J Glob Optim (2013) 57:1139–1146
DOI 10.1007/s10898-012-0013-2

A note on set-semidefinite relaxations of nonconvex quadratic programs

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Received: 5 April 2012 / Accepted: 10 November 2012 / Published online: 27 November 2012

Abstract We consider semidefinite, copositive, and more general, set-semidefinite programming relaxations of general nonconvex quadratic problems. For the semidefinite case a comparison between the feasible set of the original program and the feasible set of the relaxation has been given by Kojima and Tunçel (SIAM J Optim 10(3):750–778, 2000). In this paper the comparison is presented for set-positive relaxations which contain copositive relaxations as a special case.

Keywords Quadratic programs · Semidefinite- · Copositive- and Set-positive relaxations

Mathematics Subject Classification 90C20 · 90C22 · 90C09

1 Introduction

A constraint $x_i \in \{0, 1\}$ can equivalently be written as a quadratic constraint $x_i^2 - x_i = 0$ (or $-x_i^2 + x_i \le 0, 0 \le x_i \le 1$). By using this relation a standard lifting procedure leads to the well-known semidefinite (SDP) and copositive (COP) programming relaxations for programs with 0-1 constraints or with general quadratic constraints. So, our original problem is the global minimization problem for programs (QP) with general (nonconvex) objective and constraints. In this note we wish to analyse how sharp SDP and COP relaxations of such programs (QP) are. For the SDP relaxation this has been answered by [12]. In this note we give the corresponding result for the COP relaxations of QP and more generally for K-semidefinite (K-SD) relaxations. The results obtained are somewhat negative. They roughly speaking say that without adding extra restrictions into the relaxation we cannot expect the COP or K-SD relaxation of (nonconvex) quadratic programs to be sharp.

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To obtain sharper relaxations one has to consider additional restrictions, e.g., by adding new (quadratic) constraints which are redundant in the original QP. Recent research has revealed that for several special classes of 0–1 programs such a sharpening leads to exact COP representations (see e.g., [2–6,9,10,13–16]). The results in [6] have been extended to K-SD programs by [8]. In [7] also the exactness of an (extended) K-SD relaxation has been shown for a special class of (nonconvex) quadratic programs.

Future research should show which other classes of (non-convex) quadratic programs allow similar sharp COP (or K-SD) relaxations. Note, that exact K-SD relaxations of NP-hard problems evidently are NP-hard. However the K-SD relaxations are convex problems and one may hope that this extra structure leads to new insight and better algorithms for solving hard (non-convex) problems.

2 Notation and known results

Let in the following, $\mathcal{S}_n$ denote the set of symmetric (real-valued) $n \times n$-matrices. For a given closed cone $K \subset \mathbb{R}^n$ we define the set $\mathcal{C}_n(K)$ of K-semidefinite $n \times n$-matrices and its dual cone $\mathcal{C}_n^*(K)$ of K-positive $n \times n$-matrices (see Appendix, Lemma 2 for a proof of duality).

$$\mathcal{C}_n(K) = \left\{ A \in \mathcal{S}_n \mid z^T A z \geq 0 \; \forall z \in K \right\}, \mathcal{C}_n^*(K) = \left\{ Y = \sum_j y_j z_j z_j^T \mid y_j \geq 0, \; z_j \in K \right\}.$$

For $K = \mathbb{R}^n$ we obtain the (self-dual) cone $\mathcal{S}_n^+$ of positive semidefinite matrices and for $K = \mathbb{R}_+^n$ the cones of copositive respectively completely positive matrices.

We consider quadratic problems of the form:

$$\begin{array}{llll} QP_0: & \min \; c_0^T x & \text{s.t.} & q_j(x) \leq 0, \quad j \in J \\ & & \text{with also:} & x \in K \quad \text{in K-SD case} \end{array}$$

with quadratic functions $q_j(x) = \gamma_j + 2c_j^T x + x^T C_j x$, $C_j \in \mathcal{S}_n$, $j \in J$, and J, a finite index set. By introducing the inner product $A \bullet B = \sum_{ij} a_{ij} b_{ij}$ for $A = (a_{ij}),\; B = (b_{ij}) \in \mathcal{S}_n$, we can write $q_j(x) = \gamma_j + 2c_j^T x + x^T C_j x$ in the form

$$q_j(x) = Q_j \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \quad \text{where} \quad Q_j = \begin{pmatrix} \gamma_j & c_j^T \\ c_j & C_j \end{pmatrix}.$$

