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On the Solutions of the Rational Covariance Extension Problem Corresponding to Pseudopolynomials Having Boundary Zeros

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***Abstract*—In this note, we study the rational covariance extension problem with degree bound when the chosen pseudopolynomial of degree at most n has zeros on the boundary of the unit circle and derive some new theoretical results for this special case. In particular, a necessary and sufficient condition for a solution to be bounded (i.e., has no poles on the unit circle) is established. Our approach is based on convex optimization, similar in spirit to the recent development of a theory of generalized interpolation with a complexity constraint. However, the two treatments do not proceed in the same way and there are important differences between them which we discuss herein. An implication of our results is that bounded solutions can be computed via methods that have been developed for pseudopolynomials which are free of zeros on the boundary, extending the utility of those methods. Numerical examples are provided for illustration.**

***Index Terms*—Boundary zeros, bounded solutions, Nevanlinna–Pick interpolation, poles and zeros, rational covariance extension.**

I. INTRODUCTION

Recent years have seen significant advances in the theory of analytic interpolation on the open unit disc of the complex plane. Some major results are the parametrization of all positive real rational functions interpolating a certain positive partial covariance sequence $c_0, c_1, \ldots, c_n$, in terms of desired "spectral zeros" and the introduction of a convex optimization based approach to compute the solution [1]–[5]. However, the convex optimization approach was originally developed for the case where none of the spectral zeros lie on the unit circle. The remaining case where there are spectral zeros on the unit circle is important not only for the sake of completeness, but also due to the fact that placing or forcing a zero on the unit circle is desirable in the design of some filters. In this note, we derive some new theoretical results for this special case based on convex optimization. An alternative treatment based on solving nonlinear equations has been given in [6]. However, there are important new insights gained with the current approach. For example, we are able to derive a necessary and sufficient condition for a solution to be bounded (have no poles on the unit circle). We also assert, and demonstrate by numerical examples, that bounded solutions can be computed using methods that have been developed for pseudopolynomials free of zeros on the unit circle. Hence, those earlier algorithms can be used as complement to the algorithm of [6]. This could be advantageous in situations where one knows in advance that the solution is bounded and in view of the current lack of theoretical convergence results for the latter algorithm.

Manuscript received September 29, 2004; revised May 31, 2005 and October 18, 2005. Recommended by Associate Editor J. Berg. The work of the first author was supported by National ICT Australia, Ltd. (NICTA). National ICT Australia is funded by the Australian Government's Department of Communications, Information Technology and the Arts, and the Australian Research Council through Backing Australia's Ability and the ICT Centre of Excellence Program.

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Digital Object Identifier 10.1109/TAC.2005.863503

More recently in [7], a theory of generalized interpolation with a complexity constraint has emerged as an extensive generalization of the convex optimization approach first presented in [3]. The focus of [7] is on theoretical development (rather than numerical development as in [6]) and applies to a general, possibly abstract, class of interpolation problems with a complexity constraint (a generalization of the notion of degree constraint). In particular, it covers the case where the parametrizing pseudopolynomial has zeros on the unit circle. Our analysis, which is also based on convex optimization, proceeds in a different manner from [7]. In Section V, we discuss important differences between our work and [7].

This note is motivated by the problem of approximation of stochastic systems with noncoercive, possibly nonrational, spectral densities which arise in the study of wind gust, turbulence, laser scintillation [8], and adaptive optics [9]. This is discussed in Section V.

II. Notation and Definitions

In this section, we introduce the main notations and definitions which are used throughout the note.

- $\overline{A}$ and ∂A denote the closure and boundary of a set A, respectively.
- $\mathbb{R}, \mathbb{C}, \mathbb{D}$, and $\mathbb{T}$ denote the set of real numbers, complex numbers, the open unit disc $= \{z \in \mathbb{C} : |z| < 1\}$ and the unit circle, respectively.
- $\mathrm{col}(a_1, \ldots, a_n) = [a_1 \ldots a_n]^\top$.
- M^* denotes the conjugate transpose of a complex matrix M.
- f_* denotes the *parahermitian conjugate* of a complex function f, defined by $f_*(z) = f(z^{*-1})^*$.
- $\mathcal{C}$ denotes the Carathèodory class $\{f \in \mathcal{H} : \Re\{f(z)\} \geq 0 \ \forall z \in \mathbb{D}\}$ and $\mathcal{C}_+$ denotes the subset $\{f \in \mathcal{H} : \operatorname*{ess\,inf}_{z\in\mathbb{D}} \Re\{f(z)\} > 0\}$ of $\mathcal{C}$ where $\mathcal{H}$ denotes the set of functions holomorphic in $\mathbb{D}$.
- $\mathcal{H}^\infty$ denotes the Hardy class of functions in $\mathcal{H}$ which are essentially bounded on $\mathbb{T}$.

By a *pseudopolynomial* we mean a complex function of the form $f(z) = a_0 + \sum_{k=1}^{n} \left(a_k^* z^{-k} + a_k z^k\right)$, where $0 \leq n < \infty$, $a_n \neq 0$ and $(a_0, a_1, \ldots, a_n) \in \mathbb{R} \times \mathbb{C}^n$. We say that n is the *order* of the pseudopolynomial f (the order is zero if f is a constant function). $\mathfrak{Q}(n, A)$ denotes the set of all pseudopolynomials of order *at most* n with $(a_0, a_1, \ldots, a_n) \in \mathbb{R} \times A^n$ where $A \subseteq \mathbb{C}$. We induce a topology on this set by the maximum norm: $\|f\|_\infty = \max_{z\in\mathbb{T}} |f(z)|$. We also define $\mathfrak{Q}_+(n, A)$ to be the set of all elements of $\mathfrak{Q}(n, A)$ which are strictly positive (> 0) on $\mathbb{T}$.

