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Contractible subgraphs, Thomassen's conjecture and the dominating cycle conjecture for snarks

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Abstract

We show that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is Hamiltonian), by Thomassen (every 4-connected line graph is Hamiltonian) and by Fleischner (every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle), which are known to be equivalent, are equivalent to the statement that every snark (i.e. a cyclically 4-edge-connected cubic graph of girth at least five that is not 3-edge-colorable) has a dominating cycle.

We use a refinement of the contractibility technique which was introduced by Ryjáček and Schelp in 2003 as a common generalization and strengthening of the reduction techniques by Catlin and Veldman and of the closure concept introduced by Ryjáček in 1997.

Keywords: Dominating cycle; Contractible graph; Cubic graph; Snark; Line graph; Hamiltonian graph

1. Introduction

In this paper we consider finite undirected graphs. All the graphs we consider are loopless (with one exception in Section 3); however, we allow the graphs to have multiple edges. We follow the most common graph-theoretic terminology and notation, and for concepts and notation not defined here we refer the reader to [2]. If F, G are graphs then $G - F$ denotes the graph $G - V(F)$ and by an a, b*-path* we mean a path with end vertices a, b. A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the claw $K_{1,3}$.

In 1984, Matthews and Sumner [8] posed the following conjecture.

Conjecture A (*[8]*). *Every 4-connected claw-free graph is Hamiltonian.*

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Since every line graph is claw-free (see [1]), the following conjecture by Thomassen is a special case of Conjecture A.

Conjecture B ([12]). *Every 4-connected line graph is Hamiltonian.*

A closed trail T in a graph G is said to be *dominating*, if every edge of G has at least one vertex on T, i.e., the graph $G - T$ is edgeless (a closed trail is defined as usual, except that we allow a single vertex to be such a trail). The following result by Harary and Nash-Williams [6] shows the relation between the existence of a dominating closed trail (abbreviated DCT) in a graph G and Hamiltonicity of its line graph $L(G)$.

Theorem 1 ([6]). *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G contains a DCT.*

Let k be an integer and let G be a graph with $|E(G)| > k$. The graph G is said to be *essentially k-edge-connected* if G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ are nontrivial (i.e. containing at least one edge). If G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ contain a cycle, G is said to be *cyclically k-edge-connected*.

It is well-known that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected. Thus, the following statement is an equivalent formulation of Conjecture B.

Conjecture C. *Every essentially 4-edge-connected graph contains a DCT.*

By a *cubic* graph we will always mean a regular graph of degree 3 without multiple edges. It is easy to observe that if G is cubic, then a DCT in G becomes a dominating cycle (abbreviated DC), and that every essentially 4-edge-connected cubic graph must be triangle-free, with a single exception of the graph K_4. To avoid this exceptional case, we will always consider only essentially 4-edge-connected cubic graphs on at least five vertices.

Since a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected (see [5], Corollary 1), the following statement, known as the Dominating Cycle Conjecture, is a special case of Conjecture C.

Conjecture D. *Every cyclically 4-edge-connected cubic graph has a DC.*

Restricting to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Fleischner [4].

Conjecture E ([4]). *Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a DC.*

In [10], a closure technique was used to prove that Conjectures A and B are equivalent. Fleischner and Jackson [5] showed that Conjectures B–D are equivalent. Finally, Kochol [7] established the equivalence of these conjectures with Conjecture E. Thus, we have the following result.

Theorem 2 ([5,7,10]). *Conjectures A–E are equivalent.*

A cyclically 4-edge-connected cubic graph G of girth $g(G) \geq 5$ that is not 3-edge-colorable is called a *snark*. Snarks have turned out to be an important class of graphs, for example in the context of nowhere zero flows. For more information about snarks see the paper [9]. Restricting our considerations to snarks, we obtain the following special case of Conjecture E.

Conjecture F. *Every snark has a DC.*

The following theorem, which is the main result of this paper, shows that Conjecture F is equivalent to the previous ones.

Theorem 3. *Conjecture F is equivalent to Conjectures A–E.*

The proof of Theorem 3 is postponed to Section 4.

As already noted, every cyclically 4-edge-connected cubic graph other than K_4 must be triangle-free. Thus, the difference between Conjectures E and F consists in restricting to graphs which do not contain a 4-cycle. For the proof of the equivalence of these conjectures in Section 4 we first develop in Section 2 a refinement of the technique of contractible subgraphs that was developed in [11] as a common generalization of the closure concept [10] and Catlin's collapsibility technique [3], and in Section 3 a technique that allows us to handle the (non)existence of a DC while replacing a subgraph of a graph by another one.

2. Weakly contractible graphs

In this section we introduce a refinement of the contractibility technique from [11] under a special assumption which is automatically satisfied in cubic graphs. We basically follow the terminology and notation of [11].

For a graph H and a subgraph $F \subset H$, $H|_F$ denotes the graph obtained from H by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges (i.e. edges with one vertex of degree 1). Note that $H|_F$ may contain multiple edges and $|E(H|_F)| = |E(H)|$. For a subset $X \subset V(H)$ and a partition $\mathcal{A}$ of X into subsets, $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1 and a_2 are in the same element of $\mathcal{A}$, and $H^{\mathcal{A}}$ denotes the graph with vertex set $V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$ (here the sets $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e. if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $H^{\mathcal{A}}$).

Let F be a graph and $A \subset V(F)$. Then F is said to be *A-contractible*, if for every even subset $X \subset A$ (i.e. with $|X|$ even) and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. In particular, the case $X = \emptyset$ implies that an A-contractible graph has a DCT containing all vertices of A.

If H is a graph and $F \subset H$, then a vertex $x \in V(F)$ is said to be a *vertex of attachment of F in H* if x has a neighbor in $V(H) \setminus V(F)$. The set of all vertices of attachment of F in H is denoted by $A_H(F)$. Finally, $\mathrm{dom}_{tr}(H)$ denotes the maximum number of edges of a graph H that are dominated by (i.e. have at least one vertex on) a closed trail in H. Specifically, H has a DCT if and only if $\mathrm{dom}_{tr}(H) = |E(H)|$.

The following theorem shows that a contraction of an $A_H(F)$-contractible subgraph of a graph H does not affect the value of $\mathrm{dom}_{tr}(H)$.

Theorem 4 (*[11]*). *Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if*

$$\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$$

for every graph H such that $F \subset H$ and $A_H(F) = A$.

Specifically, F is A-contractible if and only if, for any H such that $F \subset H$ and $A_H(F) = A$, H has a DCT if and only if $H|_F$ has a DCT (the "only if" part follows by Theorem 4; the "if" part can be easily seen by the definition of A-contractibility).

Let F be a graph and let $A \subset V(F)$. The graph F is said to be *weakly A-contractible*, if for every *nonempty* even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$.

Thus, in comparison with the contractibility concept as introduced in [11], we do not include the case $X = \emptyset$. This means that we do not require that a weakly A-contractible graph has a DCT containing all vertices of A.

Clearly, every A-contractible graph is also weakly A-contractible. It is easy to see that if F is weakly A-contractible and $|A| \geq 3$, then $d_F(x) \geq 2$ for every $x \in A$.

Examples. 1. The graphs in Fig. 1 are examples of graphs that are weakly A-contractible but not A-contractible (vertices of the set A are double-circled).
2. The triangle C_3 is A-contractible for any subset A of its vertex set.
3. Let C be a cycle of length $\ell \geq 4$, let $x, y \in V(C)$ be nonadjacent and set $A = V(C)$, $X = \{x, y\}$ and $\mathcal{A} = \{\{x, y\}\}$. Then there is no DCT in C containing the edge $xy \in C^{\mathcal{A}}$ and all vertices of A. Hence no cycle C of length at least 4 is weakly $V(C)$-contractible.

If H is a graph and $F \subset H$, then H_{-F} denotes the graph with vertex set $V(H_{-F}) = V(H) \setminus (V(F) \setminus A_H(F))$ and with edge set $E(H_{-F}) = E(H) \setminus E(F)$ (equivalently, H_{-F} is the graph determined by the edge set $E(H) \setminus E(F)$).

