the degrees of the Casimirs $u_1, \dots, u_6$ are respectively $2, 5, 6, 8, 9, 12$ and thus $C_{12}^3(u)$ for example has degree 11. The terms that constitute C_{12}^3 are therefore $u_1^3 u_2$, $u_2 u_3$ and $u_1 u_5$ and only their coefficients need to be determined.

Explicit computation of the Groebner basis (using a lexicographical term ordering) shows that the quotient algebra $\mathbf{C}[x, w]/I$ is 57-dimensional, and the algebra generated by the P_{u_i} is a 6-dimensional subalgebra. The fact that it's a closed subalgebra is by no means trivial. This subalgebra is precisely the Landau–Ginzburg algebra [15].

Type F_4

Again we have used Groebner bases theory together with the grading to determine the structure constants. Explicit computation of the Groebner basis (using a lexicographical term ordering) shows that the quotient algebra $\mathbf{C}[x, w]/I$ is 78-dimensional, and the algebra generated by the P_{u_i} is a nontrivial 4-dimensional subalgebra. Just like in the other nonsimply laced cases this is not the Landau–Ginzburg algebra of type F_4, which is given in [49]. The structure constants $C_{ij}^k(\alpha_q = \delta_{q,4}, u)$ are given by* [19]:

$$(C_1^{\mathrm{T}})_j^k = \begin{pmatrix} u_1\left(\frac{250}{243}u_1^4 - \frac{10}{9}u_1u_2 - \frac{7}{3}u_3\right) & -\frac{25}{54}u_1^3 + \frac{1}{4}u_2 & -\frac{5}{3}u_1^2 & 1 \\ \begin{array}{c}\frac{100}{81}u_1^4u_2 + \frac{140}{27}u_1^3u_3 - \\ \frac{2}{3}u_1u_2^2 - \frac{4}{3}u_1u_4 - 2u_2u_3\end{array} & u_1\left(-\frac{5}{9}u_1u_2 - \frac{7}{3}u_3\right) & -6u_3 - 2u_1u_2 & 0 \\ \begin{array}{c}-\frac{2}{9}u_1u_2u_3 - \frac{2}{3}u_3^2 + \\ \frac{100}{243}u_1^4u_3 - \frac{10}{27}u_1^2u_4\end{array} & \frac{1}{6}u_4 - \frac{5}{27}u_1^2u_3 & -\frac{2}{3}u_1u_3 & 0 \\ \begin{array}{c}\frac{10}{9}u_1^2u_3^2 - \frac{1}{3}u_1u_2u_4 - \\ u_3u_4 + \frac{50}{81}u_1^4u_4\end{array} & -\frac{1}{2}u_3^2 - \frac{5}{18}u_1^2u_4 & -u_1u_4 & 0 \end{pmatrix}.$$

* To get a better lay-out, we give the transpose matrices $(C_i^{\mathrm{T}})_j^k = (C_i)_k^j$.

$$(C_2^{\mathrm{T}})_j^k = \begin{pmatrix} -\frac{25}{54}u_1^3 + \frac{1}{4}u_2 & \frac{5}{24}u_1 & \frac{3}{4} & 0 \\ u_1\left(-\frac{5}{9}u_1u_2 - \frac{7}{3}u_3\right) & \frac{1}{4}u_2 & 0 & 1 \\ \frac{1}{6}u_4 - \frac{5}{27}u_1^2u_3 & \frac{1}{12}u_3 & 0 & 0 \\ -\frac{1}{2}u_3^2 - \frac{5}{18}u_1^2u_4 & \frac{1}{8}u_4 & 0 & 0 \end{pmatrix},$$

$$(C_3^{\mathrm{T}})_j^k = \begin{pmatrix} -\frac{5}{3}u_1^2 & \frac{3}{4} & 0 & 0 \\ -6u_3 - 2u_1u_2 & 0 & -6u_1 & 0 \\ -\frac{2}{3}u_1u_3 & 0 & 0 & 1 \\ -u_1u_4 & 0 & -\frac{9}{2}u_3 & 0 \end{pmatrix},$$

$$(C_4^{\mathrm{T}})_j^k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It can be checked explicitly that these are indeed the structure constants of an associative commutative algebra.

Type G_2

Finally, we arrive at the G_2 case. Although the WDVV equations are trivially satisfied, we give the family of associative algebras to show how it fits the general pattern. Since the Groebner basis of the ideal generated by P and P_x is not so big, we can give it explicitly:

$$\begin{aligned}
&\{288u^2x^9 + 192x^{13} - 384ux^{11} - 1728x^5\mu - 12u^2x^3v - 48u^2x\mu + \\
&+ 24uvx^5 + 576u\mu x^3 + 16u^4x^5 + 3xv^2 - 112x^7u^3 + 48x^7v, \\
&- 288x^{11} + 528ux^9 - 344u^2x^7 - 90vx^5 + 2592\mu x^3 - \\
&- 54vxw - 432xu\mu + 114x^5u^3 - 24x^3uv - 10u^4x^3 + 5u^2xv + 10u^3xw, \\
&- 162vw^2 + 30u^3w^2 + 288x^{12}u - 528x^{10}u^2 - \\
&- 54vx^8 + 354u^3x^8 - 124u^4x^6 + 144vux^6 - 2592ux^4\mu + \\
&+ 24u^2x^4v + 10u^5x^4 + 432u^2x^2\mu - 27x^2v^2 + 648v\mu - 120u^3\mu\}.
\end{aligned} \tag{78}$$

The resulting structure constants with $\alpha_q = \delta_{q,2}$ are

$$C_1 = \begin{pmatrix} -\frac{2}{3}u^2 & -\frac{2}{3}uv + 16\mu \\ 1 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lie algebra G_2 constitutes the only example where due to the twisting procedure the parameter μ appears explicitly in the structure constants, making it clear once more that the direct relation between this algebra and the Landau–Ginzburg algebra is lost.

After having introduced the prepotential $\mathcal{F}$ and family of algebras $\mathcal{A}$ separately, it remains to relate the two. There are two methods known in the literature of doing this. One method exploits the existence of flat coordinates in the Landau–Ginzburg context and interprets the relation (61) as Picard–Fuchs equations [26]. It has the drawback of not being directly applicable to the nonsimply laced Lie algebras, for which flat coordinates in general do not exist. The other method is more widely applicable and uses a residue formula [36, 29]. We will explain both methods in detail below.

2.2. THE GAUSS–MANIN CONNECTION, FLAT COORDINATES AND PICARD–FUCHS EQUATIONS

This section deals only with the simply laced Lie algebras, because there is a natural connection between the structure constants of the algebra and the definition of flat coordinates for them. The nonsimply laced algebras are discussed in the next section.

Given a family of subvarieties $X \subset \mathbf{P}^n$ fibered over a moduli space $\mathcal{M}$, there is a method (dating back to Picard, Fuchs and more recently Griffiths [18]) of obtaining a set of differential equations for period integrals differentiated with respect to the moduli. Such equations are called Picard–Fuchs equations. Let X be given by an affine equation $P(x_1, \ldots, x_n) = 0$ and take a closed cycle $\Xi \subset \mathbf{P}^n$ which encloses X. We consider integrals of the type

$$\zeta^{(l)} = \int_\Xi \frac{\phi}{P^l}\Omega, \tag{79}$$

where ϕ is a polynomial and Ω is the form on $\mathbf{P}^n$ given in local coordinates by

$$\Omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \tag{80}$$

in the coordinate patch where $x_{n+1} \neq 0$. Differentiating $\zeta^{(l)}$ with respect to the moduli, we get

$$\frac{\partial \zeta^{(l)}}{\partial u_j} = \int_\Xi \left(\frac{\frac{\partial \phi}{\partial u_j}}{P^l} \quad - l \frac{\phi \frac{\partial P}{\partial u_j}}{P^{l+1}} \right) \Omega. \tag{81}$$

The main idea is to perform a series of partial integrations to reduce the powers of P occurring in the denominator: each term of the form

$$\int_\Xi l \frac{\psi \frac{\partial P}{\partial x_k}}{P^{l+1}} \Omega \tag{82}$$

equals

$$\pm \int_\Xi \mathrm{d}\left(\frac{\psi}{P^l} \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_k} \wedge \cdots \wedge \mathrm{d}x_n \right) \mp \int_\Xi \frac{\frac{\partial \psi}{\partial u_k}}{P^l} \Omega. \tag{83}$$

So we have to divide $\phi\partial P/\partial u_j$ by the various $\partial P/\partial x_k$ in order to do those partial integrations. By chosing a term ordering and constructing a Groebner basis for the ideal I generated by the $\partial P/\partial x_k$ one makes sure that the order of division is irrelevant.

