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ORIGINAL RESEARCH

The Hopf-van der Pol System: Failure of a Homotopy Method

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Abstract The purpose of this article to provide an explicit example where continuation based on the homotopy method fails. The example is a one-parameter homotopy for periodic orbits between two well-known nonlinear systems, the normal form of the Hopf bifurcation and the van der Pol system. Our analysis shows that various types of obstructions can make approximation over the whole range of the homotopy parameter impossible. The Hopf-van der Pol system demonstrates that homotopy methods may fail even for seemingly innocent systems.

Keywords Homotopy · Periodic orbits · Global bifurcations

Introduction

Recent years have seen a great number of published papers on homotopy-based methods for nonlinear systems. The basic idea behind homotopy techniques is that a known (analytical) solution of a simple problem may continuously be "deformed" into a solution of a more difficult problem. Such deformation is called a homotopy. A simple example is a linear homotopy, a continuous "interpolation" $\mathcal{H}(x, \theta) = \theta f(x) + (1 - \theta) g(x)$ between the two functions $f(x)$ and $g(x)$. The second argument of $\mathcal{H}(x, \theta)$ can thus be thought of as the deformation parameter such that for $\mathcal{H}(x, 0) = f(x)$ is a solution of the simple problem and $\mathcal{H}(x, 1) = g(x)$ is the hitherto unknown solution. Homotopy crucially depends on the implicit function theorem, which is a basic principle behind continuation and bifurcation analysis.

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Homotopy methods have been used to obtain eigenvalues starting from the identity matrix. The main challenge in a general implementation are the multiple branching points encountered when eigenvalues change from real to a complex pair. The amplitude of the limit cycle of the van der Pol system was approximated with the homotopy analysis method by López et al. [5] and Chen and Liu [1]. Less than careful applications of the so-called homotopy perturbation method were criticized by Fernández [4].

Here we report on obstruction for homotopy methods for periodic orbits. The obstruction is due to the appearance of extra equilibria such that there is a saddle-node on a invariant circle bifurcation. Unlike the eigenvalue example there is no different branch such that the continuation can be restarted from this bifurcation. The Hopf-van der Pol system (which is interesting on its own right) demonstrates that homotopy methods for global solutions cannot be used as a blackbox procedure in general.

The Hopf-van der Pol System

The van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{1}$$

has been one of the most extensively studied dynamical systems. This equation is usually written in the first-order form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{f}(x, y) = \begin{pmatrix} y \\ -x + \mu y - \mu x^2 y \end{pmatrix}. \tag{2}$$

It is well-known that the van der Pol system (2) exhibits a unique limit cycle for $\mu > 0$.

A reasonable idea to find an approximate periodic solution of (2) is to use a homotopy by starting from a system with a limit cycle with explicitly known form. Such a system can be, for example, the normal form of the Hopf bifurcation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{g}(x, y) = \begin{pmatrix} \mu x + y - (x - y)(x^2 + y^2) \\ -x + \mu y - (x + y)(x^2 + y^2) \end{pmatrix}. \tag{3}$$

A possible homotopy between the Hopf normal form and the van der Pol oscillator is

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \theta \mathbf{f}(x, y) + (1 - \theta)\, \mathbf{g}(x, y) \\ &= \begin{pmatrix} (1 - \theta)\mu x + y - (1 - \theta)(x - y)(x^2 + y^2) \\ -x + \mu y - \theta \mu x^2 y - (1 - \theta)(x + y)(x^2 + y^2) \end{pmatrix}. \end{aligned} \tag{4}$$

Equation 4 reduces to the Hopf normal form (3) and to the van der Pol oscillator (2) for $\theta = 0$ and $\theta = 1$, respectively.

