Closed-Form Expressions for Time-Frequency Operations Involving Hermite Functions

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***Abstract*—The product, convolution, correlation, Wigner distribution function (WDF) and ambiguity function (AF) of two Hermite functions of arbitrary order n and m are derived and expressed as a bounded, weighted sum of $n+m$ Hermite functions. It was already known that these mathematical operations performed on Gaussians (Hermite functions of the zeroth-order) lead to a result which can be expressed as a Gaussian function again. We generalize this reciprocity to Hermite functions of arbitrary order. The product, convolution, correlation, WDF, and AF operations performed on two Hermite functions of arbitrary order lead to remarkably similar closed-form expressions, where the difference between the operations is primarily determined by distinct phase changes of the weights of the Hermite functions in the result. The closed-form expressions are generalized to the class of square-integrable functions. A key insight from the closed-form expressions is applied to the design of orthogonal, time-frequency localized communication signals which are characterized by an AF with rotational symmetry. In addition to this application, the theoretical expressions may prove useful for signal analysis in fields ranging from communications, radar and image processing to quantum mechanics.**

***Index Terms*—Hermite functions, closed-form solutions, correlation functions, signal analysis, signal detection, ambiguity function, Wigner distribution function.**

I. Introduction

Hermite functions form an orthonormal basis for $L^2(\mathbb{R})$, the class of square-integrable functions, and consist of a Hermite polynomial, a Gaussian window and a normalization. Hermite functions can be found in various fields of science and engineering; e.g., they constitute the stationary states of the quantum harmonic oscillator [1], are eigenmodes in multi-mode optical fibers [2], eigenfunctions of the (fractional) Fourier transform and possess maximum energy concentration in terms of their second-order moments in domains linked by the Fourier transform [3]. Hermite functions find application in image processing [4], [5], quantum mechanics [1], optics [6], electroencephalograph (EEG) processing [7], spectrum estimation [8], ultra-wideband pulseshaping [9] and as a basis for multicarrier communications [10], [11].

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Autocorrelation functions of Hermite functions have first been addressed by Miller in 2003 [12]. In response to that article, Nadarajah derived the autocorrelation functions for Hermite functions [13]. Cross-correlation functions for Hermite functions have also been derived in terms of associated Laguerre polynomials [14]. Abreu derived cross-correlation functions for a generalized class of Hermite functions, called generalized Hermite wavelets [15]. In that article the author noticed that his results *"strongly suggest that the correlation functions of generalized Hermite wavelets can also be represented by generalized Hermite wavelets"* [15]. This observation is investigated in this article. However, we limit the scope to conventional Hermite functions, being an orthonormal subset of the class referred to as generalized Hermite wavelets by Abreu.

It is well known that the product, convolution and the correlation of two Gaussian functions lead to a result which is a Gaussian function again. The two-dimensional Wigner distribution function (WDF) and ambiguity function (AF) of a Gaussian signal also lead to a Gaussian function (e.g., [16]). As the Gaussian is in fact the zeroth-order Hermite function, the results of the aforementioned operations on the zeroth-order Hermite functions are well-established. We extend the Gaussian case and generalize it to Hermite functions of arbitrary order. As Hermite functions have several well-studied properties and often serve as an orthonormal signal basis, it is advantageous to express the results of mathematical operators in terms of Hermite functions. It allows for subsequent reuse of the same equations when multiple operations are performed consecutively.

Section II provides general definitions and identities which are used in later sections. In Section III the closed-form expressions for the product, convolution and correlation of two Hermite functions are derived and expressed in terms of new Hermite functions. Section IV derives the cross-WDF and cross-AF involving Hermite functions. In Section V one generalized closed-form expression is introduced and the results are generalized to the class of square-integrable functions. Section VI discusses an application of the generalized closed-form expressions for the orthogonalization of highly time-frequency localized, spectrally efficient (communication) signals. Finally, Section VII concludes this article and summarizes the main findings.

II. Definitions

Hermite functions consist of a Hermite polynomial, a Gaussian window and a normalization. Without loss of generality we primarily deal with time- and frequency-varying signals. Note that the analysis and forthcoming results apply equally well to other Fourier pairs such as position and momentum. A Hermite function $h_n(t)$ of order n is defined as

$$h_n(t) \triangleq \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \cdot H_n(t) \cdot e^{-\frac{t^2}{2}} \tag{1}$$

with H_n being the nth degree Hermite polynomial

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}. \tag{2}$$

Hermite functions form a complete orthonormal set [17]:

$$\int_{-\infty}^{\infty} h_n(t) \cdot h_m(t) dt = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases} \tag{3}$$

