Localized modes and phonon scattering of a lattice κ kink

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We consider a discrete sine-Gordon equation with a κ phase shift. We analyze the stability of a lattice κ kink which is the ground state of the system. The dependence of the localized mode frequency of a κ kink on the topological charge is analyzed numerically and analytically. We show that there is a certain range of parameter values for the coupling constant and the phase shift for an internal mode to exist. A semianalytical calculation on the existence of an internal mode of a fractional kink is presented using the characteristic that a long wavelength phonon is perfectly transmitted at the appearance of an internal mode. We also briefly discuss another type of fractional kink that exists for any value of coupling constant. This fractional kink is stable in the weak coupling limit but unstable in the opposite one.

I. INTRODUCTION

Recent technological advances in the control of the phase coherence of the Cooper-pair condensate that characterizes superconductors near a Josephson junction have promoted research on the manipulation and phase biasing of such a junction. Several examples are the experimental preparation of superconductor-ferromagnet-superconductor π junctions [1], superconductor–normal metal–superconductor junctions [2], the corner junction [3], and the zigzag junction [4]. The latter two employ unconventional high-T_c superconductors with a predominant $d_{x^2-y^2}$ pairing symmetry [5]. The aforementioned junctions are characterized by an intrinsic phase shift of π in the current-phase relation. These structures present intriguing phenomena such as the spontaneous generation of a fractional magnetic flux at the phase jump.

Recently, Goldobin *et al.* [6] have reported a successful experiment on making a long Josephson junction with an arbitrary phase shift. This inspired the investigations in the present work. Here, we consider a fractional magnetic flux in a discrete long Josephson junction with a κ phase shift, which is described by the following discrete 0-κ sine-Gordon equation:

$$\ddot{\phi}_n = C(\phi_{n+1} - 2\phi_n + \phi_{n-1}) - \sin(\phi_n + \theta_n), \qquad (1)$$

where $\phi_n(t)$ is the Josephson phase quantity, C is the coupling parameter, and θ_n is a step function representing the phase jump in the field quantity, i.e.,

$$\theta_n = \begin{cases} 0, & n \le 0, \\ -\kappa, & n \ge 1. \end{cases}$$

It follows that $|\kappa| \le 2\pi$. If $\kappa = 0$, then we obtain the known discrete sine-Gordon equation.

One possible realization of Eq. (1) in experiments, especially for $\kappa = \pi$, has been presented in [7].

In the continuous limit $C \to \infty$, Eq. (1) can be approximated by

$$\phi_{xx} - \phi_{tt} = \sin(\phi + \theta) - \frac{a^2}{12}\phi_{xxxx} + O(a^4), \qquad (2)$$

with $a \equiv 1/\sqrt{C}$, $\phi_n \approx \phi(na)$, and $\theta_n \approx \theta(na)$. The boundary conditions of $\phi(x,t)$ at the discontinuity $x=0$ are given by the continuity conditions of at least ϕ and ϕ_x.

The discrete sine-Gordon equation itself is known as a nonintegrable equation that supports a lattice 2π kink with an internal mode. Internal modes, which are also called intrinsic modes, shape modes, or quasimodes, can play a significant role in the physical dynamics of a nonlinear coherent structure [8]. They are responsible for peculiar behavior in the interactions of a lattice 2π kink with inhomogeneities, with thermal noise, or with external driving [8]. A debate on the presence of an internal mode has appeared in the study of a 2π kink supported by the continuous limit of this equation (see, e.g., [9] and references therein). Some postulate that a sine-Gordon kink has an internal mode, besides its translational or Goldstone mode, only when a perturbation is present [8,10,11]. From a mathematical point of view, whether a kink of the continuous unperturbed sine-Gordon equation possesses an internal mode or not, depends on the considered function space [12]. If one requires an eigenfunction to be bounded (decaying), then a sine-Gordon kink has (no) internal modes.

An extensive study of internal modes of a lattice sine-Gordon kink has been made by Braun, Kivshar, and Peyrard [13]. They consider the detachment of an internal mode of a kink from the continuous spectrum in the weak coupling case. Recently, Prilepsky and Kovalev [14] presented a clear theory on the mechanism of the detachment.