Failure of the Homotopy Method

Obstruction by Appearance of Equilibria

By construction, for $\mu > 0$ there is a unique stable limit cycle for both $\theta = 0$ and $\theta = 1$. Figure 1 shows the phase portraits for the Hopf-van der Pol system (4) for $\mu = 2$. The homotopy parameter is varied between 0 and 1 with increments of 0.2. The deformation of

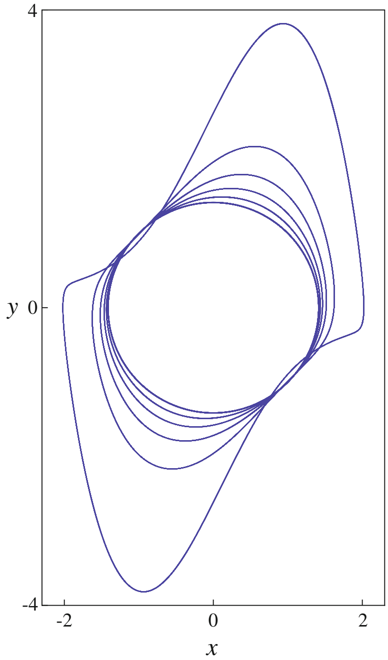

Fig. 1 Phase portraits for the Hopf-van der Pol system at $\mu = 2$. The values of the homotopy parameter θ are $\{0, 0.2, 0.4, 0.6, 0.8, 1\}$

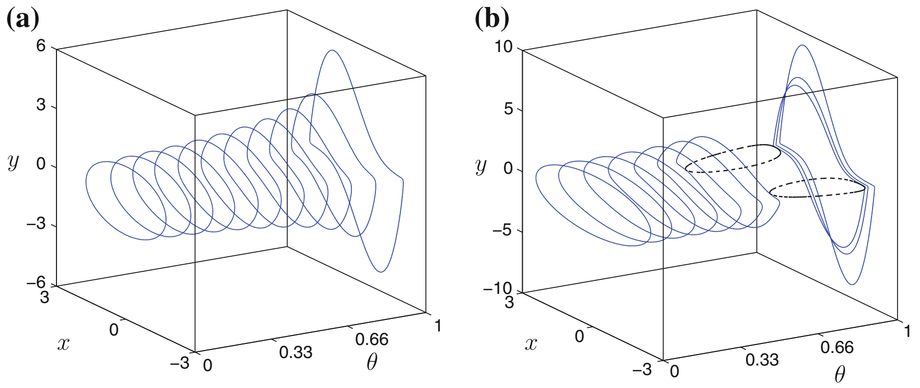

Fig. 2 One-parameter families of periodic solutions for system (4) for **a** $\mu = 3$, **b** $\mu = 6$. For $\mu = 3$ the periodic orbits for $\theta = 0$ and $\theta = 1$ belong to the same family, while for $\mu = 6$ they do not. For $\mu = 6$ we also indicate the two families of nontrivial equilibria (*dotted lines*)

the circular limit cycle of the Hopf system (3) into the characteristic van der Pol limit cycle can be observed.

From this figure one might be tempted to conclude that a homotopy-based approximation method would nicely work for any value of μ. However, this is not the case. We show this with results of numerical bifurcation analysis using MATCONT version 3p3 [2,3]. For $\mu < \mu_h \approx 4.835$ and $0 \le \theta \le 1$ a unique limit cycle exists for the Hopf-van der Pol system, and for example numerical continuation can be used to locate this periodic solution, see Fig. 2a. However, for $\mu > \mu_h$, we find two families of periodic orbits, one by continuation from $\theta = 0$, the other from $\theta = 1$. The continuation makes smaller and smaller steps in