In case X is a miniversal deformation of a singularity of ADE type [4], $\mathbf{C}[x_1, \dots, x_n]/I$ is called the Jacobian (or local) ring and its dimension the Milnor number of the singularity. The $\partial P/\partial u_i$ generate a finite-dimensional subalgebra of the Jacobian ring and one can consider the integrals

$$\zeta_i^{(l)} = \int_\Xi \frac{\frac{\partial P}{\partial u_i}}{P^l}\Omega. \tag{84}$$

Using the algebra together with the partial integrations one gets the following set of differential equations

$$\frac{\partial \zeta_i^{(l)}}{\partial u_j} - lC_{ij}^k \zeta_k^{(l+1)} + \sum_n \Gamma_{ij}^{(n)k} \zeta_k^{(l-n)} = 0. \tag{85}$$

More formally this is the equation of a flat connection, called the Gauss–Manin connection, on a cohomology bundle over $\mathcal{M}$ of which the (integrands of) $\zeta_i^{(l)}$ are sections. One can check the integrability conditions of the connection separately for each power of P in the denominator, which lead to the following identities on the structure constants

$$\begin{aligned} &[C_i, C_j] = 0, \\ &\frac{\partial C_{ij}^k}{\partial u_l} = \frac{\partial C_{lj}^k}{\partial u_i}, \end{aligned} \tag{86}$$

where C_i is the matrix with coefficients C_{ij}^k. The first of these equations expresses the associativity of the algebra, and is automatically fulfilled. The second puts an integrability condition on the structure constants, so that $C_{ij}^k = \frac{\partial^2 T^k}{\partial u_i \partial u_j}$ for some set of functions T^k. Saito [44] then goes on to construct the flat coordinates, in terms of which the connection $\Gamma_{ij}^{(0)k}$ vanishes.

As an alternative to the integrals over Ξ, one can use the higher-dimensional analogue of Cauchy's residue theorem [6] to study period integrals over closed cycles on X itself, on which $P = 0$. We will consider the family of Riemann surfaces C as subvarieties of $\mathbf{P}^2$ fibered over $\mathcal{M}$. We have indicated in Section 1.1 how to differentiate cohomology elements with respect to the moduli. We consider the subring B of the full cohomology ring, generated by $\frac{\partial \lambda_{\mathrm{SW}}}{\partial u_i}$ and $\frac{\partial^2 \lambda_{\mathrm{SW}}}{\partial u_i \partial u_j}$ with $i \leqslant j$. It is not hard to see that these are all linearly independent and therefore constitute a basis $\{\chi_i\}$ of the subring B. We will need the following lemma

LEMMA 12 ([15, 25]). *For simply laced Lie algebras, the following Picard–Fuchs equations hold in the cohomology subring B*

$$\frac{\partial^2 \lambda_{\mathrm{SW}}}{\partial u_i \partial u_j} - \sum_k C_{ij}^k(u) \frac{\partial^2 \lambda_{\mathrm{SW}}}{\partial u_k \partial u_N} + \frac{\frac{\partial w}{\partial u_i \partial u_j} - \frac{\partial Q_{ij}}{\partial x}}{\sqrt{w^2 - 4\mu}} \, \mathrm{d}x = 0, \tag{87}$$

where the structure constants $C_{ij}^k(u)$ are defined through (64), *using $\alpha_q = \delta_{q,N}$.*

Proof. Using $w = z + \mu/z$, the first-order derivative of λ_{SW} equals

$$\frac{\partial \lambda_{\mathrm{SW}}}{\partial u_i} = \frac{\partial \log(z)}{\partial u_i} \, \mathrm{d}x = \frac{1}{z} \frac{\mathrm{d}z}{\mathrm{d}w} \frac{\partial w}{\partial u_i} \, \mathrm{d}x = \frac{\frac{\partial w}{\partial u_i}}{\sqrt{w^2 - 4\mu}} \, \mathrm{d}x \tag{88}$$

and therefore the second-order derivative equals

$$\frac{\partial^2 \lambda_{\mathrm{SW}}}{\partial u_i \partial u_j} = \frac{\frac{\partial w}{\partial u_i \partial u_j}}{\sqrt{w^2 - 4\mu}} \, \mathrm{d}x - \frac{w \frac{\partial w}{\partial u_i} \frac{\partial w}{\partial u_j}}{(\sqrt{w^2 - 4\mu})^3} \, \mathrm{d}x. \tag{89}$$

Substituting the algebra (68) with $\alpha_q = \delta_{q,N}$, performing a partial integration on the part containing Q_{ij} and noting that $\frac{\partial^2 w}{\partial u_k \partial u_N} = 0$ finishes the proof of the lemma. This last fact follows from (26), which ensures that

$$\frac{\partial w}{\partial u_N} = -\frac{P_{u_N}}{P_w} = -1. \qquad \square \tag{90}$$

Denoting the basis of B by $\{\chi_i\}$ we can reformulate the Picard–Fuchs equations as

$$\frac{\partial}{\partial u_i} \chi_j + \sum_k \Gamma_{ij}^k \chi_k = 0 \tag{91}$$

thus again defining a flat connection. Since

$$\frac{\partial^2 \lambda_{\mathrm{SW}}}{\partial u_k \partial u_N} = -\frac{w \frac{\partial w}{\partial u_k} \frac{\partial w}{\partial u_N}}{(\sqrt{w^2 - 4\mu})^3} \, \mathrm{d}x \tag{92}$$

we can split up the connection $\Gamma_{ij}^k = \Gamma_{ij}^{(1)k} + \Gamma_{ij}^{(3)k}$ according to the number of powers of the square roots occurring in the denominator. For the term with three powers, the flatness condition reduces to the two identities (86) on the structure constants of the algebra.

It turns out that the flat coordinates t_i from singularity theory precisely cause $\Gamma_{ij}^{(1)k}(t) = 0$, and therefore again get the interpretation of flat coordinates. So on the one hand, there is the moduli space of the singularity $\mathbb{C}^N \setminus \Sigma$ with the discriminant removed (points for which the deformation of the singularity is still singular) and on the other hand there's the moduli space of the Seiberg–Witten curves $\mathbb{C}^N \setminus \Delta$ with the points removed that correspond to a singular curve. On both these (different!) varieties there are the flat coordinates t_i which coincide on $\mathbb{C}^N \backslash (\Sigma \cup \Delta)$. In terms of the flat coordinates, the Picard–Fuchs equations read

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_k C_{ij}^k(t) \frac{\partial^2}{\partial t_k \partial t_N} \right) \oint_\Gamma \lambda_{\mathrm{SW}} = 0, \tag{93}$$

where we integrated over an arbitrary cycle Γ. We can now prove the following theorem

THEOREM 13 ([26]). *For simply laced Lie algebras, the prepotential $\mathcal{F}$ and structure constants $C_{ij}^k(\beta, a)$ are related by*

$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} = \sum_{l,m=1}^{N} C_{ij}^l(\beta, a) \beta_m \mathcal{F}_{klm}. \tag{94}$$

Therefore the prepotential $\mathcal{F}(a_1, \ldots, a_N)$ satisfies the WDVV system.