Note that the relation $X = xx^T$ is equivalent to $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T$. In this setting the original program QP_0 takes the equivalent lifted form:

$$\begin{array}{lll} QP: & \min \; c_0^T x \quad \text{s.t.} & Q_j \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \leq 0, \; j \in J \\ & & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \\ & & \text{with also:} \quad x \in K \quad \text{in K-SD case} \end{array}$$

By replacing the (nonconvex) relation $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T$ by the SDP relaxation, $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}_{n+1}^+$, or the K-SD relaxation, $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}_{n+1}^*(\mathbb{R}_+ \times K)$, we are led to the relaxations of QP:

$$\text{SDP:} \quad \min\ c_0^T x \quad \text{s.t.} \quad Q_j \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \le 0,\ j \in J, \quad \text{and} \quad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}_{n+1}^+$$

$$\text{K-SD:} \quad \min\ c_0^T x \quad \text{s.t.}\ Q_j \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \le 0,\ j \in J, \quad \text{and} \quad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}_{n+1}^*(\mathbb{R}_+ \times K)$$

In case of a K-SD relaxation of QP we always tacitly assume that the original program QP_0 and thus QP contains the constraint $x \in K$ (explicitly or implicitly). For optimality conditions and more details on K-SD programs and their dual we refer to [11]. We introduce some notation. Let S denote the set of quadratic functions defining the feasible set of QP_0 and QP:

$$S = \{Q_j \mid j \in J\} \equiv \{q_j(x) \mid j \in J\}\ .$$

Recall that we identify a quadratic function $q(x) = \gamma + 2c^T x + x^T C x$ with the coefficient matrix $Q = \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix}$. The sets $\mathcal{F}^{\mathrm{QP}_0}$, $\mathcal{F}^{\mathrm{QP}} = \mathcal{F}^{\mathrm{QP}}(S)$, $\mathcal{F}^{\mathrm{SDP}}(S)$ and $\mathcal{F}^{\text{K-SD}}(S)$ are the feasible sets of QP_0, QP, the SDP- and the K-SD relaxation, respectively. By $\mathcal{F}_x^{\mathrm{QP}}(S)$, $\mathcal{F}_x^{\mathrm{SDP}}(S)$ and $\mathcal{F}_x^{\text{K-SD}}(S)$ we denote the projections onto the x-space $\mathbb{R}^n$. Notice that all these feasible sets defined by a set S of quadratic inequalities coincide with the feasible sets given by the conic combinations cone (S), i.e., $\mathcal{F}_x^{\mathrm{QP}}(S) = \mathcal{F}_x^{\mathrm{QP}}(\text{ cone } (S))$ etc. From these definitions we find

$$\mathcal{F}^{QP_0} = \mathcal{F}_x^{QP}(S) = \mathcal{F}_x^{QP}(\text{ cone } (S)) \subset \text{ conv } \mathcal{F}_x^{QP}(S)\ .$$

Since the objective of QP is linear, the minimum value on $\mathcal{F}_x^{QP}(S)$ and on conv $\mathcal{F}_x^{QP}(S)$ coincide. By relaxation properties we have:

$$\text{conv } \mathcal{F}_x^{QP}(S) \subset \mathcal{F}_x^{\mathrm{SDP}}(S), \qquad \text{conv } \mathcal{F}_x^{QP}(S) \subset \mathcal{F}_x^{\text{K-SD}}(S)$$

and also $\mathcal{F}_x^{\text{K-SD}}(S) \subset \mathcal{F}_x^{\mathrm{SDP}}(S)$ in case QP contains the constraint $x \in K$.

We wish to know how sharp these inclusions are. Defining the set of convex quadratic functions $\mathcal{Q}_+ := \left\{ Q = \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix} \mid C \in \mathcal{S}_n^+ \right\}$, for the SDP relaxation this question has been answered by Kojima and Tunçel in [12].

Theorem 1 ([12]) *conv* $[\mathcal{F}_x^{QP}(S)] \subset \mathcal{F}_x^{QP}(\textit{cone } (S) \cap \mathcal{Q}_+) = \mathcal{F}_x^{SDP}(S)$.