Hermite functions are the eigenfunctions of the unitary fractional Fourier transform (FrFT) operator [18], [19]:

$$\mathcal{F}^{\alpha}\{h_n(t)\}(u) \triangleq \int_{-\infty}^{\infty} K_{\alpha}(t,u) h_n(t) dt \tag{4}$$

$$= \lambda_n h_n(u), \tag{5}$$

where the eigenvalue corresponding to the Hermite function of order n equals a complex constant $\lambda_n = e^{-jn\alpha}$ with modulus 1, and K_α represents the kernel of the FrFT, i.e., [see (6) at the bottom of the page], where $\delta()$ is the Dirac delta function. A signal transformed by the FrFT leads to a signal representation over axis u making an angle α with the time-axis. The FrFT is known to lead to a rotation of the WDF by an angle α [20]. When α is equal to $\pi/2$ or $-\pi/2$ we obtain a counter-clockwise or clockwise rotation in time-frequency by 90 degrees and (4) reduces to the unitary forward and inverse Fourier transform, respectively. The benefits of using the FrFT to perform operations like convolution, filtering and multiplexing in the time-frequency domain—e.g., for optical applications—have been underlined in [6].

To derive closed-form expressions involving Hermite functions, we define the product $(\cdot)$, convolution $(*)$ and correlation $(\star)$ of two functions $f(t)$ and $g(t)$, which follow well-known definitions:

$$(f \cdot g)(t) \triangleq f(t) \cdot g(t) \tag{7}$$

$$(f * g)(\tau) \triangleq \int_{-\infty}^{\infty} f(t) \cdot g(\tau - t) dt \tag{8}$$

$$(f \star g)(\tau) \triangleq \int_{-\infty}^{\infty} \overline{f(t)} \cdot g(\tau + t) dt, \tag{9}$$

where $\overline{f(t)}$ represents the complex conjugate of $f(t)$ and τ represents the lag of the convolution and correlation integrals, respectively. The unitary Fourier transforms of the convolution and correlation integrals equal products of the Fourier transforms of the individual functions $f(t)$ and $g(t)$:

$$\mathcal{F}^{\frac{\pi}{2}}\{(f * g)(\tau)\} = \sqrt{2\pi} \cdot \mathcal{F}^{\frac{\pi}{2}}\{f(t)\} \cdot \mathcal{F}^{\frac{\pi}{2}}\{g(t)\} \tag{10}$$

$$\mathcal{F}^{\frac{\pi}{2}}\{(f \star g)(\tau)\} = \sqrt{2\pi} \cdot \overline{\mathcal{F}^{\frac{\pi}{2}}\{f(t)\}} \cdot \mathcal{F}^{\frac{\pi}{2}}\{g(t)\}. \tag{11}$$

III. The Product, Convolution and Correlation of Hermite Functions

To arrive at closed-form expressions for the convolution and correlation of two Hermite functions, this section first addresses the product of two Hermite functions h_n and h_m. The product of two Hermite functions can be written as:

$$h_n(t) \cdot h_m(t) = \frac{1}{\sqrt{2^{n+m} n! m! \pi}} \cdot H_n(t) \cdot H_m(t) e^{-t^2}. \tag{11}$$

The polynomial multiplication can be calculated using an identity, first derived by Feldheim [21] and Watson [22]:

$$H_n(t) \cdot H_m(t) = \sum_{u=0}^{\min(m,n)} \frac{m! n! 2^u}{(m-u)!(n-u)!u!} H_{n+m-2u}(t) \tag{12}$$

leading to a sum of either odd or even Hermite polynomials up to degree $n+m$. The Hermite polynomials in the right-hand side of (12) are not orthogonal when multiplied by the weighting function e^{-t^2} of (11). Therefore, as we aim for a set of orthogonal Hermite functions, the polynomials $H_{n+m-2u}(t)$ are reformulated in terms of polynomials $H_{n+m-2u}(t\sqrt{2})$, which is achieved by the following identity [23]:

$$H_n(t) = \sum_{r=0}^{\lfloor n/2 \rfloor} \left(\frac{1}{\sqrt{2}}\right)^n (-1)^r \frac{n!}{r!} \cdot \frac{H_{n-2r}(t\sqrt{2})}{(n-2r)!}. \tag{13}$$

The product of two Hermite polynomials as in (11) can now be written as a vector-matrix-vector product as shown in (14) with elements corresponding to (12) and (13), respectively.

