Numerical Functional Analysis and Optimization, 28(5–6):717–736, 2007

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560701380981

CRANK–NICOLSON SCHEME FOR ABSTRACT LINEAR SYSTEMS

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□ *This paper studies the Crank–Nicolson discretization scheme for abstract differential equations on a general Banach space. We show that a time-varying discretization of a bounded analytic C_0-semigroup leads to a bounded discrete-time system. On Hilbert spaces, this result can be extended to all bounded C_0-semigroups for which the inverse generator generates a bounded C_0-semigroup. The presentation is based on C_0-semigroup theory and uses a functional analysis approach.*

Keywords C_0-semigroup; Cayley transform; Crank–Nicolson scheme; Infinite-dimensional system.

AMS Subject Classification 34G10; 65J10; 46N20; 46N40; 93C25.

1. INTRODUCTION

Consider the abstract differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{1.1}$$

on a Banach space X. Hence $x(\cdot)$ is a Banach space valued function. This equation may represent a partial differential equation. Throughout this paper, we assume that (1.1) possesses for every initial condition x_0 a unique (weak) solution, which depends continuously on this initial condition. In other words, we assume that A is the infinitesimal generator of a C_0-semigroup; see for more information [6, 9, 15, 18, 26].

When solving the differential equation, some form of approximation/discretization will be necessary. The Crank–Nicolson scheme

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makes the following approximation, see for example [2, 22]. The time derivative is replaced by

$$\dot{x}(t) \approx \frac{x(t+\tau) - x(t)}{\tau}, \tag{1.2}$$

whereas the state at time t is replaced by

$$x(t) \approx \frac{x(t+\tau) + x(t)}{2}, \tag{1.3}$$

where τ is the time-step. Replacing the expression in (1.1) by the above approximations gives

$$x(t+\tau) = \left(I - \frac{\tau}{2}A\right)^{-1}\left(I + \frac{\tau}{2}A\right)x(t),$$

where we have assumed that $\left(I - \frac{\tau}{2}A\right)$ is invertible. Thus by defining $t = n\tau$, with $n \in \mathbb{N}$, we obtain the difference equation

$$x_d(n+1) = A_d x_d(n), \quad x(0) = x_0, \tag{1.4}$$

with $A_d = \left(I - \frac{\tau}{2}A\right)^{-1}\left(I + \frac{\tau}{2}A\right)$.

We have derived the above difference equation using ideas from numerical analysis. However, the same operators appear when applying the Cayley transform to the operator A, see, for example, Gomilko [12] or Kalton et al. [16]. Furthermore, this is also a well-known equation in system theory, see, for example, Chapter 12 in [23].

In numerical analysis, the approximation scheme (1.4) is widely used, because if A is a matrix, then the solutions of (1.1) are bounded/stable if and only if the solutions of (1.4) are bounded/stable. This result is independent of the time step τ. Of course, the approximation error $x(n\tau) - x_d(n)$ will depend on this time step. For dissipative operators A on a Hilbert space, the same result holds. On a general Banach space, there are examples showing that (1.1) can have bounded solutions, whereas the norm $\|A_d^n\|$ is unbounded in n. Thus an interesting and open problem is which conditions on A and X are necessary and sufficient such that the Crank–Nicolson discretization has the same stability properties as the original equation (1.1).

From the independent papers [3, 14] with different key ideas, we have the following result.

Theorem 1.1. *Let A and A^{-1} each be the infinitesimal generator of a bounded semigroup on the Hilbert space H. Then the operator $A_d := (I + A)(I - A)^{-1}$ is power bounded, that is,* $\sup_{n\in\mathbb{N}} \|A_d^n\| < \infty$.

It is clear that (after re-scaling), this precisely gives that the Crank–Nicolson scheme has in Hilbert spaces the property that all solutions of (1.4) are bounded provided the solutions of (1.1) and of $\dot{x}(t) = A^{-1}x(t)$ are bounded.

One of our questions is whether this results still holds when the time step τ, see (1.2) and (1.3), is nonconstant. By using the Lyapunov approach as in Guo and Zwart [14], we show the following:

Theorem 1.2. *Let A and A^{-1} each be the infinitesimal generator of a bounded C_0-semigroup on the Hilbert space H. Consider the time-varying difference equation*

$$x_d(n+1) = \left(I - \frac{\tau_n}{2}A\right)^{-1}\left(I + \frac{\tau_n}{2}A\right)x_d(n), \quad x_d(0) = x_0. \tag{1.5}$$

If $0 < \inf \tau_n \le \sup \tau_n < \infty$, then the solutions of (1.5) are bounded in n. Furthermore, if the solutions of (1.1) are stable, that is, $lim_{t\to\infty} x(t) = 0$ for all x_0, then under the above conditions on τ_n the same holds for the solutions of (1.5).

The proof of this theorem will be presented in Section 2. From the above two theorems two questions arise. Firstly, is A^{-1} always the infinitesimal generator of a bounded C_0-semigroup if A generates a bounded C_0-semigroup? Secondly, do Theorems 1.1 and 1.2 hold in a general Banach space? For analytic C_0-semigroups, one can find positive answers, see [7, 28] for the first question and [5, 20] for the second question. We present a new and very simple proof of the following theorem, see also [21].

Theorem 1.3. *Let A be the infinitesimal generator of a bounded analytic C_0-semigroup on a Banach space X. Consider the time-varying difference equation (1.5). If $0 < \inf \tau_n \le \sup \tau_n < \infty$, then the solutions of (1.5) are bounded in n.*

Solving by induction the difference equation (1.5), we see that the state at time n is given by

$$x_d(n) = \left(\prod_{k=0}^{n-1} R_0(k)\right)x_d(0),$$

where $R_0(k) = \left(I - \frac{\tau_k}{2}A\right)^{-1}\left(I + \frac{\tau_k}{2}A\right)$. Hence $\|x_d(n)\|$ is uniformly bounded (in n) if and only if the operators $\prod_{k=0}^{n-1} R_0(k)$ are uniformly bounded. One sees that the operators $R_0(k)$ mutually commute. From this the following arises as a natural question.