Proof. Changing the coordinates from t_i to a_i in Equation (93), we find

$$\begin{aligned} &\sum_{i,j} \left(\frac{\partial a_i}{\partial t_r} \frac{\partial a_j}{\partial t_s} - \sum_t^N C_{rs}^t(t) \frac{\partial a_i}{\partial t_t} \frac{\partial a_j}{\partial t_N} \right) \frac{\partial^2}{\partial a_i \partial a_j} \oint_\Gamma \lambda_{\mathrm{SW}} + \\ &\quad + \sum_i \left(\frac{\partial^2 a_i}{\partial t_r \partial t_s} - \sum_t C_{rs}^t(t) \frac{\partial^2 a_i}{\partial t_t \partial t_N} \right) \oint_\Gamma \frac{\partial \lambda_{\mathrm{SW}}}{\partial a_i} = 0. \end{aligned} \tag{95}$$

Ordinarily, the two halves of this equation need not vanish separately. However, since

$$a_i = \oint_{A_i} \lambda_{\mathrm{SW}} \tag{96}$$

we find that a_i satisfies (93) and therefore the second half of Equation (95) vanishes. Taking the cycle $\Gamma = B_k$ and defining $\beta_m = \partial a_m / \partial t_N$, the first half can be rewritten as

$$\begin{aligned} \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} &= \sum_{l,m,r,s,t} \left(\frac{\partial t_r}{\partial a_i} \frac{\partial t_s}{\partial a_j} C_{rs}^t(t) \frac{\partial a_l}{\partial t_t} \right) \left(\frac{\partial a_m}{\partial t_N} \right) \frac{\partial^3 \mathcal{F}}{\partial a_k \partial a_l \partial a_m} \\ &= \sum_{l,m} C_{ij}^k(\beta, a) \beta_m \frac{\partial^3 \mathcal{F}}{\partial a_k \partial a_l \partial a_m}. \qquad \square \end{aligned} \tag{97}$$

2.3. PICARD–FUCHS EQUATIONS FOR THE NONSIMPLY LACED ALGEBRAS

For simply laced Lie algebras, the family of associative algebras $\mathcal{A}$ is precisely the Landau–Ginzburg algebra. This gives us the direct connection between the flat coordinates and the algebra, expressed in equation (93). For nonsimply laced Lie algebras, the associative algebras are not the Landau–Ginzburg algebras [49]. For example, there is only one Landau–Ginzburg algebra of type BC whereas there are two separate algebras in the Seiberg–Witten context. Nevertheless we can show that for the classical B and C algebras, a similar relation to (94) still holds, now connecting the Landau–Ginzburg flat coordinates to the Seiberg–Witten algebras. This allows us to continue the proof.

PROPOSITION 14 ([20]). *For the nonsimply laced Lie algebras of type B_N and C_N the relation* (94) *holds. Therefore the corresponding prepotentials satisfy the WDVV equations.*

Proof. We first define the BC Landau–Ginzburg algebra. In terms of its flat coordinates the multiplication structure reads

$$\begin{aligned}
&\phi_i(t) = -\frac{\partial W^{BC}}{\partial t_i},\\
&\phi_i(t)\phi_j(t) = \hat{C}_{ij}^k(t)\phi_k(t) + Q_{ij}W_x^{BC}.
\end{aligned} \tag{98}$$

Furtermore, it is not hard to show that Q_{ij} is divisable by x and we express Q_{ij} as a linear combination

$$Q_{ij} = x\sum_k D_{ij}^k(t)\phi_k. \tag{99}$$

In [26] the following set of Picard–Fuchs equations was obtained

$$\begin{aligned}
&\Bigg(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_{k=1}^N \hat{C}_{ij}^k(t)\frac{\partial^2}{\partial t_k \partial t_N} - \sum_{k=1}^N\sum_{n=1}^N \frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee} D_{ij}^k \frac{\partial^2}{\partial t_k \partial t_n} + \\
&\quad + \sum_{k=1}^N D_{ij}^k \frac{1}{h_{\mathfrak{g}}^\vee}(1-d_k)\frac{\partial}{\partial t_k}\Bigg)\oint_\Gamma \lambda_{\mathrm{SW}} = 0,
\end{aligned} \tag{100}$$

where the $\hat{C}_{ij}^k(t)$ are the structure constants of the BC Landau–Ginzburg theory, the d_n are the degrees of the Lie algebra and $\epsilon = 1$ (-1) for B_N (C_N). Making a change of coordinates to the a_i just like we did for simply laced algebras and using the fact that the a_i satisfy (100), we get

$$\begin{aligned}
&\sum_{i,j}\Bigg[\frac{\partial a_i}{\partial t_r}\frac{\partial a_j}{\partial t_s} - \sum_t \hat{C}_{rs}^t(t)\frac{\partial a_i}{\partial t_t}\frac{\partial a_j}{\partial t_N} - \\
&\quad - \sum_{k,n} D_{rs}^t \frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee}\frac{\partial a_i}{\partial t_n}\frac{\partial a_j}{\partial t_t}\Bigg]\frac{\partial^2}{\partial a_i \partial a_j}\oint_\Gamma \lambda_{\mathrm{SW}} = 0.
\end{aligned} \tag{101}$$

Unfortunately, this is not in the form of (95) and we cannot continue as before. We do see however that the fourth term in (100) does not contribute to (101). So we go back to the first three terms of (100) and with the benefit of hindsight we introduce new objects $\gamma_{ij}^k(t)$ as

$$\hat{C}_{ij}^k(t) = \gamma_{ij}^k(t) - \sum_{k,q} D_{ij}^l \frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee} \gamma_{nl}^k. \tag{102}$$

Substituting this into the first three terms of (100) one obtains

$$\begin{aligned}&\left(\frac{\partial^2}{\partial t_i \partial t_j} - \gamma_{ij}^k(t) \frac{\partial^2}{\partial t_k \partial t_N}\right) \oint_\Gamma \lambda_{\mathrm{SW}} + \\ &\quad + \sum_{l,n} D_{ij}^l \frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee} \left(\frac{\partial^2}{\partial t_l \partial t_n} - \gamma_{ln}^k(t) \frac{\partial^2}{\partial t_k \partial t_N}\right) \oint_\Gamma \lambda_{\mathrm{SW}}.\end{aligned} \tag{103}$$

This expression consists of two parts. Making the change of coordinates to the a variables gives two equations that have to vanish separately, one for each of the two parts of (103). Each of these equations then boils down to the relation

$$\mathcal{F}_{ijk} = \gamma_{ij}^l(a) \frac{\partial a_m}{\partial t_N} \mathcal{F}_{klm} \tag{104}$$

and proves that the WDVV equations hold if the $\gamma_{ij}^k(t)$ are well-defined and if they are the structure constants of some associative algebra. This is the subject of the following lemma.

LEMMA 15. *The objects $\gamma_{ij}^k(t)$ defined through relation* (102) *exist and they are precisely the structure constants $C_{ij}^k(t)$ of the Seiberg–Witten algebra in terms of the coordinates t_i. The Seiberg–Witten algebras were defined separately for B_N and C_N in Section* 2.1.

Proof. We will restrict ourselves to the B_N case here, the proof for C_N is very similar. We will rewrite (98) in such a way that it becomes of the form

$$\phi_i(t)\phi_j(t) = \sum_{k=1}^r \gamma_{ij}^k(t)\phi_k(t) + R_{ij}\left[x\partial_x W^{BC} - W^{BC}\right]. \tag{105}$$

As a first step, we use (102):

$$\begin{aligned}\phi_i\phi_j &= \left[\hat{C}_i \cdot \overrightarrow{\phi} + D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j \\ &= \left[\left(\gamma_i - D_i \cdot \sum_{n=1}^r \frac{2nt_n}{2r-1}\gamma_n\right) \cdot \overrightarrow{\phi} + D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j \\ &= \left[\gamma_i \cdot \overrightarrow{\phi} - D_i \cdot \sum_{n=1}^r \frac{2nt_n}{2r-1}\gamma_n \cdot \overrightarrow{\phi} + D_i \cdot \overrightarrow{\phi}\, x W^{BC}\right]_j.\end{aligned} \tag{106}$$