We emphasize that in general the set $\mathcal{F}_x^{QP}(\text{cone } (S) \cap \mathcal{Q}_+)$ is strictly smaller than the set $\mathcal{F}_x^{QP}(S \cap \mathcal{Q}_+)$.

Remark 1 In [12], based on the theorem above a conceptual algorithm is analysed which generates a sequence of sets $\mathcal{F}_x^{SDP}(S_k)$ converging to the set conv $[\mathcal{F}_x^{QP}(S)]$. Starting with $S_0 = S$, in each step, by solving an SDP, a "cutting" convex, quadratic constraint $Q^k \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \gamma^k + 2(c^k)^T x + x^T C^k x \le 0$ with $Q^k \in \mathcal{Q}_+$ is constructed in such a way that for $S_{k+1} := S_k \cup \{Q^k\}$ we still have conv $[\mathcal{F}_x^{QP}(S)] \subset \mathcal{F}_x^{SDP}(S_{k+1})$ but the set $\mathcal{F}_x^{SDP}(S_{k+1})$ is strictly smaller than $\mathcal{F}_x^{SDP}(S_k)$. Note that in the context of our generalization such a procedure seems no more useful. In the case of $K = \mathbb{R}_+^n$ for example, in each step, instead of an SDP, we would have to solve a (NP-hard) "completely positive program".

3 K-SD relaxation

In this section we are interested in a comparison between the feasible set $\mathcal{F}^{\mathrm{QP}}(S)$ of the original program QP and the feasible set $\mathcal{F}^{\mathrm{K\text{-}SD}}(S)$ of the K-SD relaxation. The set $\mathcal{Q}_+$ in the SDP relaxation has now to be replaced by the set of "K-semidefinite quadratic functions":

$$\mathcal{Q}_{\mathrm{K\text{-}SD}} := \left\{ Q = \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix} \mid C \in \mathcal{C}_n(K) \right\} .$$

The following (Schur complement lemma) is well-known: For $X \in \mathcal{S}_n$, $x \in \mathbb{R}^n$ it holds,

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}^+_{n+1} \quad \Leftrightarrow \quad X - xx^T \in \mathcal{S}^+_n .$$

Unfortunately (since $\mathcal{C}^*_n(K)$ is not self-dual) this is no more true for $\mathcal{C}^*_n(K)$. We only have

Lemma 1 *Let $X \in \mathcal{S}_n$, $x \in K$ be such that $X - xx^T \in \mathcal{C}^*_n(K)$. Then, $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}^*_{n+1}$ $(\mathbb{R}_+ \times K)$.*

Proof By definition, the matrix $X - xx^T \in \mathcal{C}^*_n(K)$ can be written in the form

$$X - xx^T = \sum_{j=1}^{k} \lambda_j z_j z_j^T \quad \text{with} \quad \lambda_j \geq 0, \quad z_j \in K, \; j = 1, \ldots, k.$$

So, the decomposition

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} + \begin{pmatrix} 0 & 0^T \\ 0 & X - xx^T \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T + \sum_{j=1}^{k} \lambda_j \begin{pmatrix} 0 \\ z_j \end{pmatrix} \begin{pmatrix} 0 \\ z_j \end{pmatrix}^T$$

holds, and recalling $x \in K$, this matrix is an element of $\mathcal{C}^*_{n+1}(\mathbb{R}_+ \times K)$. □

The converse of Lemma 1 is not true in general (if $K \neq \mathbb{R}^n$). As an example we take the copositive case, i.e., $K = \mathbb{R}^n_+$, and chose $X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $x = (1, 1)^T$, $n = 2$. Then

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}^T + \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}^T \in \mathcal{C}^*_{n+1}(\mathbb{R}_+ \times \mathbb{R}^n_+) \text{ but } X - xx^T$$
$$= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \notin \mathcal{C}^*_n(\mathbb{R}^n_+).$$