Equations (1) and (2) admit a stable monotonic solitary wave which is the ground state of the system. The solitary wave, which is named a discrete κ kink and is not translationally invariant, represents a fractional magnetic flux. In the uncoupled limit $C=0$, the κ kink (mod 2π) is given by

$$\phi_n^{\kappa} = \begin{cases} 0, & n \le 0, \\ \kappa, & n \ge 1. \end{cases} \qquad (3)$$

From this structure, one can see that there is no value of ϕ_n between 0 and κ for $\kappa \ne 2\pi$ to make a solution that repre-
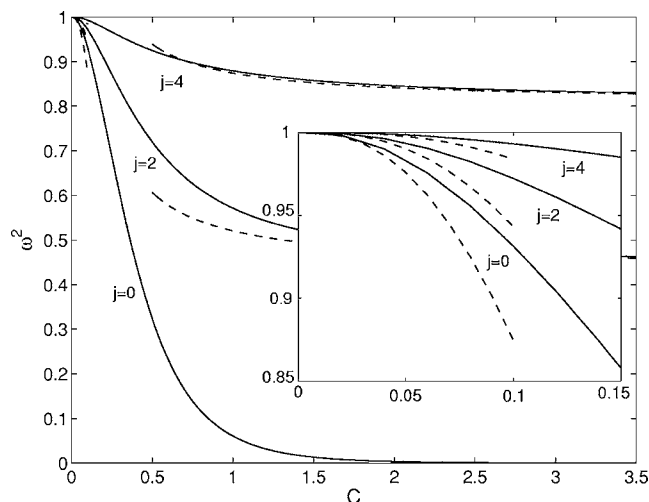
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FIG. 1. The frequency of the first localized mode of a lattice κ kink as a function of the coupling parameter C computed numerically for $\kappa = 2\pi - j\pi/4$, $j = 0, 2, 4$. Dashed lines are approximations calculated using perturbation analysis presented in the text. The inset zooms in on the region of small C.

sents a kink sitting on a site. Therefore, the proposed new definition of a Peierls-Nabarro barrier for a discrete κ kink in [7] is erroneous.

Solution (3) can be continued to the continuous limit. In that limit, the solution is nothing else but the stable fractional kink of Eq. (2) for $a = 0$ which is given by [15]

$$\phi^\kappa(x) = \begin{cases} 4\tan^{-1} e^{(x+x_0)}, & x < 0, \\ \kappa - 4\tan^{-1} e^{(-x+x_0)}, & x > 0, \end{cases} \quad (4)$$

with $x_0 = \ln\tan(\kappa/8)$.

In this paper, we will study the stability of a fractional kink. In Sec. II, we will use numerics to discuss the general case of any value of the coupling constant. In this section, we show that an internal mode might exist. We will show as well that the existence of an internal mode depends on the coupling constant and the topological charge κ. In other words, a κ kink can have a critical value of C for an internal mode of a kink to (dis)appear. This study might contribute toward the understanding of the creation and disappearance of an internal mode of a solution in a discrete system. In Sec. III, we will make an analytical approximation to the localized modes of a κ kink from the numerical calculations presented in the previous section. In Sec. IV, we will study phonon scattering by a fractional kink. The fact that the creation of an internal mode of a sine-Gordon kink corresponds to a perfect transmission of a long wavelength phonon will be exploited to make a good qualitative analysis on the existence boundary of the internal mode. In Sec. V, we will briefly discuss the presence of a fractional kink with topological charge larger than the ground state κ kink that exists for any coupling constant. We summarize the work in Sec. VI.

II. NUMERICAL ANALYSIS OF THE STABILITY OF A FRACTIONAL KINK

To obtain a static kink solution, we use a Newton-Raphson iteration method. As the initial solution for the it-
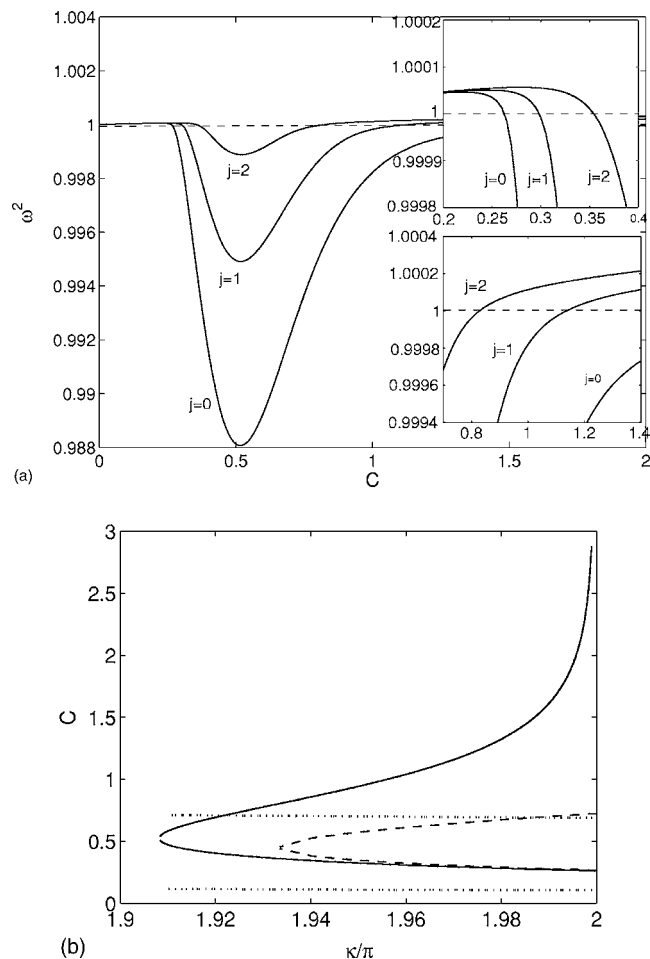