The notation $\overrightarrow{\phi}$ stands for the vector with components ϕ_k and we use a matrix notation for the structure constants. There are two things about this equation that we would like to change: the first thing is that we want the structure constants to be defined by the first term, so we would like the middle term to vanish. The second thing is that we want the third term to contain the generator $W^{BC} - xW_x^{BC}$ of the ideal J. As a first step towards resolving both these problems, we will take part of the third term and cancel it with the middle term. To do this, we will use the following equation which expresses that W^{BC} is homogeneous in the Lie algebraic grading

$$xW_x^{BC} + \sum_n 2nt_n \frac{\partial W^{BC}}{\partial t_n} = 2NW^{BC}. \tag{107}$$

Using this equation we can cancel the middle term of (106) with part of the third term at the expense of introducing new terms which then have to be canceled etcetera. This recursive process will end however and yield the desired result. First we split up the third term of (106) as follows

$$\begin{aligned}
&\left[D_i \cdot \overrightarrow{\phi}\, xW_x^{BC}\right]_j \\
&\quad = \left[-\frac{1}{2N-1} D_i \cdot \overrightarrow{\phi}\, xW_x^{BC} + \left(1 + \frac{1}{2N-1}\right) D_i \cdot \overrightarrow{\phi}\, xW_x^{BC}\right]_j \\
&\quad = \left[-\frac{D_i}{2N-1} \cdot \overrightarrow{\phi}\left(2NW^{BC} - \sum_{n=1}^{N} 2nt_n\phi_n\right) + \frac{2ND_i}{2N-1} \cdot \overrightarrow{\phi}\, xW_x^{BC}\right]_j.
\end{aligned} \tag{108}$$

Using the Landau–Ginzburg algebra (98) we rewrite the products of ϕ occurring here, thus rewriting (106) as

$$\begin{aligned}
\phi_i\phi_j = {} & \left[\gamma_i \cdot \overrightarrow{\phi} - \frac{D_i}{2N-1} \cdot \sum_n 2nt_n(\gamma_n \cdot \overrightarrow{\phi} - \hat{C}_n \cdot \overrightarrow{\phi}) - \right. \\
& \left. - \frac{D_i}{2N-1} \cdot \sum_n 2nt_n D_n \cdot \overrightarrow{\phi}\, xW_x^{BC}\right]_j + \frac{2ND_i}{2N-1} \cdot \left[xW_x^{BC} - W_{BC}\right]_j.
\end{aligned} \tag{109}$$

We now use (102) again to rewrite the second term in the first line. Then we find

$$\begin{aligned}
\phi_i\phi_j = {} & \left[\gamma_i \cdot \overrightarrow{\phi} - \frac{D_i}{2N-1} \times \right. \\
& \left. \times \sum_n 2nt_n D_n \left(-\sum_m \frac{2mt_m}{2N-1}\gamma_m \cdot \overrightarrow{\phi} + \overrightarrow{\phi}\, xW_x^{BC}\right)\right]_j + \\
& + \frac{2ND_i}{2N-1} \cdot \left[xW_x^{BC} - W_{BC}\right]_j.
\end{aligned} \tag{110}$$

Note that by cancelling one term, we automatically calculate modulo $xW_x^{BC} - W_{BC}$. We can now repeat the whole process on the term between round brackets in (110). This is a recursive process and each step will introduce an extra factor of $\sum_n 2nt_n D_n$. To see that the recursive process stops, we will prove that this is a nilpotent matrix.

The degree of Q_{ij} is $[Q_{ij}] = 2N + 1 - 2(i + j)$. Dividing by x the degree becomes $2N - 2(i + j)$. Since $[\phi_k] = 2N - 2k$ one cannot divide Q_{ij}/x by ϕ_k for $j \geqslant k$ and therefore the matrices D_i defined in (99) are strictly lower triangular. Any sum of such matrices is also lower triangular and thus nilpotent. □

2.4. THE RESIDUE FORMULA

An alternative to the approach of Picard–Fuchs equations is given by the residue formula [36], whose origins lie in the theory of integrable systems [29].

A common way of proving Riemann's bilinear relations on a Riemann surface C is to cut open the surface to obtain a fundamental $4g$-sided polygon Ψ and use Cauchy's residue theorem on Ψ. We will use the same method to obtain a residue formula for the third-order derivatives of $\mathcal{F}$.

We start by rewriting $\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k}$ and with $\omega_i = \frac{\partial \lambda_{\mathrm{SW}}}{\partial a_i}$ we find

$$\begin{aligned}
\mathcal{F}_{ijk} &= \frac{\partial}{\partial a_k}\mathcal{F}_{ij} = \frac{\partial}{\partial a_k}\sum_{m=1}^{g}\oint_{A_m}\omega_i\oint_{B_m}\omega_j \\
&= \sum_m \oint_{A_m}\frac{\partial \omega_i}{\partial a_k}\oint_{B_m}\omega_j + \sum_m \oint_{A_m}\omega_i \oint_{B_m}\frac{\partial \omega_j}{\partial a_k} \\
&= 0 + \sum_m \oint_{A_m}\omega_i \oint_{B_m}\frac{\partial \omega_j}{\partial a_k} \\
&= \sum_m \left(\oint_{A_m}\omega_i\oint_{B_m}\frac{\partial \omega_j}{\partial a_k} - \oint_{B_m}\omega_i\oint_{A_m}\frac{\partial \omega_j}{\partial a_k}\right) \\
&= \sum \operatorname{res}\left(\chi_i \frac{\partial \omega_j}{\partial a_k}\right). \qquad (111)
\end{aligned}$$

In the last line we have cut open the Riemann surface and the residues are taken inside the fundamental polygon Ψ. Since Ψ is simply connected, the holomorphic differential ω_i is exact and we denote $\omega_i = \mathrm{d}\chi_i$. In the derivation of this formula we have used

$$\oint_{A_i}\frac{\partial \lambda_{\mathrm{SW}}}{\partial a_j} = \delta_{ij}, \quad i = 1, \dots, g,\ j = 1, \dots, N. \qquad (112)$$

This relation holds for all Lie algebras due to the particular construction of cycles in Section 1.5 and Lemma 7. So it is essential that we have a complete set of A cycles (not just the subset of special cycles) with the above property in order to take the residues in the entire fundamental polygon.

We can work out $\mathcal{F}_{ijk}$ further and find

PROPOSITION 16 ([36]). *The following residue formula holds*

$$\mathcal{F}_{ijk} = \sum \operatorname{res}\left(\frac{\omega_i \otimes \omega_j \otimes \omega_k}{\mathrm{d}x \otimes \frac{\mathrm{d}z}{z}}\right) = \sum \operatorname{res}\left(\frac{P_{a_i} P_{a_j} P_{a_k}}{(zP_z)^2 P_x}\,\mathrm{d}x\right). \tag{113}$$

Proof. We can calculate $\frac{\partial \omega_j}{\partial a_k} = \frac{\partial^2 \lambda_{\mathrm{SW}}}{\partial a_j \partial a_k}$ keeping in mind that we can throw away any terms that do not contribute to the residue. Due to the second differentiation of λ_{SW}, poles arise at the zeroes of P_x. These mark the branch points of the curve, so we need precisely two factors P_x in the denominator to get a contribution to the residue. We then find up to terms that do not contribute to the residue

$$\begin{aligned}
\frac{\partial^2 \lambda_{\mathrm{SW}}}{\partial a_j \partial a_k} &= -\frac{\partial^2 x}{\partial a_j \partial a_k}\frac{\mathrm{d}z}{z} = \frac{\partial}{\partial a_j}\left(\frac{P_{a_k}}{P_x}\right)\frac{\mathrm{d}z}{z} \\
&= \frac{P_{a_j a_k}}{P_x}\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{P_{a_j}P_{a_k}}{P_x}\right)\frac{\mathrm{d}z}{zP_x} \\
&\simeq -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{P_{a_j}P_{a_k}}{P_x}\right)\frac{\mathrm{d}z}{zP_x}.
\end{aligned} \tag{114}$$

Performing a partial integration we find [36]

$$\begin{aligned}
\sum \operatorname{res}\left(\chi_i \frac{\partial \omega_j}{\partial a_k}\right) &= \sum \operatorname{res}\left(-\chi_i \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{P_{a_j}P_{a_k}}{P_x}\right)\frac{\mathrm{d}z}{zP_x}\right) \\
&= \sum \operatorname{res}\left(\frac{\mathrm{d}\chi_i}{\mathrm{d}x}\frac{P_{a_j}P_{a_k}}{P_x^2}\frac{\mathrm{d}z}{z}\right) \\
&= \sum \operatorname{res}\left(\frac{P_{a_i}P_{a_j}P_{a_k}}{(zP_z)^2 P_x}\,\mathrm{d}x\right)
\end{aligned} \tag{115}$$

and this ends the proof. □

In the proof of the residue formula, the calculation of the second-order derivatives of λ_{SW} is similar to the one for the Picard–Fuchs method. The crucial difference however is that some terms can be neglected because they do not contribute to the residue. This makes the residue formula applicable also for the nonsimply laced Lie algebras. After having obtained the above proposition, the proof that $\mathcal{F}$ satisfies the WDVV system becomes trivial.