III. Mathematical Preliminaries

A. The Rational Covariance Extension Problem

In this section, we will formally define the rational covariance extension problem (RCEP).

Definition 1: A sequence of complex numbers $c_0, c_1, \ldots, c_n$ (with $c_0 \in \mathbb{R}$) is said to be a partial covariance sequence (PCS) if the Toeplitz matrix $T = [c_{j-i}]_{i,j=1}^{n+1}$, with $c_{-|i|} = c_{|i|}^*$, is positive definite.

Problem 2 (RCEP): Given a PCS $c_0, c_1, \ldots, c_n$ ($n \geq 1$), find all rational functions $f \in \mathcal{C}$ of McMillan degree less than or equal to n such that the first $n+1$ coefficients of the Taylor series expansion of f about 0 is $(1/2)c_0, c_1, \ldots, c_n$.

The RCEP basically adds a new requirement of degree bound to the classical Carathéodory extension problem which is traditionally solved by Schur's algorithm [10]. A drawback of Schur's algorithm is that, in general, it does not give a convenient parametrization of solutions of a bounded degree. The Carathéodory extension problem is related to the classical Nevanlinna–Pick interpolation problem which was solved by Nevanlinna by an algorithm similar to Schur's [11], sometimes known as the Nevanlinna–Schur algorithm.

B. Results on the RCEP

In a series of papers [1], [2], [4], a complete parametrization of all solutions of the RCEP has been established. We state a pertinent result.

Theorem 3: For a given PCS and any polynomial $\eta \neq 0$ of degree $\leq n$ with roots in $\mathbb{C}\backslash\mathbb{D}$ and normalized by $\eta(0) = 1$, there exists a unique pair of polynomials (π, χ) of degree $\leq n$ such that $\chi(0) > 0$, $\pi + \chi$ has all its roots in $\mathbb{C}\backslash\mathbb{D}$, the pair satisfies the relation

$$\pi\chi_* + \chi\pi_* = \kappa^2 \eta\eta_* \tag{1}$$

for a fixed $\kappa > 0$, and $f = \pi/\chi$ satisfies the requirements of the RCEP.

Remark 4: This theorem is stated slightly differently from [4, Th. 2]. We have added the requirement $\chi(0) > 0$ and κ fixed so that the pair (π, χ) is unique. In [4], it is implicit that the uniqueness of (π, χ) is in the equivalence class of graph symbols.

One may also equivalently state the parametrization in terms of elements $d \in \mathfrak{Q}_+(n, \mathbb{C})\backslash\{0\}$, where $d = \kappa^2 \eta\eta_*$. Based on the theorem, we can state a more specific problem, the particular rational covariance extension problem (PRCEP).

Problem 5 (PRCEP): Given a PCS $c_o, c_1, \ldots, c_n$ ($n \geq 1$) and a pseudopolynomial $\Psi \in \mathfrak{Q}_+(n, \mathbb{C})\backslash\{0\}$, find the rational function $f = a/b \in \mathcal{C}$ of McMillan degree $\leq n$ such that the first $n+1$ coefficients of the Taylor series expansion of f about 0 is $(1/2)c_o, c_1, \ldots, c_n$ and $ab_* + ba_* = \Psi$.

Methods to compute the solution of the PRCEP for any given real valued PCS $c_0, c_1, \ldots, c_n$ and pseudopolynomial $\Psi \in \mathfrak{Q}_+(n, \mathbb{R})$ (i.e., Ψ is free of roots on $\mathbb{T}$) is given in [3], [5], and was adapted to solve the Nevanlinna–Pick interpolation problem with degree constraint in [12]. However, a specialized aspect of the theory which has received relatively less attention is the case of solving the PRCEP when the pseudopolynomial has zeros on the boundary. In this work, we extend the method of [3] and [5] to the case where the pseudopolynomial has zeros on the boundary. It turns out that this leads to interesting new theoretical insights, including a necessary and sufficient condition for a $\mathcal{H}^\infty$ solution, as shown in the next section. An alternative treatment of the problem was recently given in [6] based on solving nonlinear equations. There the orientation is toward computation of any real solution of the RCEP.

IV. Main Results

In this section, we derive some properties of the solutions of the RCEP when the parametrizing pseudopolynomial has zeros on $\mathbb{T}$. In particular we show a necessary and sufficient condition for a solution to be in $\mathcal{H}^\infty$ and establish sequential continuity of the map from Ψ to the minimizer of a certain functional $\mathbb{J}_\Psi$ (to be defined below).

Define the mapping $Q : \mathbb{R} \times \mathbb{C}^n \to \mathfrak{Q}(n, \mathbb{C})$ by

$$Q(q_0, q_1, q_2, \ldots, q_n)(z) = q_0 + \sum_{k=1}^{n} \frac{1}{2}(q_k^* z^{-k} + q_k z^k). \tag{2}$$

Clearly, Q is a bijection.

Remark 6: For shorthand, we shall write the integral $(1/2\pi)\int_{-\pi}^{\pi} f\left(e^{i\theta}\right) g\left(e^{i\theta}\right)^* d\theta$ as $\langle f, g\rangle$.