$$\begin{aligned} &H_n(t) \cdot H_m(t) \\ &= \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_{\min(m,n)} \end{bmatrix}^T \\ &\cdot \begin{bmatrix} \beta_{0,0} & \beta_{0,1} & \beta_{0,2} & \cdots & \beta_{0,\lfloor (m+n)/2 \rfloor} \\ 0 & \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,\lfloor (m+n)/2 \rfloor} \\ 0 & 0 & \beta_{2,2} & \cdots & \beta_{2,\lfloor (m+n)/2 \rfloor} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \beta_{\min(m,n),\lfloor (m+n)/2 \rfloor} \end{bmatrix} \\ &\cdot \begin{bmatrix} H_{n+m}(\sqrt{2}t) \\ H_{n+m-2}(\sqrt{2}t) \\ H_{n+m-4}(\sqrt{2}t) \\ \vdots \\ H_0(\sqrt{2}t) \text{ or } H_1(\sqrt{2}t) \end{bmatrix} \end{aligned} \tag{14}$$

$$K_\alpha(t,u) \triangleq \begin{cases} \sqrt{\frac{1 - j\cot(\alpha)}{2\pi}} \cdot e^{j\frac{u^2+t^2}{2}\cot(\alpha)} \cdot e^{-jut\csc(\alpha)} & \text{if } \alpha \neq p \cdot \pi \\ \delta(t-u) & \text{if } \alpha = p \cdot 2\pi \\ \delta(t+u) & \text{if } \alpha + \pi = p \cdot 2\pi,\ p \in \mathbb{Z}, \end{cases} \tag{6}$$

$$\text{with } \alpha_u = \frac{m!n!2^u}{(m-u)!(n-u)!u!}$$
$$\beta_{u,k} = \left(\frac{1}{\sqrt{2}}\right)^l (-1)^r \cdot \frac{l!}{r!(l-2r)!}\Bigg|_{\substack{r=k-u\\ l=n+m-2u}}$$
$$= \left(\frac{1}{\sqrt{2}}\right)^{n+m-2u} (-1)^{(k-u)} \cdot \frac{(n+m-2u)!}{(k-u)!(n+m-2k)!}$$

which can be written as:

$$H_n(t)\cdot H_m(t) = \sum_{k=0}^{\lfloor\frac{m+n}{2}\rfloor} \chi_{k,n,m}\cdot H_{n+m-2k}(\sqrt{2}t)$$
$$\text{with } \chi_{k,n,m} \triangleq \sum_{u=0}^{\min(k,n,m)} \alpha_u\cdot\beta_{u,k}$$
$$= \frac{(-1)^k n!m!}{(\sqrt{2})^{(n+m)}(n+m-2k)!} \cdot \sum_{u=0}^{\min(k,n,m)} \frac{(-4)^u(n+m-2u)!}{u!(m-u)!(n-u)!(k-u)!} \quad (15)$$

This reformulation allows to write the polynomial product as a weighted sum of $\lfloor m+n\rfloor/2$ Hermite polynomials as shown by (15). Substituting the result of (15) in (11), (10) and (11) leads us to the product, convolution and cross-correlation functions of two Hermite functions of degree n and m. The results are shown in (16a), (16b) and (16c), respectively.

$$h_n(t)\cdot h_m(t) = \frac{1}{\sqrt{2^{n+m}n!m!\pi}} \cdot \sum_{k=0}^{\lfloor\frac{m+n}{2}\rfloor} \chi_{k,n,m}\cdot H_{n+m-2k}(\sqrt{2}t)e^{-t^2}$$
$$= \sum_{k=0}^{\lfloor\frac{m+n}{2}\rfloor} (-1)^k\cdot c_{k,n,m}\cdot h_{n+m-2k}(\sqrt{2}t) \quad (16a)$$
$$(h_n(t)*h_m(t))(\tau) = \mathcal{F}^{-\frac{\pi}{2}}\left(\sqrt{2\pi}j^{(n+m)}h_n(\omega)\cdot h_m(\omega)\right)$$
$$= \sqrt{\pi}\sum_{k=0}^{\lfloor\frac{m+n}{2}\rfloor} c_{k,n,m}\cdot h_{n+m-2k}\left(\frac{1}{\sqrt{2}}\tau\right) \quad (16b)$$
$$(h_n(t)\star h_m(t))(\tau) = \mathcal{F}^{-\frac{\pi}{2}}\left(\sqrt{2\pi}j^{(n-m)}h_n(\omega)\cdot h_m(\omega)\right)$$
$$= \sqrt{\pi}(-1)^m\sum_{k=0}^{\lfloor\frac{m+n}{2}\rfloor} c_{k,n,m}\cdot h_{n+m-2k}\left(\frac{1}{\sqrt{2}}\tau\right) \quad (16c)$$
$$\text{with } c_{k,n,m} \triangleq \frac{\sqrt{n!m!}}{2^k\sqrt{2^{(n+m)}(n+m-2k)!\sqrt{\pi}}} \cdot \sum_{u=0}^{\min(k,n,m)} \frac{(-4)^u(n+m-2u)!}{u!(m-u)!(n-u)!(k-u)!}$$

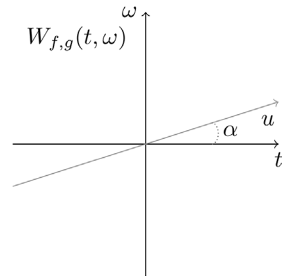

Fig. 1. Coordinate system of the Wigner distribution function (WDF).