COROLLARY 17. *The relation* (94) *follows from the definition of the algebra together with the residue formula of Proposition* 16. *Therefore we conclude again that the prepotential* $\mathcal{F}$ *satisfies the WDVV system, using now the residue formula instead of the Picard–Fuchs equations.*

2.5. REPRESENTATION INDEPENDENCE OF THE FAMILY OF ASSOCIATIVE ALGEBRAS

We have shown in Section 1.5 that the period integrals of λ_{SW} over the first N cycles of type A and the first N cycles of type B are independent of the representation ρ of $\mathfrak{g}$ chosen to define the family of spectral curves. Therefore also the prepotential $\mathcal{F}$ and the proof of the WDVV equations are representation independent.

Since a family of associative algebras is connected to a function satisfying the WDVV equations, this strongly suggests that the family $\mathcal{A}$ defined in Section 2.1 exists for any representation and is independent of it. If so, then the spectral equation

$$P(x, w, u_i) = 0 \tag{116}$$

implicitly defines a one-variable Landau–Ginzburg superpotential $w(x, u_i)$. This provides us with a straightforward calculation of an arbitrary number of different one-variable Landau–Ginzburg superpotentials (see also [14], Chapter 4).

PROPOSITION 18. *For any irreducible representation ρ the family $\mathcal{A}$ of algebras* (69) *is defined and is independent of ρ. Therefore the implicitly defined function $w(x, u_i)$ is a one-variable Landau–Ginzburg superpotential for any ρ.*

Proof. Since the period integrals of λ_{SW} are representation independent, the derivation of the residue formula (113) is representation independent. Since the WDVV equations hold, we find that

$$\sum \mathrm{res}\left(\frac{\omega_i \otimes \omega_j \otimes \omega_m}{\mathrm{d}x \otimes \frac{\mathrm{d}z}{z}}\right) = \sum_{k,l} C_{ij}^k(a)\frac{\partial a_l}{\partial t_N} \sum \mathrm{res}\left(\frac{\omega_k \otimes \omega_l \otimes \omega_m}{\mathrm{d}x \otimes \frac{\mathrm{d}z}{z}}\right)$$

thus showing that the algebra (69)

$$\omega_i \otimes \omega_j = \sum_{k,l} C_{ij}^k(a)\frac{\partial a_l}{\partial t_N}\omega_k \otimes \omega_l \quad \mathrm{mod}\, \frac{\mathrm{d}z}{z} \tag{117}$$

is representation independent. □

As an example, we will consider the Lie algebra A_4 in the 5- and 10-dimensional representations. The spectral curves are given by

$$P_5 = w + x^5 + u_1x^3 + u_2x^2 + u_3x + u_4, \tag{118}$$

$$\begin{aligned} P_{10} \;=\; & w^2 + \big(-11x^5 - 4u_1x^3 - 7u_2x^2 + (-u_1^2 + 4u_3)x + 2u_4 - u_1u_2\big)w - \\ & - x^{10} - 3x^8u_1 + x^7u_2 + (-3u_1^2 + 3u_3)x^6 + (-11u_4 + 2u_1u_2)x^5 + \\ & + (u_2^2 + 2u_1u_3 - u_1^3)x^4 + (-4u_2u_3 - 4u_4u_1 + u_1^2u_2)x^3 + \\ & + (-7u_4u_2 + u_2^2u_1 - u_1^2u_3 + 4u_3^2)x^2 + (-u_2^3 + 4u_4u_3 - u_4u_1^2)x - \\ & - u_4^2 + u_2^2u_3 - u_4u_1u_2. \end{aligned} \tag{119}$$

Defining the ideal $I = \langle P, P_x \rangle \subset \mathbf{C}[x, w]$, explicit computations show that indeed the subalgebras of $\mathbf{C}[x, w]/I$ generated by the P_{u_i} have *precisely* the same structure constants (72).

Appendix A. Holomorphic Differentials

We will give a procedure to calculate a basis for the space of holomorphic differential forms on an arbitrary Riemann surface, see, e.g., [6, 28].

The projective plane $\mathbf{P}^2$ is given by equivalence classes $[x, y, z]$ in $\mathbb{C}^3 \setminus (0, 0, 0)$, the equivalence relation being $(x, y, z) \simeq (\lambda x, \lambda y, \lambda z)$ for any $\lambda \in \mathbb{C}^*$. The projective plane is a 2-dimensional complex manifold with three coordinate patches U_x, U_y, U_z given by $x \neq 0$, $y \neq 0$ and $z \neq 0$ respectively and the local coordinates are given by $(\frac{y}{x}, \frac{z}{x})$, $(\frac{x}{y}, \frac{z}{y})$ and $(\frac{x}{z}, \frac{y}{z})$. Once an object (function, differential form) is given in one coordinate patch, one simply uses the transformations between them to find out what it looks like in another patch. For instance, the transformation from the patch $x \neq 0$ to the patch $y \neq 0$ is given by $(y, z) \to (\frac{1}{x}, \frac{z}{x})$.

Riemann surfaces can be embedded in $\mathbf{P}^2$, the downside to this being that the embedded curves are singular whereas the Riemann surface is not. The advantage for our purposes however is that it is relatively easy to calculate a basis of the space of holomorphic differential forms. So let an arbitrary Riemann surface be given by an affine equation of degree d

$$P(x, z) = 0 \tag{120}$$

which expresses the part of the curve in $\mathbf{P}^2$ in the patch $y \neq 0$. The singular points are given by $P = P_x = P_z = 0$, and outside of these points using the implicit function theorem one expresses locally $x(z)$ if $P_x \neq 0$ or $z(x)$ if $P_z \neq 0$. In another patch, say $x \neq 0$, the curve is given by the numerator of

$$P\left(\frac{1}{y}, \frac{z}{y}\right) = 0. \tag{121}$$

One should check that besides being holomorphic in each coordinate patch, the differential form is also holomorphic in the singular points. Starting with the coordinate patches, we suggest the following forms given in the patch $y \neq 0$

$$\omega = \frac{\phi(x, z)\,\mathrm{d}x}{P_z(x, z)} \tag{122}$$

with ϕ a polynomial. It may seem that ω is singular in the branch points for which $P_z = 0$. However, since $P_x \neq 0$ for those branch points one can use the implicit function theorem to express $x(z)$ locally. So one should really use z as a local coordinate and one finds

$$\omega = \frac{\phi(x, z)\frac{\mathrm{d}x}{\mathrm{d}z}\,\mathrm{d}z}{P_z} = -\frac{\phi(x, z)\,\mathrm{d}z}{P_x} \tag{123}$$

and therefore ω is nonsingular at the branch points.