For any $\Psi \in \mathfrak{Q}_+(n, \mathbb{C})\backslash\{0\}$ we consider the functional $\mathbb{J}_\Psi : Q^{-1}\left(\mathfrak{Q}_+(n, \mathbb{C})\right) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathbb{J}_\Psi(q) = \Re\Big\{c^* q - \langle \Psi, \log Q(q)\rangle\Big\}. \tag{3}$$

Note that $\mathbb{J}_\Psi$ can be viewed as an extension to $\overline{Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))}$ of the functional φ that was defined in [5, eq. (4.1)] for the special case where $\Psi \in \mathfrak{Q}_+(n,\mathbb{R})$ and $c_0, c_1, \ldots, c_n$ is real-valued. It then follows by close inspection of the proofs that most results in [5] can be easily extended to the current setting where $c_0, c_1, \ldots, c_n$ is complex-valued and $\Psi \in \overline{\mathfrak{Q}_+(n,\mathbb{C})}\backslash\{0\}$. In particular, we state the analogues of [5, Lemma 4.2, Lemma 4.3, Prop. 4.6] in the following theorem.

Theorem 7: $\mathbb{J}_\Psi$ has the following properties for any $\Psi \in \overline{\mathfrak{Q}_+(n,\mathbb{C})}\backslash\{0\}$.

- $\mathbb{J}_\Psi$ is finite and continuous at any $q \in \overline{Q^{-1}\left(\mathfrak{Q}_+(n,\mathbb{C})\right)}$, except at zero. The functional is infinite, but continuous, at $q = 0$. Moreover, $\mathbb{J}_\Psi((1-t)q_0 + tq_1)$ is a C^∞ function w.r.t. t for any $q_0, q_1 \in Q^{-1}\left(\mathfrak{Q}_+(n,\mathbb{C})\right)$.
- $\mathbb{J}_\Psi$ is strictly convex on the closed, convex domain $\overline{Q^{-1}\left(\mathfrak{Q}_+(n,\mathbb{C})\right)}$.
- For all $r \in \mathbb{R}$, $\mathbb{J}_\Psi^{-1}(-\infty, r]$ is compact. Thus $\mathbb{J}_\Psi$ is proper (i.e., $\mathbb{J}_\Psi^{-1}(A)$ is compact whenever A is compact) and bounded from below.
- The functional $\mathbb{J}_\Psi$ has a unique minimum on $\overline{Q^{-1}\left(\mathfrak{Q}_+(n,\mathbb{C})\right)}$.

We now state the first result on a solution of the RCEP corresponding to a pseudopolynomial having zeros on $\mathbb{T}$.

Theorem 8: If $q_{\min} \in \overline{Q^{-1}\left(\mathfrak{Q}_+(n,\mathbb{C})\right)}$ is a minimum for $\mathbb{J}_\Psi$ then the solution of the PRCEP is: $f = a/b$ where $bb_* = Q(q_{\min})$ and $ab_* + ba_* = \Psi$. Conversely, suppose that $f = a/b$ is the solution to the PRCEP with b being an antistable polynomial (i.e., having roots strictly in $\overline{\mathbb{D}}^c$) and $ab_* + ba_* = \Psi$. Then $q_{\min} = Q^{-1}(bb_*)$ is a unique minimum for $\mathbb{J}_\Psi$.

Proof: By inspection of the proofs of [5, Th. 4.7 and 4.8] and using the directional derivative to replace the ordinary derivative, it follows those proofs remain valid if the polynomial $\sigma = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_{n-1} z + \sigma_n$ of degree n defined in [5, eq. (2.18)] is complex and not *Schur* (i.e., having roots in $\mathbb{D}$), but merely *stable* (i.e., having roots in $\bar{\mathbb{D}}$). Also note that $\sigma_*\sigma$ can be a pseudopolynomial of degree less than n if $\exists m$ satisfying $1 \le m \le n$, such that $\sigma_k = 0$ for all $k \ge m$. The main idea is that the minimizer of $\mathbb{J}_\Psi$ may be an interior point even when $\Psi = \sigma_*\sigma \in \partial\mathfrak{Q}_+(n,\mathbb{C}) \setminus \{0\}$. ■

The minimizer of $\mathbb{J}_\Psi$ may then be found by a Newton descent type algorithm which has been outlined in [3], [5], and [12]. We illustrate this in the following example.

Example 9: Let the given partial covariance sequence be $\{0.2115, 0.0728, -0.0396\}$. We choose the pseudopolynomial $\Psi(z) = z + 2 + z^{-1}$ which has two zeros on the unit circle, i.e., both at $z = -1$, and seek a solution of the RCEP of degree 2. By using a Newton gradient descent algorithm we obtain $q_{\min} = \mathrm{col}(8.6250, 3.5000, 2.0000)$. It can be checked that $q_{\min}$ is in the interior of $\overline{Q^{-1}\left(\mathfrak{Q}_+(n,\mathbb{R})\right)}$, and the solution of the PRCEP is $f(z) = (0.098\,77 + 0.1111z + 0.012\,34z^2)/(8 + 2z - z^2)$.

An interesting question now is: What could happen if the minimum of $\mathbb{J}_\Psi$ lies on the boundary of $Q^{-1}\left(\mathfrak{Q}_+(n,\mathbb{C})\right)$? We first look at an insightful example.