Let a sequence of operators $\{R(j)\}_{j\in\mathbb{N}}$ be such that

- they mutually commute, that is, $R(j)R(k) = R(k)R(j)$,
- there exists an M such that $\sup_{j,k\in\mathbb{N}} \|R(j)^k\| \le M$.

Are these conditions sufficient for the sequence $\|\prod_{j=1}^{k} R(j)\|$ to be uniformly bounded?

Unfortunately, the answer to this question is no. Consider, for instance, the following example. As Banach space X we take the space ℓ^{∞}, the space of bounded sequences with norm $\|x\| = \sup_n |x(n)|$, where $x = (x(1), x(2), \dots, x(n), \dots)$. As operators we define $R(1) = I$, and for $j \geq 2$ we define $R(j)x = y$, with

$$\begin{aligned} y(1) &= x(1) + x(j), \\ y(2) &= x(2), \\ &\vdots \\ y(j-1) &= x(j-1), \\ y(j) &= 0, \\ y(j+1) &= x(j+1), \\ &\vdots \end{aligned}$$

For any $j \in \mathbb{N}$ it is clear that $\|R(j)^k\| \leq 2$ for all $k \in \mathbb{N}$. Furthermore, $R(k)R(j) = R(j)R(k)$ for any k, j. So this sequence of operators satisfies the two conditions. However, if we define x_m as $x_m(n) = 1$ for all n, then $\|x_m\| = 1$, but $\|R(1)R(2)\dots R(k)x_k\| = k$ and so $\|\prod_{j=1}^{k} R(j)\|$ is not uniformly bounded.

The above example shows that care should be taken when going from time-invariant operators to time-varying ones.

The proof of Theorem 1.2 is presented in the following section. In Section 3, we study the properties of the inverse of the generator for a general Banach space. Among others we show that A^{-1} always generates a once integrated semigroup. In Section 4, we return to the discretization of differential equations. In Guo and Zwart [14] it is shown that if A generates a bounded C_0-semigroup on the Hilbert space H, then for every $x_0 \in D(A)$ the solution of (1.4) is bounded. In Section 4, we show that this no longer holds in a general Banach space. Especially, we show that it does not hold in L^p for $p \neq 2$. At the end of this section, one may find the proof of Theorem 1.3.

Let $\mathscr{B}(X)$ denote a Banach algebra of all bounded operators from X to X, and let $\mathscr{C}(X)$ denote the set of densely defined, closed operator on X. It is well-known that every infinitesimal generator of a C_0-semigroup is an element of $\mathscr{C}(X)$. If A generates a bounded C_0-semigroup, then $(0, \infty) \subset \rho(A)$ (the resolvent set of A), and hence $I - \frac{\tau}{2}A$ has an inverse that is an element of $\mathscr{B}(X)$. Throughout this paper, we assume that zero is not a point spectrum of A, and that $A^{-1} \in \mathscr{C}(X)$. In other words, zero lies in the resolvent set, or in the continuous spectrum of A. From Goldstein [10, Theorem 1.8.20], we have that if A generates a bounded C_0-semigroup on a reflexive Banach space X and if zero is not a point spectrum of A, then the range of A is dense in X.

2. TIME-VARYING DISCRETIZATION

The aim of this section is to prove Theorem 1.2. For this proof, we need two lemmas.

Lemma 2.1. *Consider on the Hilbert space H for $n \geq 0$ the difference equation*

$$x_d(n+1) = A_{d,n+1}x_d(n), \quad x_d(0) = x_{d,0}. \tag{2.1}$$

If for all $r \in (0,1)$ there exists positive operators $R(r), \widetilde{R}(r) \in \mathcal{B}(H)$ such that for all $n \in \mathbb{N}$

$$r^2 A_{d,n}^* R(r) A_{d,n} - R(r) \leq -I, \tag{2.2}$$

$$r^2 A_{d,n} \widetilde{R}(r) A_{d,n}^* - \widetilde{R}(r) \leq -I \tag{2.3}$$

and

$$M := \sup_{r\in(0,1)} (1-r)\|R(r)\| < \infty, \quad \widetilde{M} := \sup_{r\in(0,1)} (1-r)\|\widetilde{R}(r)\| < \infty, \tag{2.4}$$

then

$$\sup_{n\in\mathbb{N}} \|x_d(n)\| \leq e\sqrt{M\widetilde{M}}\|x_d(0)\|. \tag{2.5}$$

If (2.2)–(2.4) *hold, and if*

$$\lim_{r\uparrow 1}(1-r)\langle x_{d,0}, R(r)x_{d,0}\rangle = 0,$$

then the solution of (2.1) *for the initial condition $x_{d,0}$ converges to zero, that is,* $\lim_{n\to\infty} x_d(n) = 0$.

Proof. The proof is divided into several steps. We define for $K > k \geq 0$ the operator $\Phi(r; k, K)$ as

$$\Phi(r; k, K) = r^{K-k} A_{d,K} \cdot \ \ldots \ \cdot A_{d,k+2} A_{d,k+1}. \tag{2.6}$$

For $K = k$, we define $\Phi(r; K, K) = I$. It is clear that Φ is the state transition map from time k to time K multiplied by r^{K-k}, that is, $x_d(K) = r^{-(K-k)}\Phi(r; k, K)x_d(k)$.