We now are able to extend (partially) the result of Theorem 1 to the K-SD relaxation of QP. Let us first present an instructive example. Let $\mathcal{F}^{\mathrm{QP}}_x(\{Q\})$ be the feasible set defined by only one inequality $q(x) = Q \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \leq 0$, $Q = \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix}$ (and $x \in K$) then:

$$\text{if } C \notin \mathcal{C}_n(K) \text{ (i.e., } q \text{ is not "K-semidefinite")} \quad \Rightarrow \quad \mathcal{F}^{\mathrm{K\text{-}SD}}_x(\{Q\}) = K .$$

To see this, note that for $C \notin \mathcal{C}_n(K)$ there exists a vector $d \in K$ such that $d^T C d < 0$. So, for any fixed $x \in K$ with $X := \lambda dd^T + xx^T$ it holds

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \bullet Q = \gamma + 2c^T x + \lambda d^T C d + x^T C x < 0 \quad \text{for } 0 < \lambda, \;\; \lambda \text{ large} .$$

Since $X - xx^T = \lambda dd^T \in \mathcal{C}_n^*(K)$, Lemma 1 implies $x \in \mathcal{F}_x^{\text{K-SD}}(\{Q\})$. So, the K-SD relaxation does not provide any restriction apart from $x \in K$. Generally, the following holds.

Theorem 2 *conv* $[\mathcal{F}_x^{QP}(S)] \subset$ *conv* $[\mathcal{F}_x^{QP}(\textit{cone}\ (S) \cap \mathcal{Q}_{K\text{-}SD})] \subset \mathcal{F}_x^{K\text{-}SD}(S)$.

Proof The first inclusion holds trivially. To prove the second, we begin by showing

$$\mathcal{Q}_{\text{K-SD}}^* = \left\{ \begin{pmatrix} 0 & 0^T \\ 0 & B \end{pmatrix} \mid B \in \mathcal{C}_n^*(K) \right\} . \tag{1}$$

In fact, $Z = \begin{pmatrix} \beta & b^T \\ b & B \end{pmatrix} \in \mathcal{Q}_{\text{K-SD}}^*$ holds if and only if for all $Q = \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix} \in \mathcal{Q}_{\text{K-SD}}$, i.e., for all $\gamma \in \mathbb{R}$, $c \in \mathbb{R}^n$, $C \in \mathcal{C}_n(K)$ we have

$$Z \bullet Q = \beta\gamma + 2c^T b + C \bullet B \geq 0 .$$

This obviously implies $\beta = 0, b = 0$ and $B \in \mathcal{C}_n^*(K)$. On the other hand for any $Z = \begin{pmatrix} 0 & 0^T \\ 0 & B \end{pmatrix}$, $B \in \mathcal{C}_n^*(K)$ it follows $Z \bullet Q = B \bullet C \geq 0$ since $C \in \mathcal{C}_n(K)$. Now, to compare the feasible sets we can write

$$\mathcal{F}_x^{\text{K-SD}}(S) = \left\{ x \mid \exists X \text{ such that } \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in -S^* \cap \mathcal{C}_{n+1}^*(\mathbb{R}_+ \times K) \right\}$$

and by using the relations $(\text{ cone } (S))^* = S^*$, $(K_1 \cap K_2)^* = K_1^* + K_2^*$ (for closed convex cones) and (1) we obtain

$$\begin{aligned}
\mathcal{F}_x^{\text{QP}}(\text{ cone } (S) \cap \mathcal{Q}_{\text{K-SD}}) &= \left\{ x \mid \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \bullet Q \leq 0 \ \ \forall Q \in \text{ cone } (S) \cap \mathcal{Q}_{\text{K-SD}} \right\} \\
&= \left\{ x \mid \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \in -(\text{ cone } (S) \cap \mathcal{Q}_{\text{K-SD}})^* \right\} \\
&= \left\{ x \mid \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \in -(S^* + \mathcal{Q}_{\text{K-SD}}^*) \right\} \\
&= \left\{ x \mid \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \in -S^* - \begin{pmatrix} 0 & 0^T \\ 0 & \mathcal{C}_n^*(K) \end{pmatrix} \right\}
\end{aligned}$$

Consequently, $x \in \mathcal{F}_x^{\text{QP}}(\text{ cone } (S) \cap \mathcal{Q}_{\text{K-SD}})$ holds if and only if with some $H \in \mathcal{C}_n^*(K)$ we have $\begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} + \begin{pmatrix} 0 & 0^T \\ 0 & H \end{pmatrix} \in -S^*$. But since $xx^T + H - xx^T \in \mathcal{C}_n^*(K), x \in K$, by Lemma 1 it follows

$$\begin{pmatrix} 1 & x^T \\ x & H + xx^T \end{pmatrix} \in -S^* \cap \mathcal{C}_{n+1}^*(\mathbb{R}_+ \times K).$$

So (with $X = H + xx^T$), the vector x is contained in the set $\mathcal{F}_x^{\text{K-SD}}(S)$, and since this set is convex the second inclusion follows.