Example 10: Consider the Carathèodory function

$$f(z) = \frac{1}{2}\frac{1+\frac{z}{2}}{1-\frac{z}{2}}. \tag{4}$$

The associated PCS is $1, 1/2, 1/4, \ldots$. We choose the pseudopolynomial $\Psi(z) = z + 2 + z^{-1}$ having a double root at $z = -1$. By Newton gradient descent we find $q_{\min} \approx \mathrm{col}(2, 0.667\,49, -1.3324)$. The roots of $Q(q_{\min})$ are $\{2.0013, -1.0061, -0.993\,96, 0.499\,67\}$ and the approximate solution is $\hat{f} = (0.3326 + 0.4978z + 0.1652z^2)/(2.0135 + 0.9952z - z^2)$. Note how two roots of $Q(q_{\min})$ are close to $z = -1$. Assuming that were it not for numerical discrepancies that both roots would be exactly -1 and cancel the two corresponding roots of Ψ, we find:$\Psi(z)/Q(q_{\min})(z) = 1.5001/(2.5010 - (z + 1/z))$ which is the power spectral density of the Carathèodory function $\hat{f}(z) = 0.499\,48((1 + 0.4997z)/(1 - 0.499\,67z))$, a function close to the true function f given in (4). Observe that we have deliberately chosen Ψ such that $q_{\min}$ is intuitively expected to lie on the boundary, in contrary to Example 9 in which $q_{\min}$ is in the interior. To see this, note that f maybe written as $f = a/b$ with $a = (1 + z/2)(z + 1)$ and $b = 2(1 - z/2)(z + 1)$ so that $a_*b + b_*a = \Psi$ and b_*b share a common double root at $z = -1$. In fact, the purpose of this example is to illustrate a case where $q_{\min}$ is at the boundary and also seems to be a stationary point, and to motivate the next theorem. We will consider this example again in Section V.

Remark 11: When $q_{\min}$ is close to or on the boundary, numerical problems can arise when Newton descent is used to find $q_{\min}$. To improve the situation for $q_{\min}$ close to the boundary, the optimization problem can be reformulated and numerically solved by a continuation method [13]. In certain circumstances, the same also applies when $q_{\min}$ is at the boundary. This is discussed in Section V.

As it turns out, the generality of the observation in Example 10 can be formally proven. It is the content of the next theorem.

Theorem 12: The solution of the PRCEP is in $\mathcal{H}^\infty$ if and only if $\mathbb{J}_\Psi$ has a stationary point in the interior or boundary of its domain. If $Q(q_{\min}) \in \partial\mathfrak{Q}_+(n,\mathbb{C})$ and $q_{\min}$ is stationary, then every root of $Q(q_{\min})$ on $\mathbb{T}$ will also be a root of Ψ on $\mathbb{T}$, and the solution of the PRCEP is of order less than n. In this case the solution is given by: $f = a/b$ where $bb_* = Q_+(q_{\min})$, $ab_* + b_*a = \tilde{\Psi}$, and

1) $Q_+(q_{\min}) \in \mathfrak{Q}_+(n,\mathbb{C})$ denotes the pseudopolynomial that is left behind after all factors $\left(z^{\pm 1} - e^{i\phi}\right)$ corresponding to the roots of $Q(q_{\min})$ on $\mathbb{T}$ have been removed from $Q(q_{\min})$;
2) $\tilde{\Psi}$ denotes the pseudopolynomial that is left behind after all factors $\left(z^{\pm 1} - e^{i\phi}\right)$ corresponding to the roots of $Q(q_{\min})$ on $\mathbb{T}$ have been removed from Ψ.

Proof: See the appendix . ■

Therefore, stationarity of the minimizer of $\mathbb{J}_\Psi$ is essentially *a trademark for the boundedness of the solution*: If it is stationary then the solution is bounded, otherwise it is not. We may also show the following sequential continuity result.

Theorem 13: Let $\Psi \in \overline{\mathfrak{Q}_+(n,\mathbb{C})}\backslash\{0\}$ and let $\{\Psi_k\}_{k\ge 1} \subset \overline{\mathfrak{Q}_+(n,\mathbb{C})}\backslash\{0\}$ be a sequence such that $\lim_{k\to\infty} \|\bar{\Psi} - \Psi_k\|_\infty = 0$. If $q_{\min} = \underset{q\in \overline{Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))}}{\arg\min} \mathbb{J}_\Psi(q)$ and $q_{\min,k} = \underset{q\in \overline{Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))}}{\arg\min} \mathbb{J}_{\Psi_k}(q)$, then $\lim_{k\to\infty} \|q_{\min} - q_{\min,k}\| = 0$ and $\lim_{k\to\infty} \|Q(q_{\min}) - Q(q_{\min,k})\|_\infty = 0$.

Proof: See the Appendix . ■

Although one may view the last theorem as a corollary to [6, Th. 3.1] when Ψ and the PCS $c_0, c_1, \ldots, c_n$ are real, it is an interesting result in its own right. Notice that its proof is based solely on properties of $\mathbb{J}_\Psi$ (see Theorem 7) and is independent of Theorem 3. On the other hand, [6, Th. 3.1] was derived based on Theorem 3. In fact, we claim that it is possible to show the converse: Theorem 3 and [6, Th. 3.1] can be derived from Theorems 7 and 13. This interesting ramification of Theorem 13 presents an alternative analysis of the RCEP, *including unbounded solutions*. The complete treatment is given in a separate work [14].