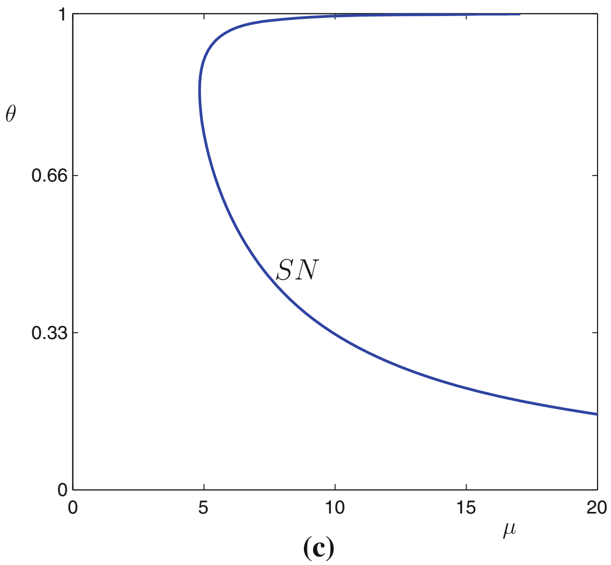

Fig. 3 Saddle-node bifurcations in system (4). The first fold appears at $\mu = \mu_h \simeq 4.835$

the homotopy parameter θ and the period of the periodic orbit grows (to infinity) indicating the presence of a homoclinic bifurcation, see Fig. 2b. Figure 3 shows the bifurcation curve with two (symmetric) branches of equilibria touching the two families of periodic orbits that appear for $\mu > \mu_h$. With μ fixed the equilibria appear and disappear in two saddle-node bifurcations between $\theta = 0$ and $\theta = 1$ and this fold curve SN corresponds to a saddle-node bifurcation on a periodic orbit. Due to symmetry these bifurcation occur on both equilibrium branches for the same parameter, so that it is actually a heteroclinic connection that appears.

Obstruction by Rotation Reversal

Consider a slightly different version of the above defined Hopf-van der Pol system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} (1-\theta)\mu x + y - (1-\theta)(x+y)(x^2+y^2) \\ -x + \mu y - \theta\mu x^2 y + (1-\theta)(x-y)(x^2+y^2) \end{pmatrix}. \tag{5}$$

Note that for $\theta = 0$, the rotation direction reverses when μ crosses 1, i.e., for $(\mu, \theta) = (1, 0)$ there is an invariant circle, but no true periodic orbit. As for the Hopf-van der Pol system above, for $\mu > 0$, there is a single stable limit cycle for both $\theta = 0$ and $\theta = 1$. For $\mu < \mu_h \approx .9605$ we can simply numerically continue the periodic solution from $\theta = 0$ to $\theta = 1$, see Fig. 4a. For $\mu > \mu_h$, we find two families of periodic orbits, one by continuation from $\theta = 0$, the other from $\theta = 1$. Again, the numerical continuation shows that the homotopy parameter θ approaches some limit, while the period of the periodic orbit grows to infinity, see Fig. 4b–c. The presence of a heteroclinic bifurcation is again corroborated by computing two (symmetric) curves of equilibria touching the two families of periodic orbits. Note that for the family from $\theta = 0$ the saddle-node bifurcation occurs on the periodic orbit, while the family from $\theta = 1$ undergoes a simple heteroclinic bifurcation.

The situation above can be explained with the two-parameter bifurcation diagram shown in Fig. 4d. For $\mu < \mu_h$ we indeed find no nontrivial equilibria and the first appears for $\mu = \mu_h$. For $\mu_h < \mu < 1$ the equilibria disappear in an other saddle-node bifurcation between $\theta = 0$ and $\theta = 1$, while for $\mu > 1$ these equilibria go to infinity when $\theta \to 1$. Thus branches SN_1