To see the difference with the SDP case (in Theorem 1) let us chose $x \in \mathcal{F}_x^{\text{K-SD}}(S)$, i.e., with some $X \in \mathcal{S}_n$ the relation

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \bullet Q \leq 0 \quad \text{for all} \quad Q = \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix} \in S$$

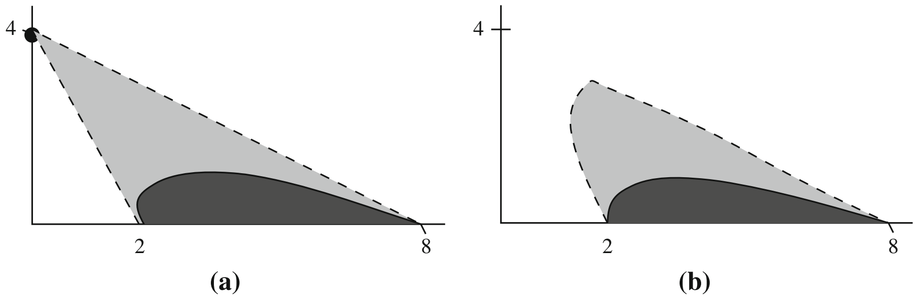

Fig. 1 **a** Sets for $\rho = 0$. **b** Sets for $\rho = 0.2$

must hold. Then we also obtain

$$\begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \bullet Q = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \bullet Q + \begin{pmatrix} 0 & 0^T \\ 0 & xx^T - X \end{pmatrix} \bullet Q \le C \bullet (xx^T - X) \; .$$

Unfortunately, the converse of Lemma 1 is not generally true. So here, even with $Q \in \operatorname{cone}(S) \cap \mathcal{Q}_{\text{K-SD}}$, i.e., with $C \in \mathcal{C}_n(K)$, the relation $C \bullet (xx^T - X) \le 0$ need not hold and x need not satisfy the corresponding original constraint $C \bullet xx^T + 2c^T x + \gamma \le 0$. We give some examples to illustrate the statement of Theorem 2 and to show that in general (for $K \ne \mathbb{R}^n$) the situation is more complicated than in the SDP case (for $K = \mathbb{R}^n$).

Example 1 We chose $K = \mathbb{R}^n_+$, i.e., the completely positive relaxation. Let us take the special case $S \subset \mathcal{Q}_{\text{K-SD}}$. In contrast to the SDP relaxation the set $\mathcal{F}_x^{\text{QP}}(S)$ need not be convex. So, an inclusion $\mathcal{F}_x^{\text{K-SD}}(S) \subset \mathcal{F}_x^{\text{QP}}(S)$ is not true in general. Even $\mathcal{F}_x^{\text{K-SD}}(S) \subset \operatorname{conv}[\mathcal{F}_x^{\text{QP}}(S)]$ need not to hold as we shall show. Theorem 2 only assures the converse $\operatorname{conv}[\mathcal{F}_x^{\text{QP}}(S)] \subset \mathcal{F}_x^{\text{K-SD}}(S)$. Even in the case $S = \{Q\}$ with $Q = \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix} \in \mathcal{Q}_{\text{K-SD}}$ the latter inclusion can be strict. Take for example

$$C = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}, \quad c = (-2.5, \; -2 + \rho), \quad \gamma = 8.$$

The feasibility conditions read:

$$\text{for } \mathcal{F}_x^{\text{QP}}(\{Q\}): \; \frac{1}{2}(x_1^2 + x_2^2) + 2x_1x_2 - 5x_1 - (2-\rho)x_2 + 8 \le 0, \quad \text{and } x \ge 0$$

$$\text{for } \mathcal{F}_x^{\text{K-SD}}(\{Q\}): \; \frac{1}{2}(X_{11} + X_{22}) + 2X_{12} - 5x_1 - (2-\rho)x_2 + 8 \le 0, \; \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}_3^*(\mathbb{R}^3_+)$$