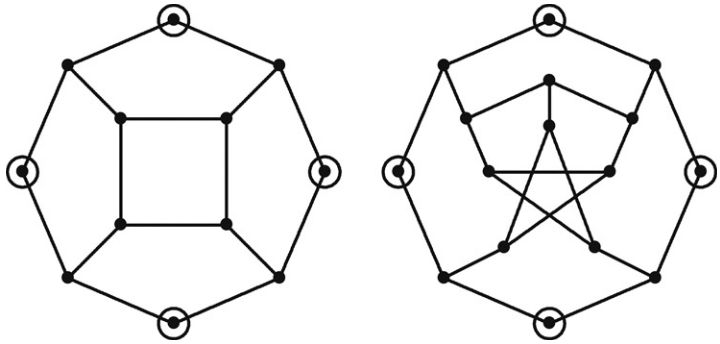

Fig. 1.

Our next theorem shows that, in a special situation, weak contractibility is sufficient to obtain the equivalence of Theorem 4.

Theorem 5. *Let F be a graph and let $A \subset V(F)$, $|A| \geq 2$. Then F is weakly A-contractible if and only if*

$$\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$$

for every graph H such that $F \subset H$, $A_H(F) = A$, $d_{H_{-F}}(a) = 1$ for every $a \in A$, and $|V(K) \cap A| \geq 2$ for at least one component K of H_{-F}.

Proof. The proof of Theorem 5 basically follows the proof of Theorem 2.1 of [11].

Let F be a graph and let H be a graph satisfying the assumptions of the theorem. Then every closed trail T in H corresponds to a closed trail in $H|_F$, dominating at least as many edges as T. Hence immediately $\mathrm{dom}_{tr}(H) \leq \mathrm{dom}_{tr}(H|_F)$.

Suppose that F is weakly A-contractible and let T' be a closed trail in $H|_F$ such that T' dominates $\mathrm{dom}_{tr}(H|_F)$ edges and, subject to this condition, T' has maximum length. If $v_F \notin V(T')$, then T' is also a closed trail in H, implying $\mathrm{dom}_{tr}(H|_F) \leq \mathrm{dom}_{tr}(H)$, as requested. Hence we can suppose $v_F \in V(T')$.

If T' is nontrivial, i.e. contains an edge, then the edges of T' determine in H a system of trails $\mathcal{P} = \{P_1, \ldots, P_k\}$, $k \geq 1$, such that every $P_i \in \mathcal{P}$ has end vertices in A (note that all trails in $\mathcal{P}$ are open since $d_{H_{-F}}(a) = 1$ for all $a \in A$). Since $d_{H_{-F}}(a) = 1$ for all $a \in A$, every $x \in A$ is an end vertex of at most one trail from $\mathcal{P}$, and we set $X = \{x \in A_H(F) | x \text{ is an end vertex of some } P_i \in \mathcal{P}\}$ and $\mathcal{A} = \{A_1, \ldots, A_k\}$, where A_i is the (two-element) set of end vertices of P_i, $i = 1, \ldots, k$.

If T' is trivial (i.e., a one-vertex trail), then we consider a component K of H_{-F} for which $|V(K) \cap A_H(F)| \geq 2$. Let $x_1, x_2 \in V(K) \cap A_H(F)$. If $V(K) \setminus \{x_1, x_2\} \neq \emptyset$ then, since K is connected, K contains a path of length at least 2 with end vertices x_1, x_2, but then we have a contradiction with the maximality of T'. Hence $V(K) = \{x_1, x_2\}$ and $E(K) = \{x_1x_2\}$, and we set $P_1 = x_1x_2$, $\mathcal{P} = \{P_1\}$, $X = \{x_1, x_2\}$ and $\mathcal{A} = \{\{x_1, x_2\}\}$. Note that in both cases the set X is nonempty.

By the weak A-contractibility of F, $F^{\mathcal{A}}$ has a DCT Q, containing all vertices of A and all edges of $E(\mathcal{A})$. The trail Q determines in F a system of trails $Q_1, \ldots, Q_k$ such that every Q_i has its two end vertices in two different elements of $\mathcal{A}$. Now, the trails Q_i together with the system $\mathcal{P}$ form a closed trail in H, dominating at least as many edges as T'. Hence $\mathrm{dom}_{tr}(H|_F) \leq \mathrm{dom}_{tr}(H)$, implying $\mathrm{dom}_{tr}(H|_F) = \mathrm{dom}_{tr}(H)$.

Next suppose that F is not weakly A-contractible (possibly even disconnected). Then, for some nonempty $X \subset A$ and a partition $\mathcal{A}$ of X into two-element sets, $F^{\mathcal{A}}$ has no DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Let $\mathcal{A} = \{\{x'_1, x''_1\}, \ldots, \{x'_k, x''_k\}\}$, and construct a graph H with $F \subset H$ by replacing the edges of $E(\mathcal{A})$ by k vertex disjoint x'_i, x''_i-paths P_i of length at least 3, $i = 1, \ldots, k$, and by attaching a pendant edge to every vertex in $A \setminus X$. Since $X \neq \emptyset$, at least one component K of H_{-F} is a path with end vertices in A, implying $|V(K) \cap A| \geq 2$. Since $F^{\mathcal{A}}$ has no DCT containing all vertices of A and all edges of $E(\mathcal{A})$, H has no DCT. However, clearly $H|_F$ has a DCT and we have $\mathrm{dom}_{tr}(H) < \mathrm{dom}_{tr}(H|_F)$. ■

In the special case of cubic graphs, we have the following corollary.

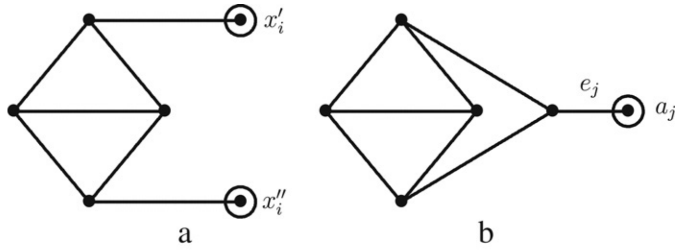

Fig. 2.

Corollary 6. *Let F be a graph with $\delta(F) = 2$, $\Delta(F) \leq 3$ and $|A| \geq 2$, where $A = \{x \in V(F) \mid d_F(x) = 2\}$. Then F is weakly A-contractible if and only if*

$$\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$$

for every cubic graph H such that $F \subset H$, $A_H(F) = A$, and $|V(K) \cap A| \geq 2$ for at least one component K of H_{-F}.

Proof. Clearly $d_{H_{-F}} = 1$ for every $a \in A$, since H is cubic. If F is weakly A-contractible, then $\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$ immediately by Theorem 5. For the rest of the proof, it is sufficient to modify the last part of the proof of Theorem 5 such that the constructed graph H is cubic. To achieve this, it is sufficient to use a copy of the graph in Fig. 2(a) instead of each of the paths P_i, and a copy of the graph in Fig. 2(b) instead of each of the pendant edges attached to the vertices $a_j \in A \setminus X$. Then there is a component K of H_{-F} with $|V(K) \cap A| \geq 2$ since X is nonempty. The graph $H|_F$ has a closed trail dominating all edges except for the edges different from e_j in the copies attached to the vertices in $A \setminus X$, while in H there is no such closed trail. ∎

We say that a subgraph $F \subset H$ is a *weakly contractible subgraph of* H if F is weakly $A_H(F)$-contractible. We then have the following corollary.

Corollary 7. *Let H be a cubic graph and let F be a weakly contractible subgraph of H with $\delta(F) = 2$. Then H has a DC if and only if $H|_F$ has a DCT.*

Proof. First note that in a cubic graph every closed trail is a cycle and that a cubic graph with a DC must be essentially 2-edge-connected. Since H is cubic and $\delta(F) = 2$, $A_H(F) = \{x \in V(F) \mid d_F(x) = 2\}$ and the weak contractibility assumption implies F is connected. If every component of H_{-F} contains one vertex from $A_H(F)$, then clearly neither H nor $H|_F$ is essentially 2-edge-connected (since H is cubic) and hence neither H nor $H|_F$ has a DCT. The rest of the proof follows from Corollary 6. ∎

Example. Let H be the graph obtained from three vertex-disjoint copies F_1, F_2, F_3 of the graph F_i from Fig. 2(a) by adding edges $x_1'x_2'$, $x_1'x_3'$, $x_2'x_3'$, $x_1''x_2''$, $x_1''x_3''$, $x_2''x_3''$. Then H is cubic, $F_1 \subset H$ is weakly contractible, $H|_{F_1}$ has a DCT, but H has no DC. This example shows that the assumption $\delta(F) = 2$ in Corollaries 6 and 7 cannot be omitted.