FIG. 2. (a) The frequency of the lowest antisymmetric mode of a lattice κ kink for $\kappa = 2\pi - 0.1j$, $j = 0, 1, 2$. An internal mode exists when the frequency $\omega^2 < 1$. The insets zoom in on the regions of the creation and destruction of an internal mode (see the text). (b) The existence boundary of an intrinsic mode. Dotted and dashed lines are from perturbation analysis Eq. (22) and semianalytical calculation presented in Sec. IV, respectively.

eration, one can use either Eq. (3) or Eq. (4). Because the computational domain as well as the fabrication of junction lattices is, of course, limited to a finite number of sites, one has to deal with boundary conditions. A physically reasonable choice is to take free boundary conditions. Throughout the whole paper, we use 200 sites which is already reasonably large for moderate values of C.

After obtaining a kink solution, we then proceed to considering the stability of the solution. Linearizing Eq. (1) about $\phi_n = \phi_n^\kappa$ gives

$$\ddot{u}_n = C(u_{n+1} - 2u_n + u_{n-1}) - \cos(\phi_n^\kappa + \theta_n)u_n. \quad (5)$$

Substituting the stability ansatz $u_n = e^{i\omega t}v_n$ yields the eigenvalue problem

$$-\omega^2 v_n = C(v_{n+1} - 2v_n + v_{n-1}) - \cos(u_n^\kappa + \theta_n)v_n. \quad (6)$$

The *continuous spectrum* of a fractional kink ϕ_n^κ is obtained by substituting $v_n = e^{i\kappa n/\sqrt{C}}$ for $n \to \pm\infty$ in Eq. (6). After a

simple algebraic calculation, one will obtain the dispersion relation for such linear waves, i.e.,

$$\omega^2 = 1 + 4C \sin^2\left(\frac{\kappa/\sqrt{C}}{2}\right). \tag{7}$$

Hence, the continuous spectrum ω ranges in the interval $\pm[1, \sqrt{1+4C}]$.

We present in Fig. 1 the frequency of the *first localized mode* or Peierls-Nabarro mode of a lattice κ kink as a function of the coupling parameter C. One can see that unless $\kappa = 2\pi$, a κ kink will not have a zero mode. This is in agreement with the fact that a fractional fluxon is attached to the discontinuity point. As soon as the coupling is turned on, the frequency of the first localized mode detaches from the boundary of the continuous spectrum. But this is not the case for the second localized mode.

In Fig. 2, we present the next smallest eigenvalue that corresponds to the lowest antisymmetric mode of a κ kink calculated numerically for $\kappa = 2\pi$, $2\pi - 0.1$, $2\pi - 0.2$. We enlarge the region where the eigenvalue detaches from and reattaches to the boundary of the continuous spectrum. In the region where the eigenvalue is smaller than one, this antisymmetric mode becomes localized. Hence, an internal mode is present.

A lattice 2π kink is known to have an internal mode. As has been discussed by Prilepsky and Kovalev [14], there is a critical coupling for the creation of an internal mode of a lattice 2π kink. In Fig. 2, the critical coupling is the point where the eigenvalue curve crosses the line $\omega^2 = 1$. From Fig. 2, we find that a lattice κ kink with $|\kappa| < 2\pi$ can possess an internal mode. However, from [15], we know that in the continuum limit, a fractional kink has only one eigenvalue. Therefore, we can conclude that a lattice κ kink has a critical coupling parameter $C_{cr}^{a,d}$ for the appearance and the disappearance of an internal mode. $C_{cr}^a \to \approx 0.26$ and $C_{cr}^d \to \infty$ as $\kappa \to 2\pi$ [14].

We also find that there is a critical topological charge κ_{cr} below which a lattice κ kink has no internal mode for any coupling constant. Numerically, $\kappa_{cr} \approx 6$.

III. STABILITY ANALYSIS OF A FRACTIONAL KINK IN STRONGLY AND WEAKLY COUPLED SYSTEMS

A. Strong coupling limit

The linear stability of a fractional kink (4) satisfying Eq. (2) for $a = 0$ has been calculated in [15]. Let ϕ_a^κ be the solution of Eq. (2), with $\phi_a^\kappa \equiv \phi^\kappa$ when $a = 0$. Linearizing Eq. (2) about the solution ϕ_a^κ, writing $\phi(x,t) = \phi_a^\kappa + e^{i\omega t}v(x)$, and retaining the term linear in v gives

$$v_{xx} + [\omega^2 - \cos(\phi_a^\kappa + \theta)]v + \frac{a^2}{12}v_{xxxx} = 0. \tag{8}$$

The smallest eigenvalue of Eq. (4) for $a = 0$ is given by [15]

$$\omega_0^2 = \frac{1}{2}\cos\frac{\kappa}{4}\left(\cos\frac{\kappa}{4} + \sqrt{4 - 3\cos^2\frac{\kappa}{4}}\right), \tag{9}$$

with the corresponding eigenfunction

$$v = \begin{cases} v^-(x) = e^{\mu(x+x_0)}[\tanh(x+x_0) - \mu], & x < 0, \\ v^+(x) = v^-(-x), & x > 0, \end{cases} \tag{10}$$

where $\mu = \sqrt{1 - \omega_0^2}$.