V. Discussion, Extensions, and Application of Results

Our convex optimization based approach is reminiscent of the extensive and abstract generalization of [3] and [5] given in [7], but it may be inspected that the two treatments are not identical and there are two important differences which we shall now discuss. First, the objectives of the two works are different. In [7], the objective is to extend the convex optimization technique to generalize Theorem 3 to the setting of a general class of interpolation problems with a so-called complexity constraint, whereas in the present work we do not attempt to rederive Theorem 3, but rather to use the theorem and/or properties of $\mathbb{J}_\Psi$ when Ψ has zeros on $\mathbb{T}$ (to the best of our knowledge, we are the first to do this) to derive Theorems 8, 12, and 13. Second, our treatment is centered on analysis of boundary properties of the functional $\mathbb{J}_\Psi$ when Ψ may have zeros on $\mathbb{T}$. Although a generalized version of $\mathbb{J}_\Psi$ was formulated in [7], its properties when Ψ has zeros on $\mathbb{T}$ were not investigated. Instead, an alternative route was taken whereby the case $\Psi \in \partial\mathfrak{Q}_+(n,\mathbb{C})\backslash\{0\}$ is treated via analysis of a functional $\mathbb{K}_\Psi$ (see [7, eq. (2.16)]) defined on a set of Schur functions (i.e., functions in $\mathcal{H}$ bounded in magnitude by one) satisfying a certain constraint. In particular, it has been shown that the unique extremal point of $\mathbb{K}_\Psi$ (which, in this case, is a maximizer) is always stationary (see the penultimate part of the proof of [7, Th. 1] on uniqueness of a solution [7, p. 13]). On the other hand, this is *not the case* for $\mathbb{J}_\Psi$. As we have shown, the extremal point of $\mathbb{J}_\Psi$ (which is a minimizer) *need not be stationary*. In fact, it is precisely this unique property of $\mathbb{J}_\Psi$ over $\mathbb{K}_\Psi$ which led us to a characterization of $\mathcal{H}^\infty$ solutions of the RCEP as stated in Theorem 12. Continuing further, we note that for Ψ positive definite on $\mathbb{T}$, $\mathbb{K}_\Psi$ is obtained from a transformation of the functional $\mathbb{I}_\Psi$, the dual of $\mathbb{J}_\Psi$ (see [7, eq. (2.14)]). To derive our results within the development of [7], some results relating $\mathbb{K}_\Psi$ and $\mathbb{J}_\Psi$ need to be established for Ψ nonnegative but not positive definite. Then one should show that the maximizer f of $\mathbb{K}_\Psi$ satisfies $\operatorname*{ess\,inf}_{z\in\mathbb{T}} |1+f(z)| > 0$ (this is equivalent to the RCEP having a bounded solution) if and only if the minimizer of $\mathbb{J}_\Psi$ is stationary. Clearly, these relations have not been considered in [7]. In light of these facts, our results do not obviously follow from [7]. On the contrary, it may be possible to generalize them to the setting of [7] by further analysis of the generalized version of $\mathbb{J}_\Psi$. Indeed, we should keep in mind that our results are specialized to the RCEP, while those of [7] apply to a more general, possibly abstract, class of interpolation problems with a complexity constraint.

We now discuss some practical implications of Theorems 8 and 12. From Theorem 8, we see that in the case where Ψ has zeros on $\mathbb{T}$ and the minimizer of $\mathbb{J}_\Psi$ is in the interior of $Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$ and away from the boundary, the solution can be computed rather quickly and easily by Newton descent. We have illustrated this in Example 9. When the minimizer is close to the boundary, the continuation method of [13] can be applied for good numerical results. For cases where Theorem 12 is applicable, it ought to also be possible to compute solutions by the continuation method. Example 10 shows that even a standard Newton descent method can yield an approximate solution, albeit a crude one. Therefore, it is reasonable to expect the continuation method to give good numerical results for such cases, or cases almost like it (i.e., almost cancellations of insignificant poles lying close to the boundary). Indeed, to support this claim we rework Example 10 using the continuation method.

Example 14: Let c_0, c_1, c_2 and Ψ be as given in Example 10. Applying the continuation method with step length parameter $\varepsilon = 0.01$ (see [13, p. 1196]) yields $b(z) = 1.1547 + 0.5773z - 0.5774z^2$, $a(z) = 0.5774 + 0.8660z + 0.2887z^2$, and the corresponding solution is

$$f(z) = \frac{0.5774 + 0.8660z + 0.2887z^2}{1.1547 + 0.5773z - 0.5774z^2}.$$

Therefore, it could be a worthwhile first step to find a solution by the continuation method of [13]. Since convergence is better understood for that method, this can be beneficial because at present there are no theoretical convergence results for the more general numerical algorithm of [6]. Moreover, the Hessian of the modification of $\mathbb{J}_\Psi$ given in [13] can be inverted in a fast and efficient manner because it has a nice Toeplitz+Hankel (T+H) structure. This kind of structure does not seem to be present in the latter algorithm. There can also be circumstances where one wants to place zeros of Ψ on $\mathbb{T}$, yet one has *a priori* knowledge that the solution is bounded or is only interested in $\mathcal{H}^\infty$ solutions. An example of such a circumstance can be found in [15] and [14]. Thus, our results have extended the utility of the earlier methods of [5], [13].

Our work has largely been motivated by approximation of stochastic systems with noncoercive (i.e., can have zeros on $\mathbb{T}$), possibly nonrational, spectral densities arising in practice. These processes appear in applications such as aircraft control under the influence of turbulence [8] and control of adaptive optics [9]. Specifically, we are interested in new algorithms for computing canonical spectral factors of spectral densities of the type mentioned above. Many spectral factorization algorithms, such as the Bauer and Schur algorithms, which are based on Cholesky decomposition of a semi-infinite Toeplitz matrix [16], [17] are known to converge slowly when the spectral density has zeros close to or on $\mathbb{T}$. Based on solutions of the RCEP (which can be specifically chosen to correspond to pseudopolynomials with roots on $\mathbb{T}$), a new approach to spectral factorization has been proposed in [15], [18].

The main results of this note are readily extendable to the Nevanlinna–Pick interpolation problem with degree constraint by suitable modifications (see [12], [19]) of the proofs presented in this note.