Step 1. Using equation (2.2), we find that for $K \geq 0$

$$\begin{aligned}\Phi(r;0,K)^*\Phi(r;0,K) &\leq \Phi(r;0,K)^*[-r^2A_{d,K+1}^*R(r)A_{d,K+1} + R(r)]\Phi(r;0,K)\\ &= -\Phi(r;0,K+1)^*R(r)\Phi(r;0,K+1)\\ &\quad + \Phi(r;0,K)^*R(r)\Phi(r;0,K). \end{aligned}\tag{2.7}$$

Step 2. For all $n \in \mathbb{N}$, we have that

$$\begin{aligned}\sum_{k=0}^{n} r^{2k}\|x_d(k)\|^2 &= \sum_{k=0}^{n} r^{2k}\langle x_d(k), x_d(k)\rangle\\ &= \sum_{k=0}^{n}\langle x_d(0), \Phi(r;0,k)^*\Phi(r;0,k)x_d(0)\rangle\\ &\leq \sum_{k=0}^{n}\langle x_d(0), (-\Phi(r;0,k+1)^*R(r)\Phi(r;0,k+1)\\ &\qquad + \Phi(r;0,k)^*R(r)\Phi(r;0,k))x_d(0)\rangle\\ &= -\langle x_d(0), \Phi(r;0,n+1)^*R(r)\Phi(r;0,n+1)x_d(0)\rangle\\ &\quad + \langle x_d(0), R(r)x_d(0)\rangle,\end{aligned}$$

where we have used (2.7). Because $R(r) \geq 0$, we conclude that for $n \geq 0$

$$\sum_{k=0}^{n} r^{2k}\|x_d(k)\|^2 \leq \langle x_d(0), R(r)x_d(0)\rangle. \tag{2.8}$$

Step 3. Let $n \in \mathbb{N}$ and consider the backward, dual system of (2.1), that is, for $k = 0, \ldots, n-1$,

$$\tilde{x}_d(k) = A_{d,k+1}^*\tilde{x}_d(k+1), \quad \tilde{x}_d(n) = \tilde{x}_{d,n}. \tag{2.9}$$

It is not hard to see that $\tilde{x}_d(k) = r^{k-n}\Phi(r;k,n)^*\tilde{x}_{d,n}$. Using equation (2.2), we find similarly as in Step 1, that for $k \geq 1$

$$\begin{aligned}\Phi(r;k,n)\Phi(r;k,n)^* &\leq -\Phi(r;k-1,n)\widetilde{R}(r)\Phi(r;k-1,n)^*\\ &\quad + \Phi(r;k,n)\widetilde{R}(r)\Phi(r;k,n)^*.\end{aligned}\tag{2.10}$$

If we define $A_{d,0} = A_{d,1}$, then we have that (2.10) holds for $k \geq 0$. Hence for $\tilde{x}_d(k)$, the following inequality holds:

$$\begin{aligned}
\sum_{k=0}^{n} r^{2(n-k)} \|\tilde{x}_d(k)\|^2 &\leq \sum_{k=0}^{n} \langle \tilde{x}_d(n), \Phi(r;k,n)\Phi(r;k,n)^* \tilde{x}_d(n)\rangle \\
&\leq \sum_{k=0}^{n} \langle \tilde{x}_d(n), (-\Phi(r;k-1,n)\widetilde{R}(r)\Phi(r;k-1,n)^* \\
&\qquad + \Phi(r;k,n)\widetilde{R}(r)\Phi(r;k,n)^*)\tilde{x}_d(n)\rangle \\
&= -\langle \tilde{x}_d(n), \Phi(r,-1,n)\widetilde{R}(r)\Phi(r;-1,n)^* \tilde{x}_d(n)\rangle \\
&\qquad + \langle \tilde{x}_d(n), \widetilde{R}(r)\tilde{x}_d(n)\rangle \\
&\leq \langle \tilde{x}_d(n), \widetilde{R}(r)\tilde{x}_d(n)\rangle,
\end{aligned}$$

because $\widetilde{R}(r) \geq 0$.

Step 4. Let $n \in \mathbb{N}$. Then following [25], we have that

$$\begin{aligned}
|(n+1)r^n \langle \tilde{x}_d(n), x_d(n)\rangle| &= \left| \sum_{k=0}^{n} \langle \tilde{x}_d(n), \Phi(r,0,n)x_d(0)\rangle \right| \\
&= \left| \sum_{k=0}^{n} \langle \Phi(r,k,n)^* \tilde{x}_d(n), \Phi(r,0,k)x_d(0)\rangle \right| \\
&\leq \sqrt{\sum_{k=0}^{n} r^{2(n-k)} \|\tilde{x}_d(k)\|^2 \sum_{k=0}^{n} r^{2k} \|x_d(k)\|^2} \\
&\leq \sqrt{\langle \tilde{x}_d(n), \widetilde{R}(r)\tilde{x}_d(n)\rangle} \sqrt{\langle x_d(0), R(r)x_d(0)\rangle} \\
&\leq \frac{1}{1-r} \sqrt{M\widetilde{M}} \|\tilde{x}_d(n)\| \|x_d(0)\|, \qquad (2.11)
\end{aligned}$$

where we have used (2.8) and (2.10). Choosing in the above expression $r = \frac{n}{n+1}$ we find that

$$|\langle \tilde{x}_d(n), x_d(n)\rangle| \leq \left(1 + \frac{1}{n}\right)^n \sqrt{M\widetilde{M}} \|\tilde{x}_d(n)\| \|x_d(0)\|.$$

Thus

$$\|x_d(n)\| \leq e\sqrt{M\widetilde{M}} \|x_d(0)\|.$$

Step 5. From equations (2.4) and (2.11) we find that

$$|(n+1)r^n\langle\tilde{x}_d(n), x_d(n)\rangle| \le \sqrt{\frac{\widetilde{M}}{1-r}}\|\tilde{x}_d(n)\|\sqrt{\langle x_d(0), R(r)x_d(0)\rangle}.$$

Let ε be a positive number, and choose $r_\varepsilon \in (0,1)$ such that $\langle x_{d,0}, R(r)x_{d,0}\rangle \le \frac{\varepsilon}{1-r}$ for $r \in (r_\varepsilon, 1)$. Without loss of generality, we can choose r_ε such that $\frac{r_\varepsilon}{1-r_\varepsilon}$ is an integer. Then for these r's

$$|(n+1)r^n\langle\tilde{x}_d(n), x_d(n)\rangle| \le \sqrt{\frac{\widetilde{M}}{1-r}}\|\tilde{x}_d(n)\|\sqrt{\frac{\varepsilon}{1-r}}.$$

Choosing $n \ge \frac{r_\varepsilon}{1-r_\varepsilon}$ or equivalently, $r = \frac{n}{n+1}$, we obtain that

$$|\langle\tilde{x}_d(n), x_d(n)\rangle| \le \left(1+\frac{1}{n}\right)^n\sqrt{\widetilde{M}}\|\tilde{x}_d(n)\|\sqrt{\varepsilon}.$$

Thus for these time instants, we have

$$\|x_d(n)\| \le e\sqrt{\widetilde{M}}\sqrt{\varepsilon}.$$

Because we can do this for every positive ε, we have proved the assertion.