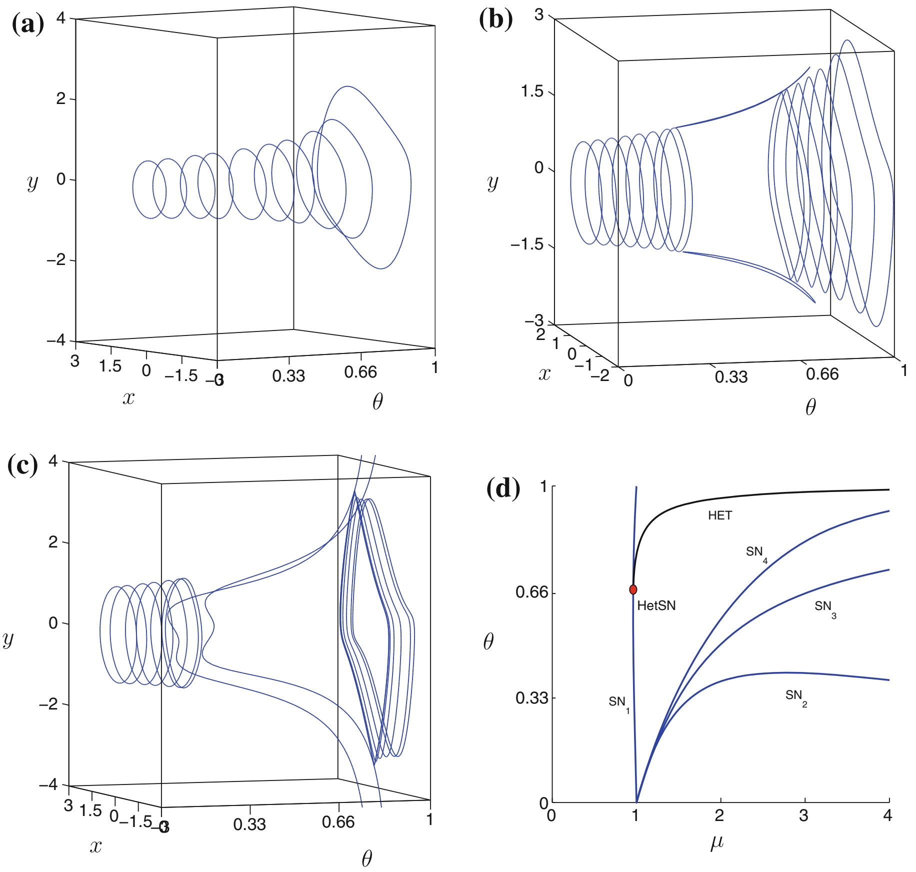

Fig. 4 One-parameter families of periodic solutions for system (5) for **a** $\mu = .5$, **b** $\mu = .97$ and **c** $\mu = 1.4$. For $\mu = .5$ the periodic orbits for $\theta = 0$ and $\theta = 1$ belong to the same family, while for $\mu = .97$ and $\mu = 1.4$ they do not. **d** This is explained by several bifurcation curves in system (5). The first fold appears at $\mu = \mu_h \approx .9605$. For higher μ there is no homotopy from $\theta = 0$ to $\theta = 1$. For $\mu_h \leq \mu < 1$, the saddle-node on an invariant curve SN_1 is encountered. For $\mu > 1$, we encounter SN_2 from below and the heteroclinic connection HET from above. Note that the HET curve emerges from the $HetSN$ point on SN_1

and SN_2 correspond to saddle-node bifurcations on a periodic orbit, while at the other curves the periodic orbit is not involved. Indeed, the family from $\theta = 1$ does not disappear near a fold curve but near the black line HET. On this curve there are two heteroclinic connection between two symmetric saddles. Decreasing θ the periodic orbit and these connections. We conclude with the observation that the main reason for the obstruction for homotopy in this example is that for $\mu < 1$ the rotation direction is the same for $\theta = 0$ and $\theta = 1$, while it is different for $\mu > 1$.

Discussion

We have shown that homotopy-based approximation of periodic orbits may fail even for seemingly innocent systems. Our analysis shows that various types of obstructions can make approximation over the whole range of the homotopy parameter impossible. Numerically another technique is more useful. If, for instance, the periodic orbit is stable, then simulations may give a reasonable approximation which can be used as a starting point for continuation. This strategy is used in [3] and can even be adapted for initializing the computation of homoclinic and heteroclinic orbits [2]. Such numerically oriented homotopy methods seem more tractable although these too need a user supplying suitable initial data.

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