VI. Conclusion

Our main contributions in this note are some new theoretical results on solutions of the RCEP corresponding to $\Psi \in \partial\mathfrak{Q}_+(n,\mathbb{C})\backslash\{0\}$, i.e., the case where the parametrizing pseudopolynomial has zeros on $\mathbb{T}$. In particular, we show that for a solution to be in $\mathcal{H}^\infty$, it is necessary and sufficient that the minimizer of $\mathbb{J}_\Psi$ is stationary. Furthermore, we have shown that some solutions for this case can be computed using methods that have been developed for Ψ which is free of zeros on $\mathbb{T}$, extending the utility of those methods. We also establish the sequential continuity of a certain map based solely on the properties of $\mathbb{J}_\Psi$ and independently of the result on complete parametrization of all solutions of the RCEP. However, full exploitation of this result will be given in [14].

We have also discussed differences between our work and [7] which is also based on convex optimization but applies to a more general class of interpolation problems. We point out some interesting differences between the functionals $\mathbb{J}_\Psi$ and $\mathbb{K}_\Psi$, which are the main object of the analysis of, respectively, our note and [7], and argue that our results do not obviously follow from [7] and that it may be possible to generalize them to the setting of [7].

Although we have specifically treated the RCEP, the results presented here readily extends to the Nevanlinna–Pick interpolation with degree constraint as described in [12].

Appendix

Proof of Theorem 12

We need only prove the initial statement that the solution of the PRCEP is bounded if and only if $\mathbb{J}_\Psi$ has a stationary point in the interior or boundary of its domain. The remaining statements of the theorem all follow from the proof of the initial statement. Let q be such that $Q(q) \in \partial\mathfrak{Q}_+(n,\mathbb{C})$ and such that all the roots of $Q(q)$ on $\mathbb{T}$ are also the roots of Ψ on $\mathbb{T}$. Let the set of all $q \in \mathbb{C}^{n+1}$ satisfying the previous two conditions be denoted by $\mathcal{M}_{n,\Psi}$. First we show that for any $q \in \mathcal{M}_{n,\Psi} \cup Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$, the directional derivatives of $\mathbb{J}_\Psi$ exist in all feasible directions. To this end, for any $q_0 \in Q^{-1}(\overline{\mathfrak{Q}_+(n,\mathbb{C})})$ we define the directional derivative

$$\nabla \mathbb{J}_{q_0-q}(q) = \lim_{h \downarrow 0} \frac{\mathbb{J}_\Psi(q + h(q_0 - q)) - \mathbb{J}_\Psi(q)}{h}.$$

It is easy to check that if $q + h(q_0 - q) \in \partial \mathfrak{Q}_+(n,\mathbb{C})$ for all $0 \leq h < \zeta$ and some $\zeta > 0$, then $Q(q)$ and $Q(q_0)$ must share a root on $\mathbb{T}$. Since all roots of $Q(q)$ on $\mathbb{T}$ are also roots of Ψ on $\mathbb{T}$, it follows that $\Psi/Q(q + h(q_0 - q))$ is uniformly bounded a.e. on $\mathbb{T}$ for all $q_0 \in Q^{-1}(\mathfrak{Q}(n,\mathbb{C}))$ and for all $h > 0$. From the mean-value theorem of calculus, it follows that

$$\Psi(e^{i\theta}) \frac{\log Q(q + h(q_0 - q))(e^{i\theta}) - \log Q(q)(e^{i\theta})}{h} = \frac{\Psi(e^{i\theta}) Q(q_0 - q)(e^{i\theta})}{Q(q)(e^{i\theta}) + \eta(h, e^{i\theta}) Q(q_0 - q)(e^{i\theta})}$$

where $0 < \eta(h, e^{i\theta}) < h$, for all θ except for a finite number for which $Q(q)(e^{i\theta}) = 0$. Since the right hand side of the last equality is uniformly bounded for almost all $(h, e^{i\theta}) \in [0,1] \times \mathbb{T}$, we have that

$$\lim_{h \downarrow 0} \left\langle \Psi, \frac{\log Q(q + h(q_0 - q)) - \log Q(q)}{h} \right\rangle = \left\langle \Psi, \lim_{h \downarrow 0} \frac{\log Q(q + h(q_0 - q)) - \log Q(q)}{h} \right\rangle$$

by the Lebesque Dominated Convergence Theorem [20]. Therefore, for any $q = \mathrm{col}(q_0, \ldots, q_n) \in \mathcal{M}_{n,\Psi} \cup Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$ and any $q_0 = \mathrm{col}(q_{00}, \ldots, q_{0n}) \in Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$ we get

$$\begin{aligned}\nabla_{q_0-q} \mathbb{J}(q) &= \Re\left\{ c^*(q_0 - q) - \sum_{k=0}^{n} \left\langle \frac{\Psi}{Q(q)}, g_{k*} \right\rangle (q_{0k} - q_k) \right\} \\ &= \Re\left\{ \sum_{k=0}^{n} \left(c_k - \left\langle \frac{\Psi}{Q(q)}, g_k \right\rangle \right)^* (q_{0k} - q_k) \right\}\end{aligned}$$

where $g_k(z) = z^k$. Furthermore, we also observe that the directional derivatives do not exist in any feasible direction for all $q \notin \mathcal{M}_{n,\Psi} \cup Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$.

Now, we are ready to prove **necessity**. By Theorem 3 and since the solution of the PRCEP is bounded by hypothesis, we know that there is a unique $\Omega \in \mathfrak{Q}_+(n,\mathbb{C})$ such that $\langle \Psi/\Omega, g_k \rangle = c_k$, for $k = 0, 1, \ldots, n$ and $Q^{-1}(\Omega)$ lies in $\mathcal{M}_{n,\Psi} \cup Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$. Setting $q = Q^{-1}(\Omega)$ then we have that $\nabla \mathbb{J}_{q_0-q}(q) = 0$. Hence, that particular choice of q is a stationary point and it is the unique minimizer of $\mathbb{J}_\Psi$. This establishes the necessity.