3. Replacement of a subgraph

In this section we develop a technique to replace certain subgraphs by others without affecting the (non)existence of a DCT.

Let G be a graph and let $F \subset G$ be a subgraph of G. Let F' be a graph such that $V(F') \cap V(G) = \emptyset$, let $A' \subset V(F')$ be such that $|A'| = |A_G(F)|$ and let $\varphi : A_G(F) \to A'$ be a bijection. Let H be the graph obtained from G_{-F} and F' by identifying each $x \in A_G(F)$ with its image $\varphi(x) \in A'$. We say that the graph H is obtained by *replacement* (*in* G) *of* F *by* F' *modulo* φ and denote $H = G[F \overset{\varphi}{\to} F']$.

Note that if $H = G[F \overset{\varphi}{\to} F']$ then also clearly $G = H[F' \overset{\varphi^{-1}}{\longrightarrow} F]$.

Let F be a graph and let $A = \{a_1, \ldots, a_k\} \subset V(F)$. Let $\overline{A}$ be a set with $\overline{A} \cap V(F) = \emptyset$, $|\overline{A}| = |A|$, and set $\overline{A} = \{\overline{a}_1, \ldots, \overline{a}_k\}$. Then $\overline{F}^A$ denotes the graph with vertex set $V(\overline{F}^A) = V(F) \cup \overline{A}$ and with edge set $E(\overline{F}^A) = E(F) \cup \{a_i\overline{a}_i | i = 1, \ldots, k\}$ (i.e., $\overline{F}^A$ is obtained from F by attaching a pendant edge to every vertex of A).

The following observation shows that, under certain conditions, the replacement in a graph G of a weakly contractible subgraph by another one affects neither the existence nor the nonexistence of a DCT in G.

Proposition 8. *Let G be a graph with $\delta(G) \geq 1$ and let $F \subset G$ be a weakly contractible subgraph of G such that $|E(F)| \geq 1$, $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Let F', $|E(F')| \geq 1$, be a weakly A'-contractible graph for an $A' \subset V(F')$, and let $\varphi : A_G(F) \to A'$ be a bijection. Then G has a DCT if and only if $G[F \overset{\varphi}{\to} F']$ has a DCT.*

Proof. Set $H = G[F \overset{\varphi}{\to} F']$. For $|A_G(F)| = 0$ the assumptions $G \not\simeq \overline{F}^{A_G(F)}$ and $\delta(G) \geq 1$ imply that G is disconnected and neither G nor H has a DCT. If $|A_G(F)| = 1$ or if $|A_G(F)| \geq 2$ and $|V(K) \cap A_G(F)| = 1$ for every component K of G_{-F}, then neither G nor H can have a DCT since $|E(F)| \geq 1$, $|E(F')| \geq 1$, $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Thus, we can assume that $|A_G(F)| \geq 2$ and there is a component K of G_{-F} such that $|V(K) \cap A_G(F)| \geq 2$. Then, by Theorem 5, G has a DCT if and only if $G|_F$ has a DCT. Similarly, H has a DCT if and only if $H|_{F'}$ has a DCT, but the graphs $G|_F$ and $H|_{F'}$ are, up to the number of pendant edges at v_F ($v_{F'}$), isomorphic. ■

In the special case of cubic graphs, we obtain the following consequence.

Corollary 9. *Let G be a cubic graph and let $F \subset G$ be a weakly contractible subgraph of G with $\delta(F) = 2$. Let F' be a graph with $\delta(F') = 2$ and $\Delta(F') \leq 3$, let $A' = \{x \in V(F') | d_{F'}(x) = 2\}$ and suppose that F' is weakly A'-contractible. Let $\varphi : A_G(F) \to A'$ be a bijection. Then the graph $H = G[F \overset{\varphi}{\to} F']$ is cubic and G has a DC if and only if H has a DC.*

Proof. Clearly $A_G(F) = \{x \in V(F) | d_F(x) = 2\}$ and since G is cubic, we have $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Since φ is a bijection, H is cubic. By Proposition 8, G has a DCT if and only if H has a DCT, but in cubic graphs every DCT is a DC. ■

Now we consider a similar question if F and/or F' are not contractible. We restrict our observations to cubic graphs.

A connected graph F without multiple edges with $\Delta(F) \leq 3$ will be called a *cubic fragment*. For any cubic fragment F and $i = 1, 2$ we set $A_i(F) = \{x \in V(F) | d_F(x) = i\}$ and $A(F) = A_1(F) \cup A_2(F)$ (note that if $F \subset H$, F is connected and H is cubic, then F is a cubic fragment and $A_H(F) = A(F)$). A cubic fragment F is said to be *essential* if $|V(F) \setminus A_1(F)| \geq 2$. It is easy to observe that if F is an essential cubic fragment, the set $V(F) \setminus A_1(F)$ induces (in F) a connected subgraph with at least one edge.

For a cubic fragment F we now introduce the concept of an F-linkage. An F-linkage will be allowed to contain loops. A loop on a vertex v is considered as an edge joining v to itself, and is denoted by an element vv of the edge set. Edges of an F-linkage that are not loops will be referred to as *open edges*.

Let F be a cubic fragment and let B be a graph with $V(B) \subset A(F)$, $E(B) \cap E(F) = \emptyset$, and with components $B_1, \ldots, B_k$. We say that B is an *F-linkage*, if $E(B)$ contains at least one open edge and, for any $i = 1, \ldots, k$,

(i) every B_i is a path (of length at least one) or a loop,
(ii) if B_i is a path of length at least two, then all interior vertices of B_i are in $A_1(F)$,
(iii) if B_i is a loop at a vertex x, then $x \in A_2(F)$.

Let F be a cubic fragment and let B be an F-linkage. Then F^B denotes the graph with vertex set $V(F^B) = V(F)$ and edge set $E(F^B) = E(F) \cup E(B)$. Note that $E(B)$ and $E(F)$ are assumed to be disjoint, i.e. if $h_1 = x_1x_2 \in E(F)$ and $h_2 = x_1x_2 \in E(B)$, then h_1, h_2 are parallel edges of the graph F^B.

Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. For any F_1-linkage B, $\varphi(B)$ denotes the graph with vertex set $V(\varphi(B)) = \{\varphi(x) | x \in V(B)\}$ and edge set $E(\varphi(B)) = \{\varphi(x)\varphi(y) | xy \in E(B)\}$ (note that the sets $E(F_2)$ and $E(\varphi(B))$ are again considered to be disjoint, and we admit $x = y$ in which case $\varphi(x)\varphi(x)$ is a loop at $\varphi(x)$). Note that $\varphi(B)$ is an F_2-linkage.

Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. We say that φ is a *compatible mapping* if

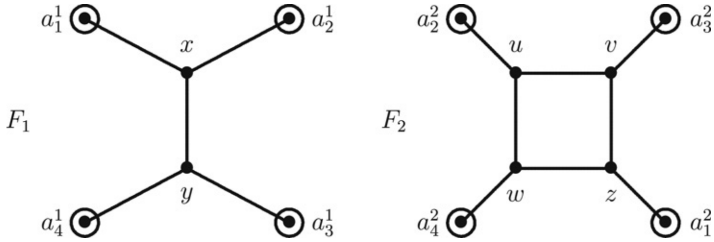

Fig. 3.

(i) $\varphi(A_i(F_1)) = A_i(F_2)$, $i = 1, 2$,

(ii) if B is an F_1-linkage such that F_1^B has a DC containing all open edges of B, then $F_2^{\varphi(B)}$ has a DC containing all open edges of $\varphi(B)$.

For a compatible mapping $\varphi : A(F_1) \to A(F_2)$ we will simply write $\varphi : F_1 \to F_2$.