We have computed the feasible sets. For $\rho = 0$ the set $\mathcal{F}_x^{\text{QP}}(\{Q\})$ consists of the point $(0, 4)$ together with the convex (black) set (see Fig. 1a). The set $\mathcal{F}_x^{\text{K-SD}}(\{Q\})$ equals the (grey) triangle $\operatorname{conv}[\mathcal{F}_x^{\text{QP}}(\{Q\})]$. For $\rho > 0$ (small) the point $(0, 4)$ is no more feasible for $\mathcal{F}_x^{\text{QP}}(\{Q\})$ (black) and the (convex) set $\mathcal{F}_x^{\text{K-SD}}(\{Q\})$ (grey) (depending continuously on ρ) is as sketched in Fig. 1b (for $\rho = 0.2$). Obviously in this example $\rho = 0.2$ we have

$$\mathcal{F}_x^{\text{QP}}(\{Q\}) = \operatorname{conv}[\mathcal{F}_x^{\text{QP}}(\{Q\})] \subsetneq \mathcal{F}_x^{\text{K-SD}}(\{Q\}) \; .$$

For the special case $S \cap \mathcal{Q}_{\text{K-SD}} = \emptyset$ we have:

$$\operatorname{conv}[\mathcal{F}_x^{\text{QP}}(\operatorname{cone}(S) \cap \mathcal{Q}_{\text{K-SD}}] \subset \mathcal{F}_x^{\text{K-SD}}(S) \subset \operatorname{conv}[\mathcal{F}_x^{\text{QP}}(S \cap \mathcal{Q}_{\text{K-SD}}] = \mathbb{R}^n_+ \; .$$

The equality on the right-hand side follows by the assumption $S \cap \mathcal{Q}_{\text{K-SD}} = \emptyset$, so that the feasibility condition for $\mathcal{F}_x^{\text{QP}}(S \cap \mathcal{Q}_{\text{K-SD}})$ reduces to $x \in K = \mathbb{R}^n_+$. It is not difficult to construct examples where both inclusions are strict.

Acknowledgment We would like to thank the referee's for their valuable comments.

Appendix

For completeness we provide the explicit description of $\mathcal{C}_n^*(K)$ and a generalization of a Schur complement result.

Lemma 2 *For any closed set $K \subset \mathbb{R}^n$ the dual of $\mathcal{C}_n(K) = \{A \in \mathcal{S}_n \mid z^T Az \geq 0 \ \forall z \in K\}$ is: $\mathcal{C}_n^*(K) = \{Y = \sum_j y_j z_j z_j^T \mid y_j \geq 0, z_j \in K\}$.*

Proof We show that with $\mathcal{C} := \{Y = \sum_j y_j z_j z_j^T \mid y_j \geq 0, z_j \in K\}$ we have $\mathcal{C}_n(K) = \mathcal{C}^*$. By using $C^{**} = C$ (for closed convex cones C) we find the identity claimed in the lemma. "$\subset$": If $A \in \mathcal{C}_n(K)$ then for all $Y \in \mathcal{C}$ we obviously have $A\bullet Y = \sum_j y_j A\bullet z_j z_j^T \geq 0$, i.e., $A \in \mathcal{C}^*$.
"$\supset$": Suppose $A \notin \mathcal{C}_n(K)$, i.e., $z^T Az < 0$ for some $z \in K$. Then for $Y = zz^T \in \mathcal{C}$ it follows $Y\bullet A < 0$, so that $A \notin \mathcal{C}^*$. □

Lemma 3 [generalized Schur complement lemma, see e.g., [1] for the case $K = \mathbb{R}^n_+$] *It holds*

$$\begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix} \in \mathcal{C}_{n+1}(\mathbb{R}_+ \times K) \quad \Leftrightarrow \quad \begin{array}{l} \gamma \geq 0,\ C \in \mathcal{C}_n(K) \quad \textit{and} \\ x^T(\gamma C - cc^T)x \geq 0 \ \forall x \in K \ \textit{with}\ c^T x \leq 0 \ . \end{array}$$