It has also been shown in [15] that the above eigenvalue is the only eigenvalue of Eq. (4).

Next, we will calculate the effect of discreteness on the eigenvalue above. First, we need to find the spatially localized correction to the kink shape ϕ^κ (4) due to the perturbation term. Looking for the kink solution $\phi_a^\kappa(x)$ of Eq. (2) in the form of a perturbation series

$$\phi_a^\kappa(x) = \phi^\kappa + a^2 u(x) + O(a^4)$$

gives that u satisfies

$$u_{xx} - \cos(\phi^\kappa + \theta)u = -\frac{a^2}{12}\phi_{xxxx}^\kappa. \tag{11}$$

From [12,16], we know the homogeneous solution of Eq. (11). One can then use the variation of constants method to obtain the general solution of Eq. (11) (see the Appendix), i.e.,

$$u = A \operatorname{sech}\tilde{x} + B(\tilde{x}\operatorname{sech}\tilde{x} + \sinh\tilde{x}),$$

$$A = A_0 + \frac{1}{24}\frac{-3\cosh(\tilde{x})\sinh(\tilde{x}) + 7\cosh^3(\tilde{x})\sinh(\tilde{x}) - 3\tilde{x} + \tilde{x}\cosh^2(\tilde{x}) - 2\tilde{x}\cosh^4(\tilde{x})}{\cosh^4(\tilde{x})},$$

$$B = B_0 + \frac{1}{24}\frac{3 - \cosh^2(\tilde{x}) - 2\cosh^4(\tilde{x})}{\cosh^4(\tilde{x})}, \tag{12}$$

$$B_0 = \frac{1}{12}.$$

with $\tilde{x} = \pm x + x_0$ for $x < 0$ and $x > 0$, respectively.

The integration constants A_0 and B_0 are determined by the conditions for u. Applying $\lim_{x \to -\infty} u = 0$ and $u(0) = 0$ yields

$$A_0 = \frac{1}{12}\frac{x_0 + x_0 e^{2x_0} - 3e^{2x_0} + 3}{e^{2x_0} + 1},$$

The second condition we apply to u is from the continuity condition of ϕ_a^κ. The continuity of $\partial_x u$ is automatically satisfied by the symmetry of $\partial_x u$ with respect to $x = 0$.

Now we can proceed to the calculation of the eigenvalue of the perturbed kink ϕ_a^κ. One can do the same procedure as above by expanding all the quantities in Eq. (8) in terms of
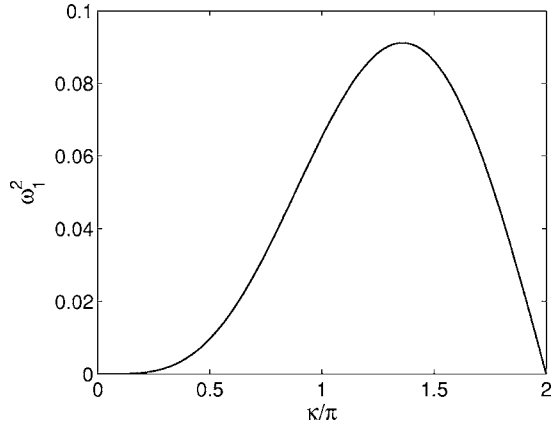
FIG. 3. Plot of ω_1^2, which is the multiplication factor of the a^2 term in the expansion of the smallest eigenvalue of a fractional kink about $a=0$.

Taylor series expansion about $a=0$. After some lengthy calculations, the smallest eigenvalue of ϕ_a^κ up to order $O(a^4)$ is given by

$$\omega^2 = \omega_0^2 + a^2 \omega_1^2, \qquad (13)$$

with

$$\omega_1^2 = \frac{-\mu}{192[2\mu\cos(\kappa/4) + 1 + \mu^2]}$$
$$\times [\cos(\kappa/4)(-10 + 32\mu^4 - 48\mu^2) + 9\cos(5\kappa/4)$$
$$+ \cos(\kappa/2)(-16\mu + 32\mu^3) + 36\mu\cos(\kappa)$$
$$+ \cos(3\kappa/4)(1 + 48\mu^2) + 16\mu^5 - 16\mu^3 - 20\mu]. \quad (14)$$

A plot of ω_1^2 is presented in Fig. 3 from which we know that discreteness increases the smallest eigenvalue. Interestingly, there is a value of κ at which ω_1^2 achieves its maximum, namely,

$$\kappa = 4\arctan\left(\frac{1}{6}(5 + \sqrt{13})\sqrt{\sqrt{13} - 2}\right) \approx 4.27.$$

In the limit $\kappa \to 2\pi$, $\omega_1^2 \to 0$ as it should, because the correction term representing discreteness in Eq. (2) still supports a kink with translational invariance when $\kappa = 2\pi$. A comparison of this analytical result and the numerics is presented in Fig. 1.