In the other coordinate patches we need only consider points with $y = 0$. Making a transformation to the patch $x \neq 0$ for example we find that

$$\omega = \frac{\phi(\frac{1}{y}, \frac{z}{y})\,\mathrm{d}(\frac{1}{y})}{P_z(\frac{1}{y}, \frac{z}{y})}. \tag{124}$$

Expanding ω around $y = 0$ one finds

$$\omega = \frac{-y^{-[\phi]}y^{-2}\,\mathrm{d}y}{y^{d-1}} R(y, z), \tag{125}$$

where $[\phi]$ denotes the polynomial degree of ϕ and $R(0, z) \neq 0, \infty$. This expression is holomorphic at $y = 0$ if and only if $[\phi] \leqslant d - 3$. A similar calculation can be made for the coordinate patch $z \neq 0$ without further constraints on ϕ. A useful way of summarizing the effect of the change of coordinates from $y \neq 0$ to $x \neq 0$ is by introducing the homogeneous degree $d - 3$ polynomial $\bar{\phi}(x, y, z)$ such that $\bar{\phi}(x, 1, z) = \phi(x, z)$. In terms of the homogeneous polynomial $P(x, y, z)$ we then find

$$\omega = \frac{\bar{\phi}(x, 1, z)\,\mathrm{d}x}{P_z(x, 1, z)}, \qquad \omega = \frac{\bar{\phi}(1, y, z)\,\mathrm{d}y}{P_z(1, y, z)} \tag{126}$$

in the respective coordinate patches. This allows for an easy comparison of the restrictions on $\bar{\phi}$ due to the singular points in different coordinate patches.

Up to now, the only restriction on the polynomial $\bar{\phi}(x, y, z)$ is on its total degree. The other constraints all come from the singular points and we will give an algorithm how to find them. To check holomorphicity for nonsingular points, we used the implicit function theorem to identify one of the variables as independent and express the other variable in terms of a power series. In the singular points the implicit function theorem is not applicable and there is no power series. However, after an idea of Newton one can express one of the variables as a *fractional* power series in the other variable. In fact, there can be several possible fractional series and we must consider each of them. Such an expansion is called a Puiseux expansion and it allows us to find conditions on ϕ due to the singular points. We will assume that singular points only occur in the patch $z \neq 0$ and we describe only the case where the singular point is $[0, 0, 1]$. From this procedure it should be clear what to do if the singularities are elsewhere.

ALGORITHM ([28]).

1. Construct the so-called Newton polygon. In the first quadrant of $\mathbb{Z}^2$ mark points (a, b) if $P(x, y)$ has nonzero coefficient of $x^a y^b$. We call the collection of such points $\Delta(P)$, and for each line segment connecting a pair of points in $\Delta(P)$ we consider the convex subset of $\mathbb{R}^2$ obtained by taking the points to the right and above this line segment. The Newton polygon is the union of all these convex subsets.

2. If (0, β_0) is the only point of the Newton polygon then all Puiseux expansions are zero, and we are done. If not, then there will be some point in the polygon with y-coordinate smaller than β_0 because $P(x, y)$ is irreducible. Therefore all edges of the polygon have negative slope and the following procedure should be followed for all edges with finite nonzero slope $-1/\mu_0$.
3. Assigning degrees $[x] = 1$ and $[y] = \mu_0$ one can write

$$P(x, y) = P_0(x, y) + Q_0(x, y)$$

with P_0 homogeneous of degree $\mu_0\beta_0$ and P_1 of higher degree. P_0 has at least two terms and therefore substituting $y = tx^{\mu_0}$ yields a polynomial equation in t which has a nonzero root, say t_0. Different nonzero roots may lead to different Puiseux expansions, so we keep track of all possible values. As a first term in the Puiseux expansion, we get $y_0 = t_0 x^{\mu_0}$.
4. Make the substitutions $x = x_1^{q_0}$ and $y = x^{\mu_0}(t_0 + y_1) = x_1^{p_0}(t_0 + y_1)$ where $\mu_0 = p_0/q_0$ and then

$$P(x, y) = P_0(x, y) + Q_0(x, y) = 0 + Q_0(x, y) = x_1^{p_0\beta_0} P_1(x_1, y_1).$$

5. Start again at step number 1 but now with the polynomial $P_1(x_1, y_1)$ and repeating this process over and over we obtain

$$y = t_0 x^{\mu_0} + t_1 x^{\mu_0 + \frac{\mu_1}{q_0}} + t_2 x^{\mu_0 + \frac{\mu_1}{q_0} + \frac{\mu_2}{q_0 q_1}} + \cdots .$$

Remark 19. If β_1 is the smallest number such that $y_1^{\beta_1}$ has nonzero coefficient in $P_1(x_1, y_1)$ then we have $\beta_1 \leqslant \beta_0$. Therefore, at some point we must reach $\beta_n = 1$ and then $q_n = q_{n+1} = \cdots = 1$. So indeed, we get a fractional power series.

Having all power series $y = \sum_{k=1}^{\infty} a_k x^{k/r}$ for all singular points, one can make a choice of local parameter t in such a way that $x = t^r$ and $y = \sum_{k=1}^{\infty} a_k t^k$. Substituting this in our differential form ω the constraints on its coefficients can be calculated explicitly.

A.1. HOLOMORPHIC q-DIFFERENTIALS

A holomorphic differential ω on C can be thought of as a holomorphic section of the canonical line bundle K over C. In each coordinate patch U, ω is given by a function $f(x)$ in terms of a local coordinate x. On the overlap with a chart with coordinate z, the transition function of the canonical bundle is given by $\partial z/\partial x$. The space of holomorphic q-differentials Ω^q (see, e.g., [16]) is the space of holomorphic sections of $K^{\otimes q}$, the qth tensor product of the canonical bundle with itself. Its transition functions are given by $(\frac{\partial z}{\partial x})^q$. There is a natural product $\Omega^p \times \Omega^q \to \Omega^{p+q}$ given by $(f(x)\,\mathrm{d}x^p, g(x)\,\mathrm{d}x^q) \to f(x)g(x)\,\mathrm{d}x^{p+q}$ which we sometimes denote by the tensor product, thus leading to the notation in (69). In general not all holomorphic 2-differentials are products of holomorphic 1-forms. For hyperelliptic curves

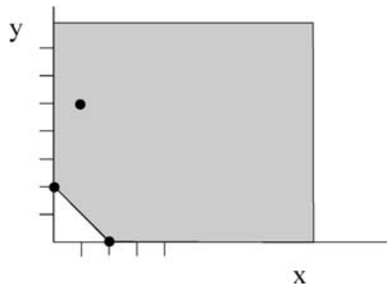

Figure 7. The Newton polygon corresponding to the singularity [0, 0, 1] of the example curve.

for example the subspace of products is only $(2g-1)$-dimensional whereas it is known that $\dim(\Omega^2) = 3g-3$ for any type of curve with genus $g > 1$.

A.2. EXAMPLE: $P(x, y) = 2xy^5 + 5y^2 - 3x^2$

The differentials are

$$\omega = \frac{\phi(x, y)\,\mathrm{d}x}{10xy^4 + 10y},$$

where ϕ is of degree less or equal to 3 and we will derive the constraints on ϕ due to the singularities. In principle we can solve the equations $\{P = 0, P_x = 0, P_y = 0\}$ by using Groebner bases but in this case it is easily seen that $x = y = 0$ is the only finite singular point. Making P homogeneous we get

$$P = 2xy^5 + 5y^2z^4 - 3x^2z^4$$

and the solutions at infinity ($z = 0$) are $[0, 1, 0]$ and $[1, 0, 0]$. Since at $[1, 0, 0]$ we have $\{P = 0, P_y = 0, P_z = 0\}$ this leaves us with a total of 2 singular points $[0, 0, 1]$, $[1, 0, 0]$ and we will discuss them separately.