□

We use the above lemma to prove that the Crank–Nicolson approximation is bounded for any time-discretization of the differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{2.12}$$

provided (2.12) has only bounded solutions and the same holds for A^{-1}.

Lemma 2.2. *Let A and A^{-1} be generators of bounded C_0-semigroups, and let $A_d(h) := (I+hA)(I-hA)^{-1}$. If $h \in [h_{\min}, h_{\max}]$ with $0 < h_{\min} \le h_{\max} < \infty$, then for every $r \in (0,1)$, there exist positive operators $R(r), \widetilde{R}(r) \in \mathcal{B}(H)$ that are independent of h and that satisfy*

$$r^2 A_d(h)^* R(r) A_d(h) - R(r) \le -I, \tag{2.13}$$

$$r^2 A_d(h)\widetilde{R}(r) A_d(h)^* - \widetilde{R}(r) \le -I, \tag{2.14}$$

and

$$M := \sup_{r\in(0,1)} (1-r)\|R(r)\| < \infty, \quad \widetilde{M} := \sup_{r\in(0,1)} (1-r)\|\widetilde{R}(r)\| < \infty. \tag{2.15}$$

Proof. From Guo and Zwart [14], we know that the boundedness of the C_0-semigroup generated by A implies the existence of positive-valued operators $Q(\sigma)$ and $\widetilde{Q}(\sigma)$ such that

$$(\sigma I - A)^* Q(\sigma) + Q(\sigma)(\sigma I - A) = I \quad \sigma > 0, \tag{2.16}$$

$$(\sigma I - A)\widetilde{Q}(\sigma) + \widetilde{Q}(\sigma)(\sigma I - A)^* = I \quad \sigma > 0, \tag{2.17}$$

$$\sup_{\sigma>0} \sigma \|Q(\sigma)\| \le M_1, \quad \sup_{\sigma>0} \sigma \|\widetilde{Q}(\sigma)\| \le \widetilde{M}_1. \tag{2.18}$$

Similarly, we have that the boundedness of the C_0-semigroup generated by A^{-1} implies the existence of positive operator valued functions $S(\sigma)$, $\widetilde{S}(\sigma)$ such that

$$(\sigma I - A^{-1})^* S(\sigma) + S(\sigma)(\sigma I - A^{-1}) = I \quad \sigma > 0, \tag{2.19}$$

$$(\sigma I - A^{-1})\widetilde{S}(\sigma) + \widetilde{S}(\sigma)(\sigma I - A^{-1})^* = I \quad \sigma > 0, \tag{2.20}$$

$$\sup_{\sigma>0} \sigma \|S(\sigma)\| \le M_2, \quad \sup_{\sigma>0} \sigma \|\widetilde{S}(\sigma)\| \le \widetilde{M}_2. \tag{2.21}$$

We use (2.16) and (2.19) to show that (2.13) holds. The proof of (2.14) will be very similar.

Using the notation $(I - hA)^{-*}$ to denote the adjoint of $(I - hA)^{-1}$, and substituting $(I - hA)^{-1}(I + hA)$ in the left-hand side of (2.13), we obtain

$$\begin{aligned}
& r^2 A_d(h)^* R(r) A_d(h) - R(r) \\
&= (I - hA)^{-*}[r^2 (I + hA)^* R(r)(I + hA) \\
&\qquad - (I - hA)^* R(r)(I - hA)](I - hA)^{-1} \\
&= (I - hA)^{-*}[(r^2 - 1)[R(r) + h^2 A^* R(r) A] \\
&\quad + (r^2 + 1)[hA^* R(r) + R(r) hA]](I - hA)^{-1} \\
&= (r^2 + 1)(I - hA)^{-*}\left[\frac{r^2 - 1}{r^2 + 1}[R(r) + h^2 A^* R(r) A] + hA^* R(r) + R(r) hA\right] \\
&\quad \times (I - hA)^{-1}.
\end{aligned} \tag{2.22}$$

Next we define σ as $\frac{1 - r^2}{2(r^2 + 1)}$ and $R(r) = 2R_1(r)$, where

$$R_1(r) = \frac{1}{h_{\min}} Q\left(\frac{\sigma}{h_{\max}}\right) + h_{\max} S(h_{\min}\sigma). \tag{2.23}$$

Substituting $R_1(r)$ in the expression within the square brackets gives

$$
\begin{aligned}
&\frac{r^2-1}{r^2+1}[R_1(r) + h^2 A^* R_1(r) A] + hA^* R_1(r) + R_1(r) hA \\
&\quad = \left[-2\sigma Q\left(\frac{\sigma}{h_{\max}}\right) - 2\sigma h^2 A^* Q\left(\frac{\sigma}{h_{\max}}\right) A + hA^* Q\left(\frac{\sigma}{h_{\max}}\right) \right. \\
&\qquad \left. + Q\left(\frac{\sigma}{h_{\max}}\right) hA \right] \frac{1}{h_{\min}} \\
&\qquad + [-2\sigma S(h_{\min}\sigma) - 2\sigma h^2 A^* S(h_{\min}\sigma) A + hA^* S(h_{\min}\sigma) + S(h_{\min}\sigma) hA] h_{\max} \\
&\quad = \left[-2\sigma Q\left(\frac{\sigma}{h_{\max}}\right) - 2\sigma h^2 A^* Q\left(\frac{\sigma}{h_{\max}}\right) A - hI + 2\frac{h\sigma}{h_{\max}} Q\left(\frac{\sigma}{h_{\max}}\right) \right] \frac{1}{h_{\min}} \\
&\qquad + [-2\sigma S(h_{\min}\sigma) - 2\sigma h^2 A^* S(h_{\min}\sigma) A - hA^* A + 2h\sigma h_{\min} A^* S(h_{\min}\sigma) A] h_{\max} \\
&\quad \le -\frac{h}{h_{\min}} I + \frac{2\sigma}{h_{\min}} \left[\frac{h}{h_{\max}} - 1 \right] Q\left(\frac{\sigma}{h_{\max}}\right) - h\, h_{\max} A^* A \\
&\qquad + 2\sigma h h_{\max} [h_{\min} - h] A^* S(h_{\min}\sigma) A \\
&\quad \le -\frac{h}{h_{\min}} I - h h_{\max} A^* A \\
&\quad \le -I - h^2 A^* A,
\end{aligned}
\tag{2.24}
$$