Let F_1, F_2 be cubic fragments and let $\varphi : A(F_1) \to A(F_2)$ be a bijection such that $\varphi(A_i(F_1)) = A_i(F_2)$, $i = 1, 2$. It is easy to observe that if F_2 is weakly $A(F_2)$-contractible then φ is compatible, and if moreover F_1 is weakly $A(F_1)$-contractible then both φ and φ^{-1} are compatible (note that B cannot contain a path of length at least 2 in this case — this is clear for $|A(F_i)| \le 2$, and for $|A(F_i)| \ge 3$ this follows from the fact that weak $A(F_i)$-contractibility of F_i then implies $A(F_i) = A_2(F_i)$).

The following example shows that the compatibility of a mapping φ does not imply φ^{-1} is compatible if the F_i's are not weakly contractible.

Example. Let F_1, F_2 be the graphs in Fig. 3 and let $\varphi : A(F_1) \to A(F_2)$ be the mapping that maps a_j^1 on a_j^2, $j = 1, 2, 3, 4$. By a straightforward check of all possible F_1-linkages B and the corresponding DC's in F_1^B and in $F_2^{\varphi(B)}$, we easily see that there are, up to symmetry, the following possibilities.

$E(B)$	DC in F_1^B	DC in $F_2^{\varphi(B)}$
$a_1^1a_4^1$	$a_1^1a_4^1yxa_1^1$	$a_1^2a_4^2wuvza_1^2$
$a_1^1a_2^1$	not existing	not existing
$a_1^1a_2^1, a_2^1a_4^1$	$a_1^1a_2^1a_4^1yxa_1^1$	$a_1^2a_2^2a_4^2wuvza_1^2$
$a_1^1a_3^1, a_3^1a_2^1$	not existing	$a_1^2a_3^2a_2^2uwza_1^2$
$a_1^1a_2^1, a_2^1a_3^1, a_3^1a_4^1$	$a_1^1a_2^1a_3^1a_4^1yxa_1^1$	$a_1^2a_2^2a_3^2a_4^2wuvza_1^2$
$a_1^1a_4^1, a_4^1a_3^1, a_3^1a_2^1$	$a_1^1a_4^1a_3^1a_2^1xa_1^1$	$a_1^2a_4^2a_3^2a_2^2uwza_1^2$
$a_1^1a_4^1, a_2^1a_3^1$	$a_1^1a_4^1ya_3^1a_2^1xa_1^1$	$a_1^2a_4^2wua_2^2a_3^2vza_1^2$
$a_1^1a_2^1, a_3^1a_4^1$	not existing	$a_1^2a_2^2uva_3^2a_4^2wza_1^2$

We conclude that $\varphi : A(F_1) \to A(F_2)$ is a compatible mapping, but there is no compatible mapping of $A(F_2)$ onto $A(F_1)$. Note that this mapping φ will play an important role in the proof of our main result in Section 4.

The following result shows that the replacement of a subgraph of a cubic graph modulo a compatible mapping does not affect the existence of a DC.

Theorem 10. *Let G be a cubic graph and let C be a DC in G. Let $F \subset G$ be an essential cubic fragment such that $G - F$ is not edgeless, and let F' be a cubic fragment such that $V(F') \cap V(G) = \emptyset$ and there is a compatible mapping $\varphi : F \to F'$. Then the graph $G' = G[F \stackrel{\varphi}{\to} F']$ is a cubic graph having a DC C' such that $E(C) \setminus E(F) = E(C') \setminus E(F')$.*

(Note that if both φ and φ^{-1} are compatible and both F and F' are essential, then G has a DC if and only if $G' = G[F \stackrel{\varphi}{\to} F']$ has a DC.)

Proof. By the compatibility of φ, $A_1(F') = \varphi(A_1(F))$ and $A_2(F') = \varphi(A_2(F))$, hence G' is cubic. Let C be a DC in G. We show that G' has a DC C' with $E(C) \setminus E(F) = E(C') \setminus E(F')$.

We first observe that $E(C) \cap E(F) \neq \emptyset$. Since F is essential, there is an edge $xy \in E(F)$ with $d_F(x) \geq 2$ and $d_F(y) \geq 2$. Then one of x, y (say, x) is on C. Since $d_F(x) \geq 2$, x has a neighbor x_1 in F, $x_1 \neq y$. Then, since $d_G(x) = 3$, the edge xy or xx_1 is in $E(C) \cap E(F)$.

Let C_F and C_{-F} denote the subgraph of C induced by the edge set $E(C) \cap E(F)$ and $E(C) \cap E(G_{-F})$, respectively. Since $E(C) \cap E(F) \neq \emptyset$ and $G - F$ is not edgeless, C_{-F} is a nonempty system of paths. Let $P_1, \ldots, P_k$ be the components of C_{-F}. Then:

- the end vertices of every P_i are in $A(F)$,
- the interior vertices of every P_i are in $A_1(F)$ or in $V(G) \setminus V(F)$,

where $i = 1, \ldots, k$.

We define an F-linkage B as follows:

(i) for every P_i, let P_i^B be the path obtained from P_i by replacing every maximal subpath of P_i with all interior vertices in $V(G) \setminus V(F)$ by a single edge (with both vertices in $A(F)$),
(ii) for every vertex $x \in A(F) \setminus V(C_{-F})$ which is on C_F (note that such a vertex x must be in $A_2(F)$), let e_x be a loop at x,
(iii) B is the graph with components $\{P_i^B | i = 1, \ldots, k\} \cup \{e_x | x \in A_2(F) \setminus V(C_{-F}) \cap V(C)\}$.

It is immediate to observe that the graph F^B has a DC C^B containing all open edges of B. By the compatibility of φ, the graph $(F')^{\varphi(B)}$ has a DC C'^B containing all open edges of the graph $\varphi(B)$.

Let $C'_{F'}$ denote the subgraph of C'^B induced by the edge set $E(C'^B) \cap E(F')$. Then $C'_{F'}$ is a system of paths, and the edges in $E(C'_{F'}) \cup E(C_{-F})$ determine a cycle C' in $G' = G[F \stackrel{\varphi}{\to} F']$ with $E(C) \setminus E(F) = E(C') \setminus E(F')$. Note that, by the construction, $V(C) \cap A(F) \subset V(C') \cap A(F')$ (this is clear for vertices x with $d_{C_{-F}}(x) \geq 1$, and for vertices x with $d_{C_{-F}}(x) = 0$ this follows from the fact that both C^B and C'^B dominate all loops in B and in $\varphi(B)$, respectively).

It remains to show that C' is a DC in G'. Thus, let $xy \in E(G')$.

If $x, y \in V(G') \setminus V(F') = V(G) \setminus V(F)$, then x or y is on C_{-F}, implying x or y is on C' since $C_{-F} \subset C'$. If $x, y \in V(F') \setminus A(F')$, then x or y is on $C'_{F'}$, implying x or y is on C' since $C'_{F'} \subset C'$.

Up to symmetry, it remains to consider the case $x \in A(F') = \varphi(A(F))$. If $x \in V(C)$, then also $x \in V(C')$ since $V(C) \cap A(F) \subset V(C') \cap A(F')$, as observed above. Hence we can suppose that $x \notin V(C)$, implying $y \in V(C)$. If $y \in A(F')$, then similarly $y \in V(C')$ and we are done; hence $y \notin A(F')$. Then either $y \in V(F') \setminus A(F')$, or $y \in V(G') \setminus V(F')$. But then, in the first case y is on $C'_{F'}$ since C' is dominating in $(F')^{\varphi(B)}$, and in the second case y is on C_{-F} since C is dominating in G. In either case this implies $y \in V(C')$. ■

The following result shows that the existence of a compatible mapping is not affected by a replacement of a subgraph by another one modulo a compatible mapping.