We proceed to prove **sufficiency**. Let q be a stationary point of $\mathbb{J}_\Psi$ by letting $\nabla \mathbb{J}_{q_0-q}(q) = 0$ for all $q_0 \in Q^{-1}(\mathfrak{Q}(n,\mathbb{C}))$. Then $q \in \mathcal{M}_{n,\Psi} \cup Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$, otherwise no directional derivative will exist at q, and we have

$$\Re\left\{ \sum_{k=0}^{n} \left(c_k - \left\langle \frac{\Psi}{Q(q)}, g_k \right\rangle \right)^* (q_{0k} - q_k) \right\} = 0. \tag{5}$$

Now, for any $q \in \mathcal{M}_{n,\Psi} \cup Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$ we may write $Q(q) = Q_+(q) Q_0(q)$ where $Q_0(q)$ is a pseudopolynomial with all its roots on $\mathbb{T}$ or is identically equal to 1 if no such roots exist, while $Q_+(q)$ is a pseudopolynomial which does not have roots on the boundary. Because all the roots of $Q(q)$ which are on the boundary are also roots of Ψ by hypothesis, we may write $\Psi = \tilde{\Psi}(q) Q_0(q)$, where $\tilde{\Psi}(q)$ is a pseudopolynomial defined by $\tilde{\Psi}(q) = \Psi/Q_0(q)$. After inserting the two identities into (5), we obtain

$$\Re\left\{ \sum_{k=0}^{n} \left(c_k - \left\langle \frac{\tilde{\Psi}(q)}{Q_+(q)}, g_k \right\rangle \right)^* (q_{0k} - q_k) \right\} = 0. \tag{6}$$

However, (6) holds for all $q_0 \in \overline{Q^{-1}(\mathfrak{Q}(n,\mathbb{C}))}$. Therefore, by inspection (e.g., see proof of [19, Lemma 5.1]) we must have

$$c_k - \left\langle \frac{\tilde{\Psi}(q)}{Q_+(q)}, g_k \right\rangle = 0 \Longleftrightarrow \left\langle \frac{\tilde{\Psi}(q)}{Q_+(q)}, g_k \right\rangle = c_k$$

for $k = 0, 1, \ldots, n$. Therefore, there is a unique Carathèodory function f such that $(f + f_*)(e^{i\theta}) = \tilde{\Psi}(q)(e^{i\theta})/(Q_+(q)(e^{i\theta}))$, f satisfies the interpolation constraints, and f is bounded. Hence, we have shown sufficiency. Note the cancellation that takes place if $Q(q)$ has roots on the boundary. In this case the solution f will be of degree less than n. ■

Proof of Theorem 13

For $r > 0$, define the compact sets $B_r(q_{\min}) = \{q \in \mathbb{R} \times \mathbb{C}^n : \|q - q_{\min}\| \leq r\}$ and $S_r(q_{\min}) = \partial B_r(q_{\min})$. Also define the compact sets $X_r(q_{\min}) = B_r(q_{\min}) \cap Q^{-1}\left(\overline{\mathfrak{Q}_+(n,\mathbb{C})}\right)$ and $Y_r(q_{\min}) = \partial X_r(q_{\min})$. We prove that given any $\epsilon > 0$ small enough such that $0 \notin X_\epsilon(q_{\min})$, there is a $K(\epsilon) \geq 1$ such that $q_{\min,k} \in B_\epsilon(q_{\min})\ \forall k > K(\epsilon)$. First, we observe that

$$\begin{aligned}|\mathbb{J}_\Psi(q) - \mathbb{J}_{\Psi_k}(q)| &\leq \langle |\Psi - \Psi_k|, |\log Q(q)| \rangle \\ &\leq \|\Psi - \Psi_k\|_\infty \langle \mathbf{1}, |\log Q(q)| \rangle, \text{ if } q \neq 0\end{aligned}$$

where $\mathbf{1} : z \mapsto 1\ \forall z \in \mathbb{T}$. If we define $D = \max_{q \in X_\epsilon(q_{\min})} \langle \mathbf{1}, |\log Q(q)| \rangle$, we have that $\forall q \in X_\epsilon(q_{\min})$

$$|\mathbb{J}_\Psi(q) - \mathbb{J}_{\Psi_k}(q)| \leq D \|\Psi - \Psi_k\|_\infty$$

or, more explicitly

$$\begin{aligned}\mathbb{J}_\Psi(q) - D\|\Psi - \Psi_k\|_\infty &\leq \mathbb{J}_{\Psi_k}(q) \\ &\leq \mathbb{J}_\Psi(q) + D\|\Psi - \Psi_k\|_\infty.\end{aligned} \tag{7}$$