where we have used (2.16), (2.19), and twice the fact that $h_{\min} \le h \le h_{\max}$. Combining (2.24) with (2.22) gives for $R(r)$

$$
\begin{aligned}
r^2 A_d(h)^* R(r) A_d(h) - R(r) &\le 2(r^2+1)(I - hA)^{-*}[-I - h^2 A^* A](I - hA)^{-1} \\
&\le -(r^2+1) I \le -I.
\end{aligned}
\tag{2.25}
$$

because $\|x\|^2 \le 2\|(I - hA)^{-1}x\|^2 + 2\|hA(I - hA)^{-1}x\|^2$.

It remains to show that

$$\sup_{r\in(0,1)} (1-r)\|R(r)\| < \infty.$$

We have

$$
\begin{aligned}
\sup_{r\in(0,1)} (1-r)\|R(r)\| &\le \sup_{r\in(0,1)} \frac{1-r^2}{2(r^2+1)} 2\|R(r)\| \\
&\le \sup_{\sigma\in(0,\frac{1}{2})} 4\sigma \left\| \frac{1}{h_{\min}} Q\left(\frac{\sigma}{h_{\max}}\right) + h_{\max} S(h_{\min}\sigma) \right\| \\
&\le 4\frac{h_{\max}}{h_{\min}} M_1 + 4\frac{h_{\max}}{h_{\min}} M_2,
\end{aligned}
$$

where we have used (2.15) and (2.21). Hence we have constructed a solution $R(r)$ of (2.13) which satisfies the first inequality in (2.15). Similarly, by using (2.17) and (2.20), we can construct a solution of (2.14) satisfying the second inequality of (2.15). Hence we have proved the lemma. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. The first part of the theorem follows directly from Lemmas 2.1 and 2.2. To prove the stability, we need that if A generates a stable C_0-semigroup, and if A^{-1} generates a bounded C_0-semigroup, then this C_0-semigroup is stable as well. Furthermore, from [14] we know that if A generates a stable C_0-semigroup, then there exists a solution of (2.16) such that for all $x \in H$

$$\lim_{\sigma \downarrow 0} \sigma \langle x, Q(\sigma)x \rangle = 0.$$

Because A^{-1} also generates a stable C_0-semigroup, we have that a similar result holds for $S(\sigma)$, see (2.19). Because we can choose

$$R(r) = \frac{2}{h_{\min}} Q\left(\frac{\sigma}{h_{\max}}\right) + 2h_{\max} S(h_{\min}\sigma)$$

we see that the assertion follows from Lemma 2.1. □

The proof of Theorem 1.3 is given at the end of Section 4.

3. SOME PROPERTIES OF INVERSE GENERATORS

In Zwart [28] and Gomilko [11], we can find the following result.

Lemma 3.1. *Let A generate an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then the following equality holds*

$$e^{A^{-1}\tau} x_0 = x_0 - \int_0^\infty \tau h_{\mathrm{ac}}(t\tau) T(t) x_0 \, dt,$$

where

$$h_{\mathrm{ac}}(t) = \frac{1}{\sqrt{t}} J_1(2\sqrt{t}),$$

with $J_1(\cdot)$ the Bessel function of the first kind and of the first order.

In this lemma, we made the assumption that A generates an exponentially stable C_0-semigroup. This implies that A^{-1} is a bounded operator, and thus it generates a C_0-semigroup. Note that we do not know whether this C_0-semigroup is bounded. In the more general situation, when A generates a bounded C_0-semigroup, it is still unknown whether its inverse generates a C_0-semigroup as well. However, we have that it generates a once integrated semigroup. The following definition is taken from [1].

Definition 3.2. Let $A \in \mathscr{C}(X)$ and $k \in \mathbb{N}$. The operator A is said to be the generator of k-times integrated semigroup if there exist an $\omega \geq 0$ and a strongly continuous function $S(\cdot) : [0, \infty) \to \mathscr{B}(X)$ such that $(\omega, \infty) \subset \rho(A)$, $\|S(t)\| \leq Me^{\omega t}$, and

$$(\lambda I - A)^{-1} = \lambda^k \int_0^\infty e^{-\lambda t} S(t)\, dt, \quad \text{for } \mathrm{Re}(\lambda) > \omega.$$

In this case, the family of operators $(S(t))_{t\geq 0}$ is called k-times integrated semigroup generated by A and we denote it as $(e_k^{At})_{t\geq 0}$.

Theorem 3.3. *Let the operator A generate the bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Assume further that A^{-1} exists as an element of $\mathscr{C}(X)$. Then A^{-1} generates 1-times integrated semigroup and for $t \geq 0$*

$$\begin{aligned} e_1^{A^{-1}t} x_0 &= t x_0 + t^2 \int_0^\infty h'_{\mathrm{ac}}(ts) T(s) x_0\, ds \\ &= t x_0 + t^2 \int_0^\infty \left[\frac{\sqrt{ts} J_0(2\sqrt{ts}) - J_1(2\sqrt{ts})}{(ts)^{3/2}} \right] T(s) x_0\, ds, \quad x_0 \in X. \end{aligned} \tag{3.1}$$