Proposition 11. *Let X, F be essential cubic fragments such that there is a compatible mapping $\psi : X \to F$. Let $F_1 \subset F$ be an essential cubic fragment, and let F_2 be a cubic fragment such that $V(F) \cap V(F_2) = \emptyset$ and there is a compatible mapping $\varphi : F_1 \to F_2$. Let $F' = F[F_1 \stackrel{\varphi}{\to} F_2]$. Then there is a compatible mapping $\psi' : X \to F'$.*

Proof. For any $x \in A(X)$ set

$$\psi'(x) = \begin{cases} \psi(x) & \text{if } x \in \psi^{-1}(A(F) \setminus A(F_1)), \\ \varphi(\psi(x)) & \text{if } x \in \psi^{-1}(A(F) \cap A(F_1)). \end{cases}$$

Then $\psi' : A(X) \to A(F')$ is a bijection, and $\psi' : A_i(X) \to A_i(F')$, $i = 1, 2$, by the compatibility of ψ and φ. Let B be an X-linkage such that X^B has a DC containing all open edges of B. By the compatibility of ψ, the graph $F^{\psi(B)}$ has a DC C containing all open edges of $\psi(B)$. We need to show that $(F')^{\psi'(B)}$ has a DC containing all open edges of $\psi'(B)$. We will construct a cubic graph H such that $F \subset H$, H has a DC that coincides with C on F, and the structure of $H - F$ implies that an application of Theorem 10 to H yields the required DC in $(F')^{\psi'(B)}$.

Let $B_1, \ldots, B_k$ be the components of $\psi(B)$, and choose the notation such that

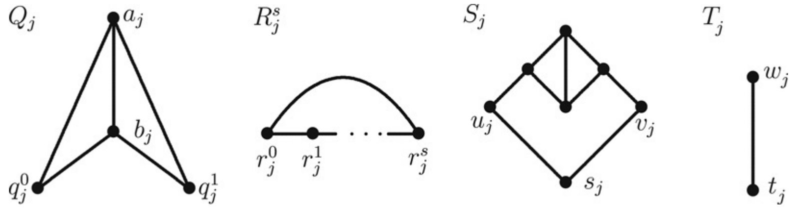

Fig. 4.

- $B_1, \ldots, B_p$ ($p \geq 1$) are paths, $V(B_j) = \{x_j^0, \ldots, x_j^{\ell_j}\}$ (i.e. B_j is of length ℓ_j), $j = 1, \ldots, p$;
- if none of $B_1, \ldots, B_k$ is a loop, then $\ell = 0$, otherwise $B_{p+1}, \ldots, B_{p+\ell}$ are loops, $V(B_{p+j}) = \{x_{p+j}\}$, $j = 1, \ldots, \ell$;
- if $A(F) \setminus V(\psi(B)) = \emptyset$, then $f = 0$, otherwise $A(F) \setminus V(\psi(B)) = \{x_{p+\ell+1}, \ldots, x_{p+\ell+f}\}$.

Thus, we have $k = p + \ell$ and $V(\psi(B)) = \cup_{j=1}^{p+\ell}(V(B_j))$.

Let Q_j, R_j^s ($s \geq 2$), S_j and T_j be the graphs shown in Fig. 4. We construct a cubic graph H containing F by the following construction:

- take the graph F with the labeling of vertices of $A(F)$ defined above;
- for each B_j with $1 \leq j \leq p$, $\ell_j = 1$, take one copy of Q_j and for $i = 0, 1$ identify $x_j^i = q_j^i$ if $x_j^i \in A_1(F)$ or add the edge $x_j^i q_j^i$ if $x_j^i \in A_2(F)$, respectively,
- for each B_j with $1 \leq j \leq p$, $\ell_j > 1$, take one copy of R_j^s for $s = \ell_j$ and
  - for $i = 0$ and $i = \ell_j$ identify $x_j^i = r_j^i$ if $x_j^i \in A_1(F)$ or add the edge $x_j^i r_j^i$ if $x_j^i \in A_2(F)$, respectively,
  - for $1 \leq i \leq \ell_j - 1$ identify $x_j^i = r_j^i$;
- for each B_j with $p + 1 \leq j \leq p + \ell$ (if $\ell > 0$) take one copy of S_j, add the edge $x_j s_j$, and if $\ell \geq 2$, then for $j \geq p + 2$ add the edge $v_{j-1} u_j$;
- for each x_j with $p + \ell + 1 \leq j \leq p + \ell + f$ (if $f > 0$) do the following:
  - if $x_j \in A_1(F)$, take one copy of S_j, identify $x_j = s_j$ and if $f \geq 2$, then for $j \geq p + \ell + 2$ add the edge $v_{j-1} u_j$ (if $x_{j-1} \in A_1(F)$), or the edge $w_{j-1} u_j$ (if $x_{j-1} \in A_2(F)$), respectively;
  - if $x_j \in A_2(F)$, take one copy of T_j, identify $x_j = t_j$ and if $f \geq 2$, then for $j \geq p + \ell + 2$ add the edge $v_{j-1} w_j$ (if $x_{j-1} \in A_1(F)$), or the edge $w_{j-1} w_j$ (if $x_{j-1} \in A_2(F)$), respectively;
  - if $x_{p+\ell+1} \in A_2(F)$, then relabel $w_{p+\ell+1}$ as $u_{p+\ell+1}$ and if $x_{p+\ell+f} \in A_2(F)$, then relabel $w_{p+\ell+f}$ as $v_{p+\ell+f}$;
- if $\ell \neq 0$, then
  - for $\ell_1 = 1$ remove the edge $q_1^0 a_1$ and add the edges $q_1^0 u_{p+1}$ and $a_1 v_{p+\ell}$,
  - for $\ell_1 > 1$ remove the edge $r_1^0 r_1^1$ and add the edges $r_1^0 u_{p+1}$ and $r_1^1 v_{p+\ell}$;
- if $f \neq 0$, then
  - for $\ell_1 = 1$ remove the edge $b_1 q_1^1$ and add the edges $b_1 u_{p+\ell+1}$ and $q_1^1 v_{p+\ell+f}$,
  - for $\ell_1 > 1$ remove the edge $r_1^{\ell_1-1} r_1^{\ell_1}$ and add the edges $r_1^{\ell_1-1} u_{p+\ell+1}$ and $r_1^{\ell_1} v_{p+\ell+f}$.

Then H is a cubic graph, $F \subset H$, $A_H(F) = A(F)$, and it is straightforward to check that H has a DC C^H such that $E(C^H) \cap E(F) = E(C) \cap E(F)$.

Let C_{-F}^H denote the subgraph of C^H induced by the edge set $E(C^H) \cap E(H_{-F})$. Then the structure of the graphs Q_j, R_j^s, S_j and T_j implies the following properties of C_{-F}^H:

- if $1 \leq j \leq p$ and $i = 0$ or $i = \ell_j$, then $d_{C_{-F}^H}(x_j^i) = 1$,
- if $1 \leq j \leq p$ and $1 \leq i \leq \ell_j - 1$, then $d_{C_{-F}^H}(x_j^i) = 2$,
- if $\ell > 0$ and $p + 1 \leq j \leq p + \ell$, then $d_{C_{-F}^H}(x_j) = 0$ and x_j has no neighbor on C_{-F}^H,
- if $f > 0$ and $p + \ell + 1 \leq j \leq p + \ell + f$, then $d_{C_{-F}^H}(x_j) = 0$ and all neighbors of x_j in H_{-F} are on C_{-F}^H.

Set $H' = H[F_1 \overset{\varphi}{\to} F_2]$. By the compatibility of φ and by Theorem 10, H' has a DC $C^{H'}$ such that $E(C^{H'}) \setminus E(F_2) = E(C^H) \setminus E(F_1)$. Specifically, $F' \subset H'$ and $E(C^{H'}) \setminus E(F') = E(C^H) \setminus E(F)$. Let $C^{H'}_{F'}$ and $C^{H'}_{-F'}$ denote the subgraph of $C^{H'}$ induced by $E(C^{H'}) \cap E(F')$ and $E(C^{H'}) \cap E(H'_{-F'})$, respectively. Then $C^{H'}_{-F'} = C^H_{-F}$, and from the above properties of C^H_{-F} we obtain the following properties of $C^{H'}_{F'}$:

- if $1 \le j \le p$ and $i = 0$ or $i = \ell_j$, then $d_{C^{H'}_{F'}}(x^i_j) = 1$,
- if $1 \le j \le p$ and $1 \le i \le \ell_j - 1$, then $d_{C^{H'}_{F'}}(x^i_j) = 0$ and all edges of F' with at least one vertex in $N_{F'}(x^i_j)$ have at least one vertex on $C^{H'}$,
- if $\ell > 0$ and $p + 1 \le j \le p + \ell$, then $d_{C^{H'}_{F'}}(x_j) = 2$,
- if $f > 0$ and $p + \ell + 1 \le j \le p + \ell + f$, then either $d_{C^{H'}_{F'}}(x_j) = 2$, or $d_{C^{H'}_{F'}}(x_j) = 0$ and all neighbors of x_j in F' are on $C^{H'}_{F'}$.