Proof The left-hand side means: $(\alpha \ \ x)^T \begin{pmatrix} \gamma & c^T \\ c & C \end{pmatrix} \begin{pmatrix} \alpha \\ x \end{pmatrix} = \gamma\alpha^2 + 2\alpha c^T x + x^T Cx \geq 0 \ \ \forall \alpha \geq 0,\ \ x \in K$. "$\Rightarrow$": The above inequality implies $\gamma \geq 0$, $x^T Cx \geq 0 \forall x \in K$ and in the case $c^T x \geq 0$ we are done. In the case $c^T x \leq 0$, $\gamma = 0$ we also obtain $c^T x = 0$. For the remaining case $c^T x \leq 0$, $\gamma > 0$ we write

$$0 \leq \gamma\alpha^2 + 2\alpha c^T x + x^T Cx = \frac{1}{\gamma}(\gamma\alpha + c^T x)^2 + \frac{1}{\gamma} x^T(\gamma C - cc^T)x \ .$$

Then the assumption $x^T(\gamma C - cc^T)x < 0$ for some $x \in K$, $c^T x \leq 0$ leads to a contradiction (with a choice $\alpha = -c^T x \geq 0$). The direction "$\Leftarrow$" is easy. □

References

1. Bomze, I.M.: Block pivoting and shortcut strategies for detecting copositivity. Linear Algebra Appl. **248**, 161–184 (1996)
2. Bomze, I.M., De Klerk, E.: Solving standard quadratic optimization problems via linear semidefinite and copositive programming. Dedicated to Professor Naum Z. Shor on his 65th birthday. J. Glob. Optim. **24**(2), 163–185 (2002)
3. Bomze, I.M., Dür, M., de Klerk, E., Roos, C., Quist, A.J., Terlaky, T.: On copositive programming and standard quadratic optimization problems. J. Glob. Optim. **18**, 301–320 (2000)
4. Bomze, I.M., Jarre, F.: A note on Burer's copositive representation of mixed-binary QPs. Optim. Lett. **4**(3), 465–472 (2010)

5. Bomze, I.M., Schachinger, W.: Multi-standard quadratic optimization: interior point methods and cone programming reformulation. Comput. Optim. Appl. **45**, 237–256 (2010)
6. Burer, S.: On the copositive representation of binary and continuous nonconvex quadratic programs. Math. Program. **120**, 479–495 (2009)
7. Burer, S., Dong, H.: Representing quadratically constrained quadratic programs as generalized copositive programs. Oper. Res. Lett. **40**(3), 203–206 (2012)
8. Burer, S.: Copositive programming. In: Anjos, M., Lasserre, J.B. (eds.) Handbook of Semidefinite, Cone and Polynomial Optimization (to appear) (2010)
9. de Klerk, E., Pasechnik, D.V.: Approximation of the stability number of a graph via copositive programming. SIAM J. Optim. **12**(4), 875–892 (2002)
10. Dür, M.: Copositive programming—a survey, in Recent Advances. In: Diehl, M., Glineur, F., Jarlebring, E., Michiels, W. (eds.) Optimization and its Applications in Engineering, pp. 320 Springer, Berlin (2010)
11. Eichfelder, G., Jahn, J.: Set-semidefinite optimization. J. Convex Anal. **15**(4), 767–801 (2008)
12. Kojima, M., Tunçel, L.: Cones of matrices and successive convex relaxations of nonconvex sets. SIAM J. Optim. **10**(3), 750–778 (2000)
13. Povh, J., Rendl, F.: A copositive programming approach to graph partitioning. SIAM J. Optim. **18**(1), 223–241 (2007)
14. Povh, J., Rendl, F.: Copositive and semidefinite relaxations of the quadratic assignment problem. Discret. Optim. **6**, –3231241 (2009)
15. Preisig, J.C.: Copositivity and the minimization of quadratic functions with nonnegativity and quadratic equality constraints. SIAM J. Control Optim. **34**, 1135–1150 (1996)
16. Quist, A.J., De Klerk, E., Roos, C., Terlaky, T.: Copositive relaxation for general quadratic programming. Optim. Methods Softw. **9**, 185–208 (1998)