For any $r > 0$, define $Z_r(q_{\min}) = Y_r(q_{\min})$ if $q_{\min} \in Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$ and $Z_r(q_{\min}) = S_r(q_{\min}) \cap \overline{Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))}$ if $q_{\min} \in \partial Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$. Notice that $Z_r(q_{\min})$ is a compact set. Choose any $\epsilon > 0$ small enough such that $0 \notin X_\epsilon(q_{\min})$ and such that $Z_\epsilon(q_{\min}) \subset Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$ if $q_{\min} \in Q^{-1}(\mathfrak{Q}_+(n,\mathbb{C}))$. Next, for any $q \neq q_{\min}$ define the unit vector $u_q = (q - q_{\min})/\|q - q_{\min}\|$, and for any $q \in Z_\epsilon(q_{\min})$ and any $0 < d < \epsilon$ define the functions $L_1(q,d) = \mathbb{J}_\Psi(q) - \mathbb{J}_\Psi(q_{\min} + d u_q)$ and $L_2(q,d) = \mathbb{J}_\Psi(q_{\min} + d u_q) - \mathbb{J}_\Psi(q_{\min})$. Clearly, from the strict convexity of $\mathbb{J}_\Psi$, $L_1(\cdot, d)$ and $L_2(\cdot, d)$ are continuous, positive-valued (> 0) functions on $Z_\epsilon(q_{\min})$. Furthermore, define $\delta_i(d) = \min_{q \in Z_\epsilon(q_{\min})} L_i(q,d)$ for $i = 1, 2$. Observe that $\delta_i(d) > 0$ for $i = 1, 2$, for if it is not then $\exists q \in Z_\epsilon(q_{\min})$ such that $L_1(q,d) = 0$ and/or $L_2(q,d) = 0$, contradicting the fact that they are positive-valued on $Z_\epsilon(q_{\min})$. Let us now choose a fixed $d \in (0, \epsilon)$. Choose $K_d(\epsilon)$ (note the dependence on d) large enough such that $\|\Psi - \Psi_k\|_\infty < \min\{\delta_1(d), \delta_2(d)\}/3D$ for all $k > K_d(\epsilon)$, then using (7) one easily gets that for any $q \in Z_\epsilon(q_{\min})$

$$\mathbb{J}_{\Psi_k}(q_{\min}) < \mathbb{J}_{\Psi_k}(q_{\min} + d u_q) < \mathbb{J}_{\Psi_k}(q)\ \forall k > K_d(\epsilon). \tag{8}$$

From (8) and the strict convexity of $\mathbb{J}_{\Psi_k}$ for all k, it follows that $q_{\min,k} \in X_\epsilon(q_{\min}) \backslash Z_\epsilon(q_{\min})$ for all $k > K_d(\epsilon)$.

Summarizing, we have shown that for every $\epsilon > 0$ such that $0 \notin X_{\epsilon/2}(q_{\min})$, $\exists K(\epsilon/2)$ such that for all $k > K(\epsilon/2)$, $q_{\min,k} \in X_{\epsilon/2}(q_{\min})$ is in the interior of $B_\epsilon(q_{\min})$, or in other words,

$\lim_{k\to\infty} \|q_{\min} - q_{\min,k}\| = 0$. From the last result, it follows immediately that $\lim_{k\to\infty} \|Q(q_{\min}) - Q(q_{\min,k})\|_\infty = 0$. This concludes the proof. ■

ACKNOWLEDGMENT

This work was initiated while the first author was at the University of Twente. The authors would like to thank the Associate Editor and anonymous reviewers for their helpful feedback.

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Adaptive Output-Feedback Tracking of Stochastic Nonlinear Systems

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***Abstract*—We address the adaptive stabilization and tracking problems for a class of output feedback canonical systems driven by Wiener noises of unknown covariance. Filtered transformation and backstepping techniques are employed in the stochastic control design. We obtain two adaptive controllers that guarantee the global stability in probability for vanishing perturbations or the input-to-state stability in probability for nonvanishing perturbations respectively. The tracking error can converge to a small residual set around the origin in the sense of mean quartic value.**

***Index Terms*—Stability in probability, stochastic adaptive stabilization, stochastic disturbance attenuation.**

I. INTRODUCTION

After a success of constructive control design for deterministic systems, stochastic nonlinear control has attracted attention recently. Some nonlinear control design methods such as Sontag's stabilization formula, backstepping techniques and nonlinear optimality were extended to the case of stochastic nonlinear systems [1]–[3], [8], [13], where the exogenous disturbances are Wiener noises. The main technical obstacle in the Lyapunov design for stochastic systems is that the Itô stochastic differentiation involves not only the gradient but also the higher order Hessian term. Pan and Basar [13] were the first to solve the stochastic stabilization problem for the class of strict-feedback systems based on a risk-sensitive cost criterion, their result guarantees global asymptotic stability in probability. Deng and Krstić [1], [2] suggested a quartic Lyapunov function to give a backstepping design for stochastic strict-feedback systems and then extended the results on inverse optimal control to the stochastic case. A continuation of these contributions is the adaptive stabilization [6] of the stochastic parametric strict-feedback systems in the presence of uncertain noises, where the unknown parameters are both system parameters and a reduced covariance parameter.

Apart from the strict-feedback model mentioned above, a class of systems in the output feedback canonical form proposed by Marino and Tomei is another well-known model. Geometric conditions which characterize the class of nonlinear systems that can be transformed into this form were given in [11]. It was shown in [10] and [12] that the class of systems which are globally stabilizable by output feedback is not much broader than this canonical form. The filtered transformation is the key technique to design an adaptive output feedback control for these systems. Krstić *et al.* employ another filtered transformation (K-filters) and use tuning functions scheme to revamp the control design.

In this note, we address the output feedback global adaptive stabilization and tracking problem for the output-feedback systems disturbed by Wiener noise of unknown covariance. For the case of vanishing perturbations, that is, the equilibrium point is preserved in the presence of perturbations (noises and other uncertain terms), our goal is the adaptive stabilization in probability. For the case of nonvanishing

Manuscript received March 14, 2003; revised May 18, 2004, January 8, 2005, and October 9, 2005. Recommended by Associate Editor R. Boel. This work was supported by the Specialized Research Fund for the Doctoral Program of Higher Education of China under Grant 20050358044.

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Digital Object Identifier 10.1109/TAC.2005.863501