This implies that $C^{H'}_{F'}$ together with the open edges of $\psi'(B)$ determines the required DC in $(F')^{\psi'(B)}$ containing all open edges of $\psi'(B)$. ■

For a cubic fragment F with $A(F) = A_2(F)$ we will simply write $\overline{F}^{A(F)} = \overline{F}$. If F_1, F_2 are cubic fragments with $A(F_i) = A_2(F_i)$, $i = 1, 2$ and $\varphi : A(F_1) \to A(F_2)$ is a bijection, then $\overline{\varphi}$ denotes the bijection $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ defined by $\overline{\varphi}(\overline{a}) = \overline{\varphi(a)}$, $a \in A(F_1)$.

In the proof of Proposition 14 we will also need the following statement showing that the existence (or nonexistence) of a compatible mapping is not affected by adding pendant edges to vertices of attachment.

Proposition 12. *Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and $A(F_i) = A_2(F_i)$, $i = 1, 2$, and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. Then φ is compatible if and only if $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ is compatible.*

Proof. Set $A(F_1) = \{a_1, \ldots, a_k\}$. Suppose first that φ is compatible and let $\overline{B}$ be an $\overline{F_1}$-linkage such that there is a DC $\overline{C}$ in $(\overline{F_1})^{\overline{B}}$ containing all open edges of $\overline{B}$. Since $A(\overline{F_1}) = A_1(\overline{F_1})$, all components of $\overline{B}$ are paths. We define an F_1-linkage B as follows:

(i) $a_i a_j \in E(B)$, $i \ne j$, if and only if $\overline{B}$ has a component which is an $\overline{a_i}, \overline{a_j}$-path,
(ii) $a_i a_i \in E(B)$ if and only if $\overline{a_i} \in A(\overline{F_1}) \setminus V(\overline{B})$.

(This means that vertices in $A(F)$ corresponding to internal vertices of paths in $\overline{B}$ will not be in $V(B)$, and vertices corresponding to vertices not in $V(\overline{B})$ will have loops in B.)

Since $\overline{C}$ dominates all edges of $\overline{F_1}$ (including the edges $a_i\overline{a_i}$ with $\overline{a_i} \notin V(\overline{B})$), it is straightforward to see that removing from $\overline{C}$ the edges of $\overline{B}$ and the pendant edges of $\{a_i\overline{a_i}, i = 1, \ldots, k\} \cap E(\overline{C})$, and adding the open edges of B results in a DC C in F_1^B, containing all open edges of B. Using the compatibility of φ we obtain a DC in $F_2^{\varphi(B)}$ containing all open edges of $\varphi(B)$, and adding the pendant edges and all edges of $\overline{\varphi}(\overline{B})$ yields a required DC in $(\overline{F_2})^{\overline{\varphi}(\overline{B})}$.

Conversely, let $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ be compatible and let B be an F_1-linkage. Since $A(F_1) = A_2(F_1)$, B contains no paths of length more than one. Suppose the notation is chosen such that $E(B) = \{a_1a_2, \ldots, a_{2p-1}a_{2p}, a_{2p+1}a_{2p+1}, \ldots, a_{2p+\ell}a_{2p+\ell}\}$, where $2p + \ell \le k$. Then we define $\overline{B}$ as the graph which has as components the path $a_1 a_{2p+\ell+1} \ldots a_k a_2$ and (if $p > 1$) the edges $a_{2i-1}a_{2i}$, $i = 2, \ldots, p$. The rest of the proof is similar to that above. ■

4. Equivalence of Conjectures A–F

Before proving our main result, Theorem 3, we first prove several auxiliary statements that describe the structure of potential counterexamples to Conjecture D.

Proposition 13. *If Conjecture D is not true, then there is an essential cubic fragment F such that*

(i) $|A_2(F)| = |A(F)| = 4$,
(ii) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*

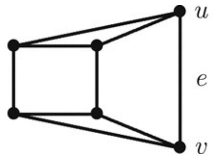

Fig. 5.

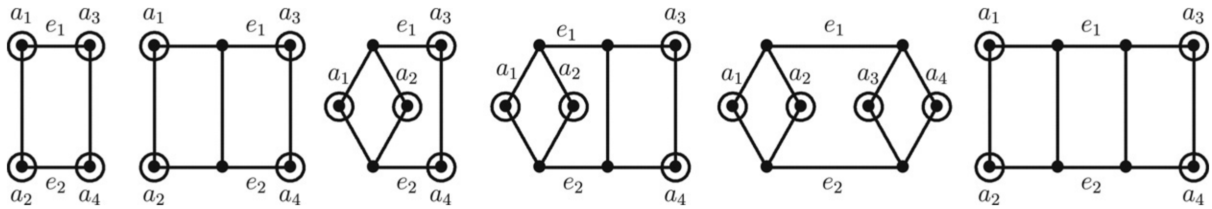

Fig. 6.

(iii) *there is no compatible mapping* $\varphi : C_4 \to F$.

Proof. Let G be a counterexample to Conjecture D, i.e. a cyclically 4-edge-connected cubic graph having no DC, let $e = uv \in E(G)$ and set $F = G - \{u, v\}$. Then F is an essential cubic fragment with $|A_2(F)| = |A(F)| = 4$. Let, to the contrary, $\varphi : C_4 \to F$ be a compatible mapping and set $G' = G[F \xrightarrow{\varphi^{-1}} C_4]$. Then G' is isomorphic to one of the graphs in Fig. 5, and hence G' has a DC. But then, by Theorem 10, the graph $G = G'[C_4 \stackrel{\varphi}{\to} F]$ has a DC, a contradiction. ■

Proposition 14. *Let F be an essential cubic fragment such that*

(i) $|A_2(F)| = |A(F)| = 4$,
(ii) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*
(iii) *there is no compatible mapping* $\varphi : C_4 \to F$,
(iv) *subject to* (i), (ii) *and* (iii), $|V(F)|$ *is minimal.*

Then F is essentially 3*-edge-connected and contains no cycle of length* 4.

Proof. Recall that a cubic graph is cyclically 4-edge-connected if and only if it is essentially 4-edge-connected (see [5]).

We first show that F is essentially 3-edge-connected. Suppose the contrary. By definition, F is connected. Denote $A(F) = \{a_1, a_2, a_3, a_4\}$, and let f_i denote the edge in $E(G) \setminus E(F)$ incident with a_i, $i = 1, 2, 3, 4$. If F has a cut edge e, then some nontrivial (i.e. containing at least one edge) component of $F - e$ contains at most two vertices a_i, but then e together with the corresponding edges f_i is an essential edge cut in G of size at most 3, a contradiction. Hence F has no cut edge. (Note that F has also no cut vertex since G is cubic.)

Thus, let $R = \{e_1, e_2\} \subset E(F)$ be an essential edge cut of F, and let F_1, F_2 be nontrivial components of $F - R$. Denote $e_i = b_i^1 b_i^2$ with $b_i^j \in V(F_j)$, $i, j = 1, 2$. If $|V(F_1) \cap A(F)| = 1$, then we set $V(F_1) \cap A(F) = \{x\}$ and observe that the edges e_1, e_2 and the only edge of G_{-F} incident to x form an essential edge cut of G of size 3, a contradiction. We obtain a similar contradiction for $|V(F_1) \cap A(F)| = 0$; hence $|V(F_1) \cap A(F)| \geq 2$. Symmetrically, $|V(F_2) \cap A(F)| \geq 2$, implying $|V(F_1) \cap A(F)| = |V(F_2) \cap A(F)| = 2$. Thus, we can suppose that the notation is chosen such that $a_1, a_2 \in V(F_1)$ and $a_3, a_4 \in V(F_2)$.