– $[0, 0, 1]$: This point can be treated immediately by the algorithm given above. The Newton polygon is given in Figure 7. Therefore $\mu_0 = 1$, $\beta_0 = 2$ and substituting $y = t_0x$ and looking at the lowest order terms we must solve $5t_0^2 - 3 = 0$ which has 2 solutions. Substituting $x = x_1$ and $y = x(t_0 + y_1)$ we get $P_1(x_1, y_1) = 2x_1^4(t_0 + y_1)^5 + 10t_0y_1 + 5y_1^2$ and therefore $\mu_1 = 4$, $\beta_1 = 1$. Since β_1 has reached its smallest value now, we find that the total solution will be

$$y = \sum_{k=1}^{\infty} a_k x^k$$

and therefore x is itself a good local parameter. We find that $y = xR(x)$ where $R(x)$ is nonzero for $x = 0$ and we have around the singularity

$$\omega = \frac{\phi(x, y)\,\mathrm{d}x}{xQ(x)},$$

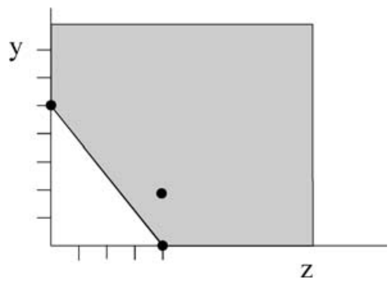

Figure 8. The Newton polygon corresponding to the singularity $[1, 0, 0]$ of the example curve.

where $Q(x)$ is nonzero at $x = 0$. There are no other edges to consider than this one, so the only restriction coming from this singularity is that $\phi(x, y)$ should not have a constant term.

– $[1, 0, 0]$: First we look at $P(1, y, z) = 2y^5 + 5y^2z^4 - 3z^4$. Now we can treat this case in a completely similar way as before by just exchanging symbols $x \to z$, $y \to y$. The Newton polygon is given in Figure 8.
So we get $\mu_0 = 4/5$ and $\beta_0 = 5$ and substituting $y = t_0 z^{\mu_0}$ and looking at the lowest-order terms we must solve $2t_0^5 - 3 = 0$ which has 5 solutions. Substituting $z = z_1^5$ and $y = z_1^4(t_0 + y_1)$ we get $P_1(y_1, z_1) = 2(t_0 + y_1)^5 - 3 + 5z_1^8(t_0 + y_1)^2$. Therefore $\mu_1 = 8$, $\beta_1 = 1$ and since β_1 has reached its smallest value now (and therefore μ_1 is an integer), we need not go further. We find that the total solution will be

$$y = \sum_{k=1}^{\infty} a_k z^{k/5}.$$

So a good local parameter is t, where $z = t^5$ and $y = t^4 R(t)$ where $R(0) \neq 0$. Substituting this into ω we get

$$\omega = -z^{(d-3)-d_\phi} \frac{\phi(1, y, z)\,\mathrm{d}z}{Q_y(1, y, z)} = \frac{\phi(1, t^4, t^5) 5t^4\,\mathrm{d}t}{t^{16} Q(t)},$$

where $Q(0) \neq 0$. So preventing a pole at $t = 0$ and keeping into account that ϕ has degree 3 means that we can only have z^3, yz^2, y^2z, y^3. Making these homogeneous of degree 3 and then going to the coordinate patch $x \neq 0$ means we have $1, y, y^2, y^3$. There are no other edges to consider than this one, so these are the only conditions for this singularity.

Combining the restrictions from the two singularities one finds $g = 3$ and the holomorphic forms

$$\omega_1 = \frac{y\,\mathrm{d}x}{10xy^4 + 10y}, \qquad \omega_2 = \frac{y^2\,\mathrm{d}x}{10xy^4 + 10y}, \qquad \omega_3 = \frac{y^3\,\mathrm{d}x}{10xy^4 + 10y}.$$

Note tbat this curve is hyperelliptic since the span of tensor products of these three forms is $(2g - 1)$-dimensional.

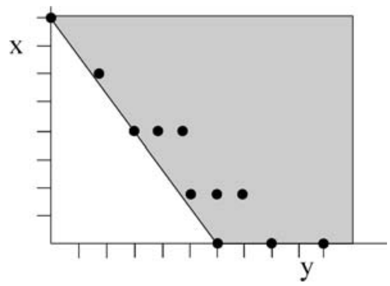

Figure 9. The Newton polygon corresponding to the singularity $[0, 0, 1]$ of the G_2 Seiberg–Witten curve.

A.3. HOLOMORPHIC 1-FORMS FOR THE G_2 SEIBERG–WITTEN CURVE

The curves are given by the degree 10 polynomial equation

$$z^2\left(3\left(z-\frac{\mu}{z}\right)^2+2(2ux^2-x^4)\left(z+\frac{\mu}{z}\right)-x^8+2ux^6-u^2x^4+vx^2\right)=0.$$

Explicit calculation shows that the only singular points $[x, y, z]$ are $[0, 0, 1]$, $[1, 0, 0]$, $[0, 1, \sqrt{\mu}]$, $[0, 1, -\sqrt{\mu}]$.

– For the first of these singularities $[0, 0, 1]$ we intend to express $x(y)$ and the corresponding Newton polygon is given in Figure 9. One easily finds that $t = y^{1/4}$ is a good local parameter and $x = t^3R(t)$ with $R(0) \neq 0$. We find

$$\omega = \frac{\bar{\phi}(x, y, 1)\,\mathrm{d}y}{P_x(x, y, 1)} = \frac{\phi(t^3, t^4)t^3\,\mathrm{d}t}{t^{21}Q(t)} \tag{127}$$

for some rational function $Q(t)$ with $Q(0) \neq 0$. For this to be holomorphic at $t = 0$, $\bar{\phi}(x, y, z)$ should be given by one of the following 18 expressions:

$y^7, zy^6, y^6x, z^2y^5, zy^5x, y^5x^2, z^2y^4x, zy^4x^2, y^4x^3,$
$z^2y^3x^2, zy^3x^3, y^3x^4, zy^2x^4, y^2x^5, zyx^5, yx^6, zx^6, x^7.$

The number of constraints following from this singular point (called its delta invariant) is therefore $36 - 18 = 18$, with 36 the number of polynomials in two variables of degree less or equal to $d - 3 = 7$.

– The second singularity $[1, 0, 0]$ can be treated similarly. We consider $z(y)$ as a fractional power series, and (see Figure 10) we find that a good local parameter is $t = z^{1/5}$ and $y = t^4R(t)$ with $R(0) \neq 0$. The form ω therefore becomes

$$\omega = \frac{\bar{\phi}(1, y, z)\,\mathrm{d}z}{P_y(1, y, z)} = \frac{\phi(t^4, t^5)t^4\,\mathrm{d}t}{t^{14}Q(t)} \tag{128}$$

with $Q(0) \neq 0$. We find that $\phi(y, z)$ should *not* be given by one of the following 5 expressions: y^2, y, yz, 1, z. The delta invariant for this singularity is

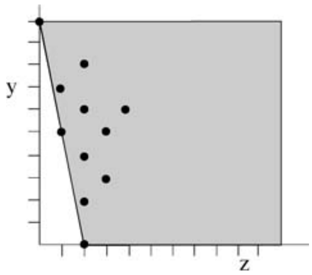

Figure 10. The Newton polygon corresponding to the singularity $[1, 0, 0]$ of the G_2 Seiberg–Witten curve.

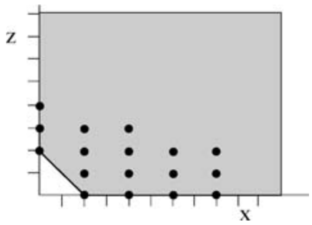

Figure 11. The Newton polygon corresponding to the singularities $[0, 1, \pm\sqrt{\mu}]$ of the G_2 Seiberg–Witten curve. We have changed coordinates so that the singularity is placed at $[0, 1, 0]$.

therefore 5 and $\bar{\phi}(x, y, z)$ is now restricted to the following set of 13:

$$y^7, zy^6, y^6x, z^2y^5, zy^5x, y^5x^2, z^2y^4x,$$
$$zy^4x^2, y^4x^3, z^2y^3x^2, zy^3x^3, y^3x^4, zy^2x^4.$$

– For both the singularities $[0, 1, \pm\sqrt{\mu}]$, there are no fractional power series required since ordinary power series suffice. In Figure 11 we have depicted the Newton polygons for $P(x, \hat{z})$ with $\hat{z} = z \pm \sqrt{\mu}$. The Puiseux expansions read $x = t, z = (\mp\sqrt{\mu} + \alpha t)R(t)$ with $(R(0) \neq 0$ and for some number α. The form ω becomes

$$\omega = \frac{\phi(x, z)\,\mathrm{d}z}{P_x(x, 1, z)} = \frac{\phi(t, \mp\sqrt{\mu} + \alpha t)\,\mathrm{d}t}{tQ(t)} \tag{129}$$

with $Q(0) \neq 0$. Terms in $\phi(x, z)$ with no dependency on x should read $(z \pm \sqrt{\mu})z^j$. The combined effect of these two singularities is therefore that from the set of 13 possible $\bar{\phi}(x, y, z)$ the terms y^7, zy^6, z^2y^5 are replaced with the single term $(z + \sqrt{\mu}y)(z - \sqrt{\mu}y)y^5$, thus confirming that the genus is this curve is 11.