Proof. By the asymptotic behavior of the Bessel functions, it is easy to see that h'_{ac} is absolutely integrable on $[0, \infty)$, see [27]. Because $T(t)x_0$ is bounded on this interval, it is easy to see that the expression on the right of (3.1) is bounded by Mt, for some positive M. Next we are going to show that $(\lambda I - A^{-1})^{-1} x_0 = \lambda \int_0^\infty e^{-\lambda t} e_1^{A^{-1}s} x_0\, ds$. Standard Laplace theory gives that the Laplace transform of t is $1/\lambda^2$. Now we concentrate on the second term in (3.1):

$$\begin{aligned} &\lambda \int_0^\infty t^2 \left[\int_0^\infty h'_{\mathrm{ac}}(ts) T(s) x_0\, ds \right] e^{-\lambda t}\, dt \\ &\quad = \lambda \int_0^\infty \left[\int_0^\infty t^2 h'_{\mathrm{ac}}(ts) e^{-\lambda t}\, dt \right] T(s) x_0\, ds \\ &\quad = \lambda \int_0^\infty \left[\frac{t^2}{s} h_{\mathrm{ac}}(ts) e^{-\lambda t} \Big|_0^\infty - \int_0^\infty h_{\mathrm{ac}}(ts) \frac{1}{s} [t^2 e^{-\lambda t}]'\, dt \right] T(s) x_0\, ds \end{aligned}$$

$$= -\lambda \int_0^\infty \frac{1}{s} \int_0^\infty h_{ac}(ts)[2te^{-\lambda t} - \lambda t^2 e^{-\lambda t}]dt\, T(s)x_0\, ds$$
$$= -\lambda \int_0^\infty \frac{1}{s}\left[\frac{2}{\lambda^2}e^{-\frac{s}{\lambda}} + \frac{-2\lambda + s}{\lambda^3}e^{-\frac{s}{\lambda}}\right]T(s)x_0\, ds$$
$$= -\frac{1}{\lambda^2}\int_0^\infty e^{-\frac{s}{\lambda}}T(s)x_0\, ds$$
$$= -\frac{1}{\lambda^2}\left(\frac{1}{\lambda}I - A\right)^{-1} x_0.$$

So we have that the Laplace transform of the right-hand side of (3.1) equals to

$$\frac{1}{\lambda}x_0 - \frac{1}{\lambda^2}\left(\frac{1}{\lambda}I - A\right)^{-1} x_0 = \left(\frac{1}{\lambda}\left(\frac{1}{\lambda}I - A\right) - \frac{1}{\lambda^2}\right)\left(\frac{1}{\lambda}I - A\right)^{-1} x_0$$
$$= (\lambda I - A^{-1})^{-1}x_0. \qquad \square$$

Remark 3.4. From [17], it follows that there is a maximal subspace $D \subset X$ such that operator A^{-1} restricted to D generates a C_0-semigroup. This means that the generator A^{-1} of the once integrated semigroup $e_1^{A^{-1}t}$ generates on D the C_0-semigroup $e^{A^{-1}t}$.

If A^{-1} is bounded, then it is clear that it generates a C_0-semigroup. Because the inverse commutes with A, we know that $A + A^{-1}$ is an infinitesimal generator as well. If A^{-1} is merely a closed, densely defined operator, then it is unknown whether it generates a C_0-semigroup. Under the assumption that it does, we study the domain and range of the generator of $(e^{At}e^{A^{-1}t})_{t\geq 0}$.

Proposition 3.5. *Assume that the operators A and A^{-1} both generate C_0-semigroups, which are denoted by $(e^{At})_{t\geq 0}$ and $(e^{A^{-1}t})_{t\geq 0}$, respectively. The generator A_1 of the C_0-semigroup $T(t) := e^{At}e^{A^{-1}t}, t \geq 0$, has the following properties:*

1. $\operatorname{ran}(A_1) \subset D(A) + \operatorname{ran}(A)$.
2. $D(A) \cap \operatorname{ran}(A) \subset D(A_1)$ *and on this set, we have that* $A_1 = A + A^{-1}$.
3. $D(A) \cap \operatorname{ran}(A) = \{x \in X \mid x = Az, z \in D(A^2)\}$ *is a core of operator* A_1, *that is, the closure of* $(A_1, D(A) \cap \operatorname{ran}(A))$ *equals* $(A_1, D(A_1))$.

Proof. Consider the following identity for any $x \in X$ and any $t > 0$,

$$\left(e^{At} - I\right)\left(e^{A^{-1}t} - I\right)x = \left(e^{At}e^{A^{-1}t} - I\right)x - \left(\left(e^{At} - I\right) + \left(e^{A^{-1}t} - I\right)\right)x. \qquad (3.2)$$

Furthermore, for any C_0-semigroup $(S(t))_{t\geq 0}$, we have that

$$S(t)x_0 - x_0 = A_S \int_0^t S(s)x_0\,ds,$$

where A_S is the corresponding infinitesimal generator. Combining these facts, we see that $(e^{At}e^{A^{-1}t} - I)x \in \mathrm{ran}(A_1)$, but the other elements of (3.2) belong to $\mathrm{ran}(A)$ and $\mathrm{ran}(A^{-1})$. Because the range of A^{-1} equals the domain of A, the first assertion follows.

If $x \in D(A) \cap D(A^{-1}) = D(A) \cap \mathrm{ran}(A)$, then (3.2) gives that $x \in D(A_1)$. Furthermore, dividing in (3.2) by t and taking the limits gives

$$0 = A_1 x - (Ax + A^{-1}x),$$

which shows the second assertion.

The third assertion follows from Corollary III.5.8 of [9]. □

In third property of the previous proposition, we see that $D(A) \cap \mathrm{ran}(A)$ plays an important role. If A is (boundedly) invertible, then the range of A is the whole Banach space, and so this intersection equals the domain of A. The following proposition gives a nice fact when this intersection equals the range of A.

Proposition 3.6. *Under the assumption of Proposition* 3.5, *we have that* $\mathrm{ran}(A) \subset D(A)$ *if and only if* $D(A^2) = D(A)$.