If $|V(F_1)| > 4$, then there is a compatible mapping $\varphi : C_4 \to F_1$ by the minimality of F. Let $\widetilde{C}$ be a copy of C_4 and set $H = F[F_1 \xrightarrow{\varphi^{-1}} \widetilde{C}]$. Then $|V(H)| < |V(F)|$ and, by the minimality of F, there is a compatible mapping $\psi : C_4 \to H$. By Proposition 11 (with $X := C_4$, $F := H$, $F_1 := \widetilde{C}$ and $F_2 := F_1$), there is a compatible mapping $\psi' : C_4 \to H[\widetilde{C} \stackrel{\varphi}{\to} F_1] = F$, a contradiction. Hence $|V(F_1)| \leq 4$ and, symmetrically, $|V(F_2)| \leq 4$.

Now, since G is cyclically 4-edge-connected, either $\{a_1, a_2\} \cap \{b_1^1, b_2^1\} = \emptyset$, or (up to symmetry), $a_1 = b_1^1$ and $a_2 = b_2^1$. Hence F_1 is a single edge or a cycle of length 4. Similarly, F_2 is a single edge or a cycle of length 4. Thus,

F is isomorphic to one of the graphs shown in Fig. 6. However, it is straightforward to check that for each of these graphs there is a compatible mapping $\varphi : C_4 \to F$, a contradiction. Thus, F is essentially 3-edge-connected.

Next we show that

(∗) F contains no subgraph $\widetilde{F}$, $\widetilde{F} \neq F$, with $|V(\widetilde{F})| > 4$ and $|A_2(\widetilde{F})| = |A(\widetilde{F})| = 4$.

Thus, let $\widetilde{F}$ be such a subgraph. By the minimality of F, there is a compatible mapping $\varphi : C_4 \to \widetilde{F}$. Let $\widetilde{C}$ be a copy of C_4 and set $H = F[\widetilde{F} \xrightarrow{\varphi^{-1}} \widetilde{C}]$. By the minimality of F, there is a compatible mapping $\psi : C_4 \to H$. By Proposition 11 (with $X := C_4$, $F := H$, $F_1 := \widetilde{C}$ and $F_2 := \widetilde{F}$), there is a compatible mapping $\psi' : C_4 \to H[\widetilde{C} \xrightarrow{\varphi} \widetilde{F}] = F$, a contradiction. Hence there is no such $\widetilde{F}$.

Finally, we show that F contains no cycle of length 4. Let, to the contrary, $Y \subset F$ be a copy of C_4 (note that possibly $V(Y) \cap A(F) \neq \emptyset$). Let $\overline{F}$ be the graph obtained from F by attaching a pendant edge to each vertex in $A(F)$, and let F_1 and F_2 be the graphs shown in Fig. 3 (recall that we already know there is a compatible mapping $\varphi : F_1 \to F_2$). Let $\overline{Y}$ be the (only) subgraph of $\overline{F}$ such that $Y \subset \overline{Y}$ and $\overline{Y}$ is isomorphic to F_2, let T be a copy of F_1 and let $\varphi : T \to \overline{Y}$ be a compatible mapping. Set $\overline{F}' = \overline{F}[\overline{Y} \xrightarrow{\varphi^{-1}} T]$ (i.e., $\overline{F} = \overline{F}'[T \xrightarrow{\varphi} \overline{Y}]$), and let F' be the graph obtained from $\overline{F}'$ by removing the four pendant edges. Then F' is a cubic fragment with $|A(F')| = |A_2(F')| = 4$.

We show that there is no compatible mapping $\psi : C_4 \to F'$. Let, to the contrary, $\psi : C_4 \to F'$ be compatible. By adding pendant edges to $A(C_4)$ and $A(F')$ and by Proposition 12, there is a compatible mapping $\overline{\psi} : \overline{C_4} \to \overline{F}'$. Thus, we have $\overline{\psi} : \overline{C_4} \to \overline{F}'$, $T \subset \overline{F}'$ and $\varphi : T \to \overline{Y}$. By Proposition 11, there is a compatible mapping $\overline{\psi}' : \overline{C_4} \to \overline{F}$. By removing the pendant edges and by Proposition 12 we obtain a compatible mapping $\psi' : C_4 \to F$, a contradiction. Thus, there is no compatible mapping $\psi : C_4 \to F'$.

By the minimality of F, the graph F' (and hence also $\overline{F}'$) cannot be a subgraph of a cyclically 4-edge-connected cubic graph. Thus, there is an edge cut R' of $\overline{F}'$ such that $|R'| \leq 3$ and at least one component X' of $\overline{F}' - R'$ contains a cycle and has minimum degree 2 (if such an R' does not exist then, identifying the vertices of degree 1 of $\overline{F}'$ with vertices of a C_4, we get a cyclically 4-edge-connected cubic graph containing $\overline{F}'$, a contradiction). However, there is no such edge cut in $\overline{F}$. Since $\overline{F}' = \overline{F}[\overline{Y} \xrightarrow{\varphi^{-1}} T]$, R' contains the edge $e = xy \in E(T)$ with $d_T(x) = d_T(y) = 3$ and some two edges $f_1, f_2 \in E(\overline{F}') \setminus E(T)$. Suppose the vertices of T are labeled such that $A_1(T) = \{a_1, a_2, a_3, a_4\}$, $E(T) = \{a_1x, a_2x, a_3y, a_4y, xy\}$ and $a_1, a_2, x \in V(X')$. Then $R'' = \{f_1, f_2, a_3y, a_4y\}$ is an edge cut in $\overline{F}'$ such that $|R''| = 4$ and $X' + e$ is a component of $\overline{F}' - R''$. Let e_1 (e_2, e_3, e_4) denote the pendant edge of $\overline{Y}$ which corresponds to the edge a_1x (a_2x, a_3y, a_4y) $\in E(T)$, respectively, in the mapping φ. Then $R = \{f_1, f_2, e_3, e_4\}$ is an edge cut of $\overline{F}$ such that the component X of $\overline{F} - R$ containing X' and Y has $|V(X)| > 4$ and $|A_2(X)| = |A(X)| = 4$.

By (∗) (and since $F \not\simeq C_4$, implying $e_1, e_2 \in E(F)$), F contains no such graph as a proper subgraph; hence $X = F$. But then $\{e_1, e_2\}$ is an edge cut of F, contradicting the fact that F is essentially 3-edge-connected. Hence F contains no cycle of length 4. ∎

Proposition 15. *If Conjecture D is not true, then there is an essential cubic fragment F such that*

(i) *F contains no cycle of length 4,*
(ii) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*
(iii) *$|A_2(F)| = |A(F)| = 4$ and $A(F)$ is independent,*
(iv) *there is a compatible mapping $\varphi : F \to C_4$.*

Proof. By Propositions 13 and 14, there is an essential cubic fragment H such that H contains no cycle of length 4, $|A_2(H)| = |A(H)| = 4$, there is a cyclically 4-edge-connected cubic graph G such that $H \subset G$, and there is no compatible mapping $\psi : C_4 \to H$. Let H be minimal with these properties. Since $A(H) = A_2(H)$, by the nonexistence of a compatible mapping $\psi : C_4 \to H$, H is not weakly $A(H)$-contractible. Hence there is a nonempty even set $X \subset A(H)$ and a partition $\mathcal{A}$ of X into two-element subsets such that $H^{\mathcal{A}}$ has no DCT containing all vertices of $A(H)$ and all edges of $E(\mathcal{A})$. Set $A(H) = \{a_1, a_2, a_3, a_4\}$ and suppose the notation is chosen such that $\mathcal{A} = \{\{a_1, a_2\}\}$ if $|X| = 2$ or $\mathcal{A} = \{\{a_1, a_2\}, \{a_3, a_4\}\}$ if $|X| = 4$. Then the graph H^B has no DC containing all open edges of B for either $E(B) = \{a_1a_2, a_3a_3, a_4a_4\}$ or $E(B) = \{a_1a_2, a_3a_4\}$.

Let H, H' be two copies of H (with a corresponding labeling $A(H') = \{a'_1, a'_2, a'_3, a'_4\}$), and let F be the cubic fragment obtained from H and H' by adding the edges $a_1a'_1$ and $a_2a'_2$. Recall that H contains no cycle of length 4.