B. Weak coupling limit

Now we consider the discrete 0-κ sine-Gordon equation (1).

By the implicit function theorem, there exists a unique continuation of a static κ kink (3) when the coupling constant is turned on [17]. Let $u_n^\kappa(C)$ be a solution of Eq. (1). Writing

$$u_n^\kappa(C) = u_n^{(0)} + \sum_{j=1}^{\infty} C^j u_n^{(j)}, \qquad (15)$$

where $u_n^{(0)}$ is the static kink solution of Eq. (1) for $C=0$, i.e., $u_n^{(0)} = \phi_n^\kappa$ [see Eq. (3)], then $u_n^{(j)}$, $j=1,2,\ldots$, is given implicitly by

$$\frac{1}{n!}\frac{d^j}{dC^j}\sin\left(\sum_{j=0}^{\infty} C^j u_n^{(j)}\right)\Bigg|_{C=0} = (u_{n+1}^{(j-1)} - 2u_n^{(j-1)} + u_{n-1}^{(j-1)}).$$

Considering only the first-order correction due to the coupling, we obtain

$$u_0^{(1)} = \kappa C, \quad u_1^{(1)} = \kappa(1 - C). \qquad (16)$$

The other sites will be $u_n^{(1)} = O(C^{1-n})$ for $n \leq -1$ and $u_n^{(1)} = \kappa + O(C^n)$ for $n \geq 2$.

Let us now consider the first eigenvalue of u_n^κ. The first localized mode corresponds to a symmetric eigenfunction, i.e., $v_n = v_{-n+1}$. Therefore, one can consider only the case of $n \geq 1$.

By considering the solution u_n with correction only up to order $O(C^2)$, the eigenvalue problem (6) is then given by

$$-\omega^2 v_1 = C(v_2 - v_1) - \cos(\kappa C)v_1 \qquad (17)$$

for site number 1 and

$$-\omega^2 v_n = C(v_{n+1} - 2v_n + v_{n-1}) - v_n \qquad (18)$$

for the remaining sites.

Next, we will follow the procedure presented in [14] (see also [18]) to make a qualitative analysis of the behavior of the first localized mode of this fractional kink as a function of the coupling parameter $C \ll 1$. According to [14], the eigenfunction can be written as $v_n = e^{-\Delta n}$ with $\Delta \geq 0$ for $n \geq 1$.

Using the ansatz, Eqs. (17) and (18) simplify to

$$\omega^2 = \frac{C + \cos(\kappa C) - \cos^2(\kappa C)}{C + 1 - \cos(\kappa C)},$$

$$\Delta = \ln\left(\frac{C + 1 - \cos(\kappa C)}{C}\right) \geq 0. \qquad (19)$$

A plot of this result in comparison with the numerically obtained eigenvalues is presented in Fig. 1.

After considering the first localized mode, we study the next localized mode or internal mode. This mode corresponds to an asymmetric eigenfunction, i.e., $v_n = -v_{-n+1}$. Again considering only corrections up to $\mathcal{O}(C^2)$, the eigenvalue problem is simplified to

$$-\omega^2 v_1 = C(v_2 - 3v_1) - \cos(u_1 - \theta_1)v_1 \qquad (20)$$

for site number 1 and

$$-\omega^2 v_n = C(v_{n+1} - 2v_n + v_{n-1}) - v_n \qquad (21)$$

for the other sites.

After some algebraic calculations, we obtain

$$\omega^2 = \frac{4C^2 - 3C + \cos(\kappa C)[4C - 1 + \cos(\kappa C)]}{C - 1 + \cos(\kappa C)},$$

$$\Delta = \ln\left(\frac{-C + 1 - \cos(\kappa C)}{C}\right). \qquad (22)$$

The eigenvalue ω^2 above has value smaller than 1 in a certain range. A plot of the coupling constant C as a function of κ at which $\omega^2 = 1$ is shown as dotted lines in Fig. 2(b).

Unfortunately, the analytical expression does not give a good quantitative approximation to the numerical result presented in Sec. II (see [14]). To make a better approximation, we will use a characteristic of the appearance of an internal mode in the interaction with a plane wave.