Proof. If $\mathrm{ran}(A) \subset D(A)$, then for any $x \in D(A)$ one has $Ax \in \mathrm{ran}(A) \subset D(A)$, that is, $x \in D(A^2)$. Conversely, if $D(A^2) = D(A)$, then for any $y \in \mathrm{ran}(A)$ there exists an $x \in D(A)$ such that $y = Ax$. Because by assumption, we have that x is also an element of $D(A^2)$, we have that $y \in D(A)$. Therefore $\mathrm{ran}\,(A) \subset D(A)$. □

We end this section with the following observation, which follows from [13]. Because $\mathrm{ran}(A)$ and $\mathrm{ran}(A^{-1})$ are dense in X, the bounded C_0-semigroups $(e^{At})_{t\geq 0}$ and $(e^{A^{-1}t})_{t\geq 0}$ are mean stable, that is,

$$\frac{1}{t}\int_0^t e^{As}x\,ds \to 0, \quad \frac{1}{t}\int_0^t e^{A^{-1}s}x\,ds \to 0 \quad \text{for any } x \in X \text{ as } t \to \infty.$$

4. CRANK–NICOLSON ON A GENERAL BANACH SPACE

In this section, we study the behavior of the Crank–Nicolson discretization as given in equation (1.4). Because the stability/boundedness properties are invariant under time-scaling, we may without loss of generality assume that $\tau = 2$. Hence operator A_d equals $(I - A)^{-1}(I + A)$. We denote by $\mathcal{Q}$ the mapping that gives to A the discretization A_d, that is,

$$\mathcal{Q}(A) = (I - A)^{-1}(I + A) := A_d. \tag{4.1}$$

Now it is easy to see that

$$\mathcal{Q}(A) = -\mathcal{Q}(A^{-1}). \tag{4.2}$$

Furthermore, we have the following equality

$$\begin{aligned} A_d^{n+1} &= A_d^n(I - A)^{-1} + A_d^n A(I - A)^{-1} \\ &= A_d^n(I - A)^{-1} - A_d^n(I - A^{-1})^{-1}. \end{aligned} \tag{4.3}$$

Combining (4.3) with (4.1) and (4.2) gives

$$(\mathcal{Q}(A))^{n+1} = (\mathcal{Q}(A))^n(I - A)^{-1} - (-1)^n(\mathcal{Q}(A^{-1}))^n(I - A^{-1})^{-1}. \tag{4.4}$$

From this observation, the following lemma follows directly.

Lemma 4.1. *If for every initial condition in the domain of A the difference equation* (1.4) *has a bounded solution, and if for every initial condition in the domain of A^{-1} the difference equation* (1.4) *has a bounded solution, then for any initial condition the solution of* (1.4) *is bounded.*

In Guo and Zwart [14], it has been shown that on Hilbert spaces, the mapping $A \to (\mathcal{Q}(A))^n(I - A)^{-1}$ is bounded uniformly in n provided A generates a bounded C_0-semigroup. In other words the assumption in Lemma 4.1 holds for A. In this section, we want to investigate whether the first assumption in Lemma 4.1 holds for bounded C_0-semigroups in a Banach space. This can be equivalently formulated as the question whether the mapping

$$\mathcal{P}_n(A) = (\mathcal{Q}(A))^n(I - A)^{-1} \tag{4.5}$$

is uniformly bounded for operator A that generates a bounded C_0-semigroup.

It is not hard to show that if A generates the bounded C_0-semigroup e^{At}, then the following equality holds,

$$\mathcal{P}_n(A) = \int_0^\infty L_n(2t)e^{-t}e^{At}\,dt, \tag{4.6}$$

where $L_n(t)$ is the *Laguerre polynomial* given by

$$L_n(t) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!}(-t)^k. \tag{4.7}$$

Furthermore, we have the following lemma.

Lemma 4.2. *Let A be the infinitesimal generator of the bounded C_0-semigroup e^{At} on a Banach space X. Let f be an element of $L^1(0,\infty)$. Then*

$$\mathcal{F}(A) = \int_0^\infty f(t) e^{At}\, dt \tag{4.8}$$

exists (as Pettis integral) and satisfies

$$\|\mathcal{F}(A)\| \le \int_0^\infty |f(t)|\, dt \sup_{t\ge 0} \|e^{At}\|. \tag{4.9}$$

Furthermore, there exists a Banach space and an infinitesimal generator A of a contraction C_0-semigroup on this Banach space such that

$$\int_0^\infty |f(t)|\, dt = \|\mathcal{F}(A)\|. \tag{4.10}$$

Hence, if $f \notin L^1(0,\infty)$, then (4.8) need not to exist as a bounded operator.

Proof. The first part follows from [15, Chapter XV], so we concentrate on the second part. As Banach space we choose $X = C_0(0,\infty)$, the continuous functions with limit zero at infinity. The norm on this space is given by the maximum-norm. As C_0-semigroup we choose the left-shift, that is, $(T(t)x)(\eta) = x(t+\eta)$. It is clear that this is a contraction C_0-semigroup. Let $f \in L^1(0,\infty)$ be a given function, and define $h(\eta) = \mathrm{sign}(f(\eta))$. For every $\varepsilon > 0$, we can find an element $x_\varepsilon \in X$ with norm one such that

$$\left| \int_0^\infty f(\eta)[h(\eta) - x_\varepsilon(\eta)]\, d\eta \right| \le \varepsilon.$$

For $\|F(A)\|$, we find

$$\begin{aligned} \|\mathcal{F}(A)(x_\varepsilon)\| \ge |(\mathcal{F}(A)(x_\varepsilon))(0)| &= \left| \int_0^\infty f(t) x_\varepsilon(t+0)\, dt \right| \\ &\ge \left| \int_0^\infty f(t) h(t)\, dt \right| - \varepsilon = \int_0^\infty |f(t)|\, dt - \varepsilon. \end{aligned}$$

Because x_ε has norm one, and because this holds for all $\varepsilon > 0$, we see that

$$\|\mathcal{F}(A)\| \geq \sup_{\varepsilon>0} \frac{\|\mathcal{F}(A)(x_\varepsilon)\|}{\|x_\varepsilon\|} \geq \int_0^\infty |f(t)|\,dt.$$

Combining this with (4.9), we see that we have proved (4.10). □

From this lemma and equation (4.6), we can conclude that $\mathcal{P}_n(A)$ is uniformly bounded in n for any infinitesimal generator of a bounded C_0-semigroup on a Banach space if and only if

$$\sup_{n\in\mathbb{N}} \int_0^\infty |L_n(2t)e^{-t}|\,dt < \infty. \tag{4.11}$$

Let us recall that $L_n(t)e^{-\frac{t}{2}}$ is an orthonormal sequence (see, e.g., [19]) in the space $L^2(0,\infty)$. So $2\int_0^\infty |L_n(2t)e^{-t}|^2\,dt = 1$, but unfortunately, we have the following result.