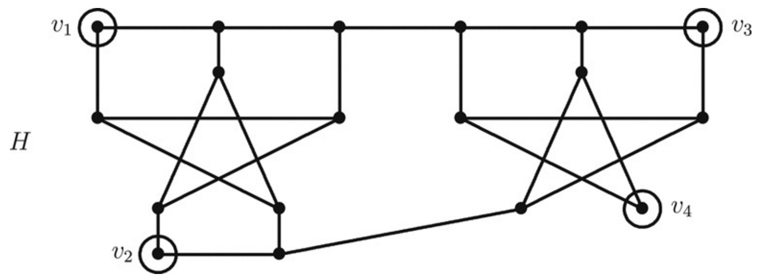

Fig. 7.

Since H is essentially 3-edge-connected by Proposition 14, the set $\{a_1, a_2, a_3, a_4\}$ (and hence also $\{a'_1, a'_2, a'_3, a'_4\}$) is independent. Hence F also contains no cycle of length 4, and the set $A(F) = \{a_3, a_4, a'_3, a'_4\}$ is independent. It remains to prove that there is a compatible mapping $\varphi : F \to C_4$.

First we show that the graph F^B has no DC containing all open edges of B for $E(B) = \{a_3a_3, a_4a_4, a'_3a'_4\}$. To the contrary, let C be such a DC. Then $(E(C) \cap E(H)) \cup \{a_1a_2\}$ is a DC in H^B containing all open edges of B for $E(B) = \{a_1a_2, a_3a_3, a_4a_4\}$, and $(E(C) \cap E(H')) \cup \{a'_1a'_2, a'_3a'_4\}$ is a DC in $H'^{B'}$ containing all open edges of B' for $E(B') = \{a'_1a'_2, a'_3a'_4\}$, which is not possible. Thus, there is no such DC in F^B. Symmetrically, $F^{B'}$ has no DC containing all open edges of B' for $E(B') = \{a'_3a'_3, a'_4a'_4, a_3a_4\}$. Let Y be a copy of C_4 with vertices labeled b_3, b_4, b'_3, b'_4 such that $b_3b_4 \notin E(Y)$ and $b'_3b'_4 \notin E(Y)$. Then it is straightforward to check that $Y^{B''}$ has a DC containing all open edges of B'' for all Y-linkages B'' except for the cases $E(B'') = \{b_3b_3, b_4b_4, b'_3b'_4\}$ and $E(B'') = \{b'_3b'_3, b'_4b'_4, b_3b_4\}$. Hence the mapping $\varphi : A(F) \to A(Y)$ that maps a_i on b_i and a'_i on b'_i, $i = 3, 4$, is a compatible mapping. ■

Note that we do not know any example of a cubic fragment with the properties given in Proposition 15. Moreover, we believe that such a graph in fact does not exist.

Now we are ready to prove the main result of this paper, Theorem 3.

Proof of Theorem 3. Clearly, Conjecture E implies Conjecture F. By Theorem 2, it is sufficient to show that Conjecture F implies Conjecture D. Thus, suppose Conjecture D is not true, and let F be an essential cubic fragment as given by Proposition 15. Let G be a counterexample to Conjecture D, i.e. a cyclically 4-edge-connected cubic graph without a DC. For any cycle C of length 4 in G, choose a compatible mapping of F on C, and let G' be the graph obtained by recursively replacing every cycle of length 4 by a copy of F. Then G' is a cubic graph of girth $g(G') \geq 5$ and, by Theorem 10, G' has no DC. Moreover, G' is cyclically 4-edge-connected since any cycle-separating edge cut in G' of size at most 3 would imply the existence of such an edge cut in G. If G' is not 3-edge-colorable, G' is a snark and we are done. Otherwise, we use the following fact and construction by Kochol [7].

Claim (*[7]*). *If a cubic graph G contains the graph H of Fig. 7 as an induced subgraph, then G is not 3-edge-colorable.*

We use the claim as follows. Let $xy \in E(G')$, let x', x'' (y', y'') be the neighbors of x (of y) different from y (x), respectively, and let G'_i, $i = 1, 2, 3$, be three copies of the graph $G' - x - y$ (where x'_i, x''_i, y'_i, y''_i are the copies of x', x'', y', y'' in G'_i), $i = 1, 2, 3$. Then the graph $\bar{G}$ obtained from G'_1, G'_2, G'_3 and H by adding the edges $x'_1v_3, x''_1v_4, y'_1x'_2, y''_1x''_2, y'_2x'_3, y''_2x''_3, y'_3v_1$ and y''_3v_2 is a cyclically 4-edge-connected graph of girth $g(\bar{G}) \geq 5$. By the claim, $\bar{G}$ is not 3-edge-colorable. It remains to show that $\bar{G}$ has no DC.

Let, to the contrary, C be a DC in $\bar{G}$. Then it is easy to check that for some $i \in \{1, 2, 3\}$, the intersection of C with G'_i is either a path with one end in $\{x'_i, x''_i\}$ and the second in $\{y'_i, y''_i\}$, or two such paths. But, in both cases, the path(s) can be easily extended to a DC in G', a contradiction. ■

5. Concluding remarks

1. Note that our proof of the equivalence of Conjecture F with Conjectures A–E is based on properties (compatible mappings) that are specific for the C_4. This means that our proof cannot be directly extended to obtain higher girth restrictions.

2. We pose the following conjecture and show it is equivalent to Conjectures A–F.

Conjecture G. *Every cyclically 4-edge-connected cubic graph contains a weakly contractible subgraph F with $\delta(F) = 2$.*

Theorem 16. *Conjecture G is equivalent to Conjectures A–F.*

Proof. We first show that Conjecture G implies Conjecture D. Suppose Conjecture G is true and let G be a minimum counterexample to Conjecture D. Hence G has no DC. Let $F \subset G$ be a weakly contractible subgraph of G with $\delta(F) = 2$ and set $A = A_G(F)$. Note that $A \neq \emptyset$ since $\delta(F) = 2$. By Corollary 7, the graph $G|_F$ has no DCT. If $|A| \leq 3$, then every edge in G_{-F} has at least one vertex in A since G is essentially 4-edge-connected. But then $G|_F$ has a (trivial) DCT, a contradiction. Hence $|A| \geq 4$.

We use the following operation (see [5]). Let H be a graph, let $v \in V(H)$ be of degree $d = d_H(v) \geq 4$, and let $x_1, \ldots, x_d$ be an ordering of the neighbors of v (allowing repetition in case of multiple edges). Let H' be the graph obtained by adding edges $x_i y_i$, $i = 1, \ldots, d$, to the disjoint union of the graph $H - v$ and the cycle $y_1 y_2 \ldots y_d y_1$. Then H' is said to be an *inflation of* H *at* v. The following fact was proved in [5].

Claim (*[5]*). *Let H be an essentially 4-edge-connected graph of minimum degree $\delta(G) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Then some inflation of H at v is essentially 4-edge-connected.*

Now let G' be an essentially 4-edge-connected inflation at v_F of the graph obtained from $G|_F$ by deleting its pendant edges. Then G' is a cubic graph having no DC (since otherwise $G|_F$ would have a DCT). Since no cycle of length $\ell \geq 4$ is weakly contractible, F is not a cycle, and since $\delta(F) = 2$, we have $|A_G(F)| < |E(F)|$. But then $|E(G')| < |E(G)|$, contradicting the minimality of G.

For the rest of the proof, it is sufficient to show that Conjecture D implies Conjecture G. Indeed, if C is a dominating cycle in G, $e = uv \in E(C)$ and $A = \{u, v\}$, then the graph F with $V(F) = V(G)$ and $E(F) = E(G) \setminus \{e\}$ is a weakly A-contractible subgraph of G. ■

It should be noted here that the last part of the proof of Theorem 16 is based on a construction with $|A| = 2$, which forces $G - F$ be empty (G_{-F} is a one edge graph) since G is cubic and cyclically 4-edge-connected. It is straightforward to observe that the following stronger statement implies Conjectures A–G. However, we do not know whether these statements are equivalent.

Conjecture H. *Every cyclically 4-edge-connected cubic graph G contains a weakly contractible subgraph F with $|A_G(F)| \geq 4$.*

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