IV. PHONON SCATTERING OF A LATTICE κ KINK

Kim, Baesens, and MacKay [19] show that the appearance of an internal mode creates a maximum transmission of phonons scattered by a 2π kink. This phenomenon is a rather general effect of the appearance of an internal mode. Therefore, we can use this characteristic to approximate the existence boundary of an internal mode of a fractional kink.

Let us consider the eigenvalue problem (6). Notice that far from the center of a fractional kink, the system is almost decoupled and hence $\cos(\phi_n^\kappa + \theta_n) \sim 1$. Therefore, for $n \to \infty$, $v_n \sim e^{ikn}$ with

$$\omega^2 = 1 - 2C[\cos(k) - 1]. \tag{23}$$

The eigenvalue problem itself can be written in matrix form as [20]

$$\begin{pmatrix} v_n \\ v_{n-1} \end{pmatrix} = \mathcal{M}_n \begin{pmatrix} v_{n+1} \\ v_n \end{pmatrix} \tag{24}$$

with

$$\mathcal{M}_n = \begin{pmatrix} 0 & 1 \\ -1 & 2 + \dfrac{\cos(\phi_n - \theta_n) - \omega^2}{C} \end{pmatrix}.$$

Applying Eq. (24) successively, the amplitude vectors at both ends of the chain are related by

$$\begin{pmatrix} v_{-N+1} \\ v_{-N} \end{pmatrix} = \mathcal{M} \begin{pmatrix} v_{N+1} \\ v_N \end{pmatrix} \tag{25}$$

with $\mathcal{M} = \Pi_{n=-N+1}^{N} \mathcal{M}_n$.

Due to the presence of a fractional kink, the continuous wave can be partly reflected and transmitted. Therefore, we seek an asymptotic solution of the form

$$v_n \underset{n \to -\infty}{\sim} e^{ikn} + r e^{-ikn}, \quad v_n \underset{n \to \infty}{\sim} t e^{-ikn}, \tag{26}$$

where t and r are the amplitude of the transmitted and the reflected wave, respectively.

Substituting this asymptotic solution in Eq. (25) and denoting M_{nm}, $n, m = 1, 2$, as matrix components of $\mathcal{M}$ gives

$$t = \frac{2i \sin(k) e^{-2ikN}}{(M_{12} - M_{21}) e^{ik} - M_{22} + M_{11} e^{2ik}},$$

or

$$T = |t|^2 = \frac{4 \sin^2(k)}{D}, \tag{27}$$

with

$$D = -4M_{11}M_{22}\cos(k)^2 - 2(M_{11} - M_{22})$$
$$\times (M_{21} - M_{12})\cos(k) + (M_{12} - M_{21})^2 + (M_{11} + M_{22})^2.$$

From Eq. (27), we know that at $k = 0$, seemingly there should be no transmission. Yet Kim, Baesens, and MacKay [19] found that perfect transmission can happen at $k = 0$ if the denominator is also vanishing, and this is at the appearance of an internal mode.

We have calculated numerically the transmission of some long wavelength phonons by a fractional kink. In Fig. 4, we present the transmission of some phonons by a $(2\pi - 0.1)$ kink. It is clear that the kink transmits long wavelength phonons at the (dis)appearance of the internal mode. The two peaks of the transmission curve will get closer to each other as κ approaches κ_{cr}.

Next, we will present a semianalytical calculation of the value of C at which the denominator of the transmission coefficient vanishes. This is based on an approximation of the lattice equations that is valid when C is small. Instead of following the asymptotic route, we take a shortcut (reminiscent of a Galerkin truncation) and consider a kink with a given number of distinguished elements and simply ignore the errors that occur elsewhere along the lattice (for a rather complete description, see [21]). Here, we consider the symmetrical four-point κ kink which is defined as (see Balmforth and co-workers [21])

$$\phi_n = 0, \quad n < -1,$$

$$C(\phi_0 - 2\phi_{-1}) = \sin \phi_{-1}, \quad C(\phi_1 - 2\phi_0 + \phi_{-1}) = \sin \phi_0,$$

$$C(\phi_2 - 2\phi_1 + \phi_0) = \sin(\phi_1 - \kappa),$$

$$C(\kappa - 2\phi_2 + \phi_1) = \sin(\phi_2 - \kappa),$$

$$\phi_n = \kappa, \quad n > 2.$$

Because the defined kink is symmetrical, the above equations can further be simplified to

$$C(-\kappa - \phi_1 + 2\phi_2) = \sin(\kappa - \phi_2),$$

$$C(-\kappa + 3\phi_1 - \phi_2) = \sin(\kappa - \phi_1), \tag{28}$$

by noting that $\phi_1 = \kappa - \phi_0$, $\phi_2 = \kappa - \phi_{-1}$.