Lemma 4.3. *For the Laguerre polynomials $L_n(t)$, the following estimate holds:*

$$\int_0^\infty |L_n(2t)e^{-t}|\,dt = O(\sqrt{n}). \tag{4.12}$$

Proof. This can be found in Lemma 1.5.4 of [24]. We present a different proof. The Laplace transform of $L_n(2t)e^{-t}$ is given by

$$g(s) = \frac{-1}{(s+1)}\left(\frac{s-1}{s+1}\right)^n.$$

Because $\frac{s-1}{s+1}$ has absolute value one on the imaginary axis, we can write $g(i\omega)$ as $\frac{-1}{i\omega+1}e^{in\phi(\omega)}$ for some real-valued function ϕ. Furthermore, $\phi''(\omega)$ is non-zero for almost all ω. From Corollary 1.5.1 of [4], we know that the induced multiplier norm of $g(s)$ on L^∞ is larger or equals $c\sqrt{n}$ for some constant c. This induced norm equals the L^1-norm of the function $L_n(2t)e^{-t}$, and so we have proved our lemma. □

Combining Lemmas 4.2 and 4.3 shows that on a general Banach space, the mapping $\mathcal{P}_n(A)$ is not uniformly bounded in n. Hence this implies that, in contrast with the Hilbert space situation, in a Banach space the solutions of the difference equation (1.4) need not be bounded when the initial condition lies in the domain of A. An example of such a Banach space can be constructed using Corollary 1.5.1 in [4]. If A is the infinitesimal generator of the shift C_0-semigroup on L^p, then $\|\mathcal{P}_n(A)\| \geq cn^{|\frac{1}{2}-\frac{1}{p}|}$, and thus unbounded.

As stated in Lemma 4.2, there exists a Banach space and an infinitesimal generator A of a bounded C_0-semigroup, such that $\|\mathscr{P}_n(A)\| = \int_0^\infty |L_n(2t)e^{-t}|dt$. One might (wrongly) conclude from this that Theorem 1.1 does not hold on Banach spaces. However, for this concrete generator and Banach space, we know that its inverse A^{-1} is not the infinitesimal generator of a bounded C_0-semigroup. Hence, we do not have a Banach space counterexample to Theorem 1.1.

As we see in Theorem 1.3, there are C_0-semigroups for which the (time-varying) discretization is stable on any Banach space. Next we present the proof of this theorem.

Proof of Theorem 1.3. We begin by noting that the operator A^{-1} generates bounded analytic C_0-semigroup, too, see [7] or [28].

Let us introduce the notation $R_0(j) = (I + \frac{\tau_j}{2}A)(I - \frac{\tau_j}{2}A)^{-1}$. Hence, similarly as in (4.4), we have the boundedness of $\prod_{j=0}^k R_0(j)$ if and only if the operators

$$\left(\prod_{j=0}^{k} R_0(j)\right)\left(I - \frac{\tau_{k+1}}{2}A\right)^{-1}, \quad k \in \mathbb{N},$$

are uniformly bounded for any generator A of a bounded analytic C_0-semigroup. Now we follow [5], and we use the representation

$$\left(\prod_{j=0}^{k} R_0(j)\right)\left(I - \frac{\tau_{k+1}}{2}A\right)^{-1} - \prod_{j=1}^{k} e^{A_n\tau_j}\left(I - \frac{\tau_{k+1}}{2}A\right)^{-1}$$
$$= \frac{1}{2\pi i}\int_\Gamma \left(\prod_{j=1}^{k}\left(\frac{1+\frac{\tau_j}{2}\lambda}{1-\frac{\tau_j}{2}\lambda}\right) - \prod_{j=1}^{k}\exp(\tau_j\lambda)\right)\left(1 - \frac{\tau_{k+1}}{2}\lambda\right)^{-1}(\lambda I - A)^{-1}d\lambda.$$

Now we divide the contour $\Gamma = \Gamma_\varepsilon \cup (\Gamma \setminus \Gamma_\varepsilon)$ as in [5]. Because $\sup_j \tau_j < \infty$, we can estimate $\left|\frac{1+\tau_j\lambda/2}{1-\tau_j\lambda/2}\right| \le \exp(-c|\lambda|)$ in the small neighborhood of zero on Γ_ε, where c is independent of τ_j and λ. Because $\inf_j \tau_j > 0$, we have for $\lambda \in \Gamma$ with large modulus that $|(1 - \tau_{k+1}\lambda/2)^{-1}| \le M/|\lambda|$ with M not depending on τ_j and λ. The final estimate of the integral uses the formula

$$\prod_{j=1}^{k} a_j - \prod_{j=1}^{k} b_j = \sum_{j=1}^{k}(a_j - b_j)\prod_{l=1}^{j} a_l \prod_{l=j+1}^{k} b_l$$

and the fact that $|\tau\lambda k e^{-c\lambda k\tau}|$ uniformly bounded for all $\lambda \in \Gamma, k \in \mathbb{N}$, and $\tau \in (0,\infty)$. □

Let us mention in this connection that in the paper [8], they considered the case when $\tau_n \to 0$ as $n \to \infty$ and $t = n\tau_n$, so they got the stability $\left\| \left(\frac{I+\tau_n A/2}{I-\tau_n A/2}\right)^n (1-A)^{-\alpha} \right\| \leq constant$, when multiplying the fraction by resolvent of power $(I - A)^{-\alpha}$ with any $\alpha > 1/2$.

ACKNOWLEDGMENTS

The authors acknowledge the hospitality of the Department of Applied Mathematics, University of Twente, and the support of a grant by NWO (B61-543) and by the Russian Foundation for Basic Research (04-01-00026).

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