In the coordinate patch $y \neq 0$ the holomorphic forms are therefore given by $\omega = \phi(x, z)\,\mathrm{d}x/P_z$ with

$$\phi(x, z) = \{x^6z, x^5z, x^4z, x^3z, x^2z^2, x^2z, x^2, xz, x, xz^2, z^2 - \mu\}. \tag{130}$$

Appendix B. Term Orderings and Groebner Bases

As preparation for the definition of the associative commutative algebras, we discuss some basic aspects of the theory of ideals in polynomial rings, see, e.g., [7]. For polynomial rings $\mathbf{C}[x]$ in one variable, ideals I are always generated by a single element. This generator is up to a constant uniquely identified as the element of the ideal with minimal degree in x. To determine whether a polynomial is in the ideal or not we divide this polynomial by the generator. If there is a zero remainder the polynomial is in I, otherwise not.

For polynomial rings in two or more variables the situation is more difficult. It can be shown that every ideal is finitely generated, but the number of generators usually exceeds one. Also, division by the generators has become less clear: in $\mathbf{C}[x]$ one divides by looking at the highest degree term in x and the rest simply follows. Here it is not clear which term has highest degree. To fix this one introduces a term ordering, a total ordering which prescribes what is the leading term of a polynomial. For instance, the lexicographical ordering in $\mathbf{C}[x, y]$ sais that one should first look at the powers of x occurring in the polynomial and if there are equal powers then further distinction is made using the powers of y.

Having introduced the term ordering, one can divide polynomials by the ideal generators to determine whether or not they are in I. However, the order of division influences the outcome: the remainder after several divisions can contain different representatives of the same equivalence class in $\mathbf{C}[x, y]$ depending on the order of division. A *Groebner basis* of generators for the ideal is a particular basis with two special properties: the first one is that the order of division is irrelevant, the outcome is always the same. The second property is that an element of the ideal gives zero remainder after division regardless of the term ordering. After the construction of a Groebner basis, membership of the ideal can therefore be decided using a straightforward division algorithm.

We will now briefly describe Buchberger's algorithm [7] to obtain a Groebner basis from a given set of generators $p_1, \ldots, p_n$. First one defines the S-polynomial $S(p_1, p_2)$ of two polynomials. Multiply p_1 and p_2 with monomials of minimal degree (with respect to the term ordering) such that their leading terms become equal. Then subtract one from the other and this gives $S(p_1, p_2)$. For instance, in the lexicographical term ordering we have

$$S\bigl(x^4 + y^2x^2 + y^2x + yx,\, yx^2 + y^3x\bigr) \tag{131}$$
$$= y\bigl(y^2x^2 + y^2x + yx + x^4\bigr) - x^2\bigl(y^3x + yx^2\bigr) \tag{132}$$
$$= -y^3x^3 + y^3x^2 + y^3x + y^2x. \tag{133}$$

The algorithm to produce a Groebner basis is now as follows: first one takes the basis $p_1, \ldots, p_n$ and divides the polynomials amongst each other in random order. If a division is possible then we replace that polynomial by its remainder. Then we add the S-polynomial of two random elements in the basis and divide it in random order by the other basis elements, again replacing it by its remainder if division is possible. We repeat this process over and over again, until every S-polynomial of basis elements is itself in the basis. The result is a Groebner basis.

Appendix C. The Seiberg–Witten Curves for $\mathbf{E_6}$ and $\mathbf{F_4}$

The E_6 curve reads

$$P_{E_6} = \frac{1}{2}x^3\left(z + \frac{\mu}{z} + u_6\right)^2 - q_1(x)\left(z + \frac{\mu}{z} + u_6\right) + q_2(x) = 0,$$

where the polynomials q_1 and q_2 are given by

$$\begin{aligned} q_1 &= 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 - \\ &\quad - 162u_1u_2x^8 + (6u_1u_3 - 27u_4)x^7 - (30u_1^2u_2 - 36u_5)x^6 + \\ &\quad + (27u_2^2 - 9u_1u_4)x^5 - (3u_2u_3 - 6u_1u_5)x^4 - \\ &\quad - 3u_1u_2^2x^3 - 3u_2u_5x - u_2^3, \\ q_2 &= \frac{1}{2x^3}(q_1^2 - p_1^2p_2), \\ p_1 &= 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + \\ &\quad + 2u_3x^4 - 5u_1u_2x^3 - u_4x^2 - u_5x - u_2^2, \\ p_2 &= 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 + \\ &\quad + u_3x^4 - 4u_1u_2x^3 - 2u_4x^2 + 4u_5x + u_2^2. \end{aligned}$$

The curve for F_4 on the other hand reads

$$P_{F_4} = -8\left(z + \frac{\mu^2}{z}\right)^3 + s_1(x)\left(z + \frac{\mu^2}{z}\right)^2 + s_2(x)\left(z + \frac{\mu^2}{z}\right) + s_3(x) = 0,$$

where the $s_i(x)$ are given by

$$\begin{aligned} s_1(x) &= -636x^9 - 300u_1x^7 - 48u_1^2x^5 - 5u_3x^3 + 2u_4x, \\ s_2(x) &= -168x^{18} - 348u_1x^{16} - 276u_1^2x^{14} + (-116u_1^3 + 14u_3)x^{12} + \\ &\quad + (-92u_4 - 20u_1^4 - 8u_1u_3)x^{10} + (-42u_1u_4 - 6u_1^2u_3)x^8 + \\ &\quad + \left(-4u_6 - \frac{10}{3}u_1^2u_4 - \frac{2}{3}u_3^2\right)x^6 + \left(\frac{1}{3}u_3u_4 - \frac{2}{3}u_6u_1\right)x^4, \\ s_3(x) &= x^{27} + 6u_1x^{25} + 15u_1^2x^{23} + (20u_1^3 + u_3)x^{21} + \\ &\quad + (5u_4 + 4u_1u_3 + 15u_1^4)x^{19} + (6u_1^2u_3 + 12u_1u_4 + 6u_1^5)x^{17} + \\ &\quad + \left(\frac{1}{3}u_3^2 + 5u_6 + 4u_1^3u_3 + \frac{26}{3}u_1^2u_4 + u_1^6\right)x^{15} + \\ &\quad + \left(\frac{4}{3}u_1^3u_4 + \frac{19}{3}u_6u_1 + u_1^4u_3 + \frac{4}{3}u_3u_4 + \frac{2}{3}u_3^2u_1\right)x^{13} + \end{aligned}$$

$$+\left(\frac{1}{3}u_1^2u_3^2-\frac{1}{3}u_1^4u_4-\frac{15}{4}u_4^2+3u_6u_1^2\right)x^{11}+$$
$$+\left(\frac{1}{3}u_6u_3-\frac{4}{9}u_1^2u_3u_4+\frac{1}{27}u_3^3-\frac{13}{6}u_4^2u_1+\frac{13}{27}u_6u_1^3\right)x^9+$$
$$+\left(-\frac{1}{9}u_3^2u_4-\frac{1}{2}u_6u_4+\frac{1}{9}u_6u_1u_3-\frac{7}{36}u_1^2u_4^2\right)x^7+$$
$$+\left(\frac{1}{12}u_4^2u_3-\frac{1}{6}u_6u_1u_4\right)x^5+\left(-\frac{1}{54}u_4^3-\frac{1}{108}u_6^2\right)x^3.$$

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