The denominator of the transmission T is then simplified to

$$D_s = (K_{11} + K_{12} - K_{21} - K_{22})^2, \tag{29}$$

with

$$K_{11} = 1 - \delta_1 \delta_0, \quad K_{12} = -\delta_0 + (\delta_0 \delta_1 - 1)\delta_2,$$

$$K_{21} = \delta_{-1} + (1 - \delta_{-1}\delta_0)\delta_1,$$

$$K_{22} = 1 - \delta_{-1}\delta_0 + [(\delta_{-1}\delta_0 - 1)\delta_1 - \delta_{-1}]\delta_2,$$

and $\delta_n = 2 + [\cos(\phi_n - \theta_n) - 1]/C$. Due to the symmetrical properties of the considered kink, $\delta_{-1} = \delta_2$ and $\delta_0 = \delta_1$.

For a given κ, one then must solve Eqs. (28) and (29) simultaneously for ϕ_1, ϕ_2, and C. A plot of C as a function of

FIG. 5. An evolution of an unstable 3π-kink with $C=0.35$.

$$\phi_n^{2\pi+\kappa} = \begin{cases} 0, & n < n_1, \\ 2\pi, & n_1 \leqslant n \leqslant 0, \\ \kappa, & 1 \leqslant n \leqslant n_2, \\ 2\pi + \kappa, & n > n_2, \end{cases}$$

with $[n_1, n_2] = [-4, 5]$ for $\kappa=6$, $[-1, 2]$ for $\kappa=5$, and $[0,1]$ for $\kappa=3$. In the limit $\kappa \to 2\pi$, $n_{1,2} \to \mp \infty$ as one might predict already [21]. This nonmonotonicity is observed until $\kappa \approx 2.41$. Below this value of κ, the solution in the limit of $C \to 0$ is monotonically given by

$$\phi_n^{2\pi+\kappa} = \begin{cases} 0, & n \leqslant 0, \\ 2\pi + \kappa, & n \geqslant 1. \end{cases}$$

The critical coupling constant C_{cr} above which the fractional kink is unstable should be able to be approximated analytically by calculating the largest eigenvalue of the kink as presented in the preceding section. In Fig. 5, we show the evolution of a 3π kink with $C=0.35$ which is already in the instability region.

VI. CONCLUSIONS

To conclude, we have discussed the stability of a fractional kink of a 0-κ sine-Gordon equation analytically and numerically. Analytical calculations have been presented in the region of strongly and weakly coupled systems. We show that a κ kink can have an internal mode. A semianalytical approximation to the existence region of this mode has been presented using the characteristics of phonon scattering when an internal mode is present.

FIG. 4. The transmission of a phonon with wave number $k=0.02, 0.01, 0.005$ by a $(2\pi-0.1)$ kink as a function of the coupling parameter C in the vicinity of (a) the appearance and (b) the disappearance of an internal mode. For $k=0$, the transmission profile will be zero everywhere except at the threshold of this (dis)appearance of the internal mode.

κ is shown in Fig. 2(b). We see that it gives a better approximation than Eq. (22) for the lower boundary of the existence of an intrinsic mode. The more sites we use to make an n-point κ kink, instead of the four-point κ kink as above, the better the approximation it gives.

V. A FRACTIONAL KINK WITH TOPOLOGICAL CHARGE $2\pi+\kappa$

Another interesting point of a 0-κ sine-Gordon equation is the presence of a static kink with topological charge larger than 2π. In the continuous limit, this kink is given by Eq. (4) with $x_0 = \ln \tan(\pi/4 + \kappa/8)$. Even though in the continuous limit this kink is unstable [15], we found that the continuation of this kink in the weakly discrete system is stable. Interestingly, the corresponding solution is not a monotonically increasing series for some values of κ. For instance, we found that in the limit $C=0$,
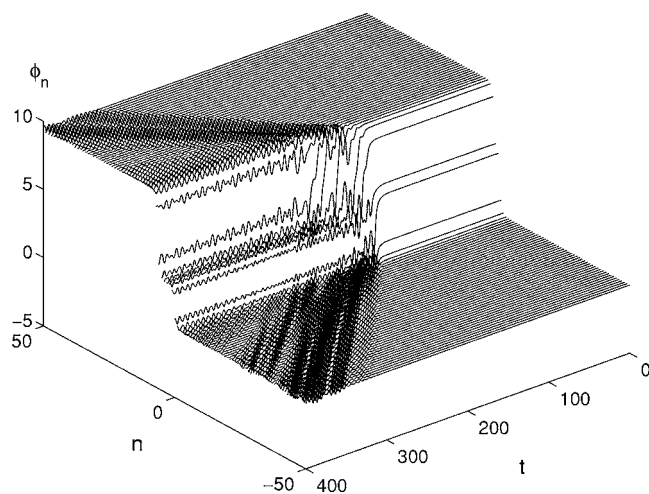
APPENDIX: VARIATION OF CONSTANTS METHOD

In this section, we will describe in brief the so-called variation of constants method. This method is frequently used to obtain the general solution of a differential equation. In this paper we have used this method to solve a differential operator of the form

$$\mathcal{L} = \partial_{xx} - \cos \Phi^F, \tag{A1}$$

with

$$\Phi^F = 4 \arctan e^x.$$

Let us consider the following differential equation:

$$\mathcal{L}\psi = \partial_{xx}\psi - \psi \cos \Phi^F = f(x). \tag{A2}$$

To solve Eq. (A2) and obtain the general solution using the variation of constants method, we need solutions of the homogeneous equation of (A2). If $f \equiv 0$, we know that [12,16]

$$\psi_b = A\partial_x\Phi^F = A/2 \operatorname{sech} x,$$

$$\psi_u = B(x \operatorname{sech} x + \sinh x), \tag{A3}$$

are two independent solutions of Eq. (A2). Notice that ψ_b is bounded and ψ_u is unbounded as $x \to \pm\infty$.

To solve Eq. (A2) for a nonzero $f(x)$, we set the constants A and B to be functions of x. Hence, we have

$$\psi = A(x)\operatorname{sech} x + B(x)(x \operatorname{sech} x + \sinh x). \tag{A4}$$

From the first derivative of ψ with respect to x we set

$$A_x \operatorname{sech} x + B_x(x \operatorname{sech} x + \sinh x) = 0 \tag{A5}$$

and from Eq. (A2) we obtain

$$-A_x \frac{\sinh x}{\cosh^2 x} + B_x \left(\frac{1}{\cosh x} - \frac{x \sinh x}{\cosh^2 x} + \cosh x \right) = f. \tag{A6}$$

From Eqs. (A5) and (A6) we can derive the expression for A_x and B_x.

After some algebraic calculations, we obtain

$$A = A_0 - \int \frac{1}{2} \frac{(x + \sinh x \cosh x)f}{\cosh x} dx,$$

$$B = B_0 + \int \frac{1}{2} \frac{f}{\cosh x} dx. \tag{A7}$$

The constants A_0 and B_0 are determined by the conditions for ψ. Hence, we have the general solution of Eq. (A2).

[1] V. V. Ryazanov, V. A. Oboznov, A. Yu. Rusanov, A. V. Veretennikov, A. A. Golubov, and J. Aarts, Phys. Rev. Lett. **86**, 2427 (2001).

[2] J. J. A. Baselmans, A. F. Morpurgo, B. J. van Wees, and T. M. Klapwijk, Nature (London) **397**, 43 (1999).

[3] D. A. Wollman, D. J. Van Harlingen, J. Giapintzakis, and D. M. Ginsberg, Phys. Rev. Lett. **74**, 797 (1995).

[4] H. Hilgenkamp, Ariando, H. J. H. Smilde, D. H. A. Blank, G. Rijnders, H. Rogalla, J. R. Kirtley, and C. C. Tsuei, Nature (London) **422**, 50 (2003).

[5] C. C. Tsuei and J. R. Kirtley, Rev. Mod. Phys. **72**, 969 (2000).

[6] E. Goldobin, A. Sterck, T. Gaber, D. Koelle, and R. Kleiner, Phys. Rev. Lett. **92**, 057005 (2004).

[7] H. Susanto and S. A. van Gils, Phys. Rev. B **69**, 092507 (2004).

[8] O. M. Braun and Yu. S. Kivshar, *The Frenkel-Kontorova Model* (Springer-Verlag, Berlin, 2004); Phys. Rep. **306**, 1 (1998).

[9] G. Kälbermann, J. Phys. A **37**, 11603 (2004).

[10] Yu. S. Kivshar, D. E. Pelinovsky, T. Cretegny, and M. Peyrard, Phys. Rev. Lett. **80**, 5032 (1998).

[11] N. R. Quintero, A. Sánchez, and F. G. Mertens, Eur. Phys. J. B **19**, 107 (2001).

[12] G. Derks, A. Doelman, S. A. van Gils, and T. Visser, Physica D **180**, 40 (2003).

[13] O. M. Braun, Yu. S. Kivshar, and M. Peyrard, Phys. Rev. E **56**, 6050 (1997).

[14] J. E. Prilepsky and A. S. Kovalev, Phys. Rev. E **71**, 046601 (2005).

[15] E. Goldobin, H. Susanto, D. Koelle, R. Kleiner, and S. A. van Gils, Phys. Rev. B **71**, 104518 (2005).

[16] E. Mann, J. Phys. A **30**, 1227 (1997).

[17] R. S. MacKay and J. A. Sepulchre, Physica D **82**, 243 (1995).

[18] C. Baesens, S. Kim, and R. S. MacKay, Physica D **113**, 242 (1998).

[19] S. Kim, C. Baesens, and R. S. MacKay, Phys. Rev. E **56**, R4955 (1997).

[20] S. W. Kim and S. Kim, Physica D **141**, 91 (2000).

[21] N. J. Balmforth, R. V. Craster, and P. G. Kevrekidis, Physica D **135**, 212 (2000).