IV. Wigner Distribution and Ambiguity Functions of Hermite Functions

The Wigner distribution function (WDF) transforms a one-dimensional time-varying signal $f(t)$ to a two-dimensional joint time-frequency description. A more general formulation for the WDF of two signals $f(t)$ and $g(t)$ has been found in the cross-WDF, being defined as [20]

$$W_{f,g}(t,\omega) \triangleq \int_{-\infty}^{\infty} f\left(t+\frac{t'}{2}\right)\cdot\overline{g\left(t-\frac{t'}{2}\right)}e^{-j\omega t'}dt' \quad (17)$$

which can be interpreted as a joint time-frequency cross-spectrum distribution function. Substitution of $t = u\cos(\alpha)$ and $\omega = u\sin(\alpha)$ changes the coordinate system to polar coordinates, where the axis in time-frequency is $u = \sqrt{t^2+\omega^2}$ and α represents the angle with the time axis as shown in Fig. 1. The slice of the WDF making an angle α with the time axis can be expressed as the correlation of the fractional Fourier transformed signals $f(t)$ and $g(t)$ [24]:

$$W_{f,g}(u\cos(\alpha),u\sin(\alpha)) = \mathcal{F}^{\pi/2}\{\sqrt{2\pi}\cdot\mathcal{F}^{\alpha-\pi/2}\{f(t)\}(u'/2) \cdot\overline{\mathcal{F}^{\alpha-\pi/2}\{g(t)\}(-u'/2)}\} \quad (18)$$

where the u' axis is making an angle $\alpha-\pi/2$ with the time-axis. The cross-WDF for two Hermite functions $h_n(t)$ and $h_m(t)$ is

$$W_{h_n,h_m}(u\cos(\alpha),u\sin(\alpha)) = \mathcal{F}^{\pi/2}\{\sqrt{2\pi}\cdot\mathcal{F}^{\alpha-\pi/2}\{h_n(t)\}(u'/2) \cdot\overline{\mathcal{F}^{\alpha-\pi/2}\{h_m(t)\}(-u'/2)}\}. \quad (19)$$

As Hermite functions form the eigenfunctions of the FrFT operator, the FrFT of h_n is straightforward and only yields a complex multiplication by the eigenvalue $\lambda_n = e^{-j\alpha n}$ as shown in (5). Substitution in (19), calculating the product of two Hermite functions in correspondence to (16a) and the inverse Fourier transform give for the WDF of two Hermite functions

$$W_{h_n,h_m}(u\cos(\alpha),u\sin(\alpha)) = \sqrt{2\pi}e^{j(n-m)\alpha+j(n+m)\frac{\pi}{2}} \cdot(-1)^m\sum_{k=0}^{\lfloor\frac{m+n}{2}\rfloor} c_{k,n,m}h_{n+m-2k}(\sqrt{2}u), \quad (20)$$

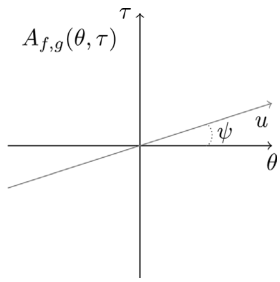

Fig. 2. Coordinate system of the ambiguity function (AF).

with $c_{k,n,m}$ as given by (16). In Cartesian coordinates, the closed-form expression is given by

$$W_{h_n(t),h_m(t)}(t,\omega) = \sqrt{\pi}e^{j\alpha(n-m)} \cdot (+j)^{n-m} \sum_{k=0}^{\lfloor \frac{m+n}{2} \rfloor} c_{k,n,m} h_{n+m-2k} \cdot (\sqrt{2}\cdot\sqrt{t^2+\omega^2}) \tag{21}$$

Whereas the WDF provides insight in the time-frequency distribution, the ambiguity function (AF) gives the two-dimensional time-frequency correlation of two functions with a relative time-lag τ and frequency-lag θ. The AF finds application in time-frequency analysis, waveform design and radar signal processing [16]. The symmetric and narrowband cross-AF of two functions $f(t)$ and $g(t)$ is defined as [20]:

$$A_{f,g}(\theta,\tau) \triangleq \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right)\cdot \overline{g\left(t-\frac{\tau}{2}\right)} e^{j\theta t} dt, \tag{22}$$

where θ and τ represent the radial frequency and time shift, respectively.

We use a similar approach for the AF as for the WDF. Substitution of $\theta = u\cos(\psi)$ and $\tau = u\sin(\psi)$ changes the coordinate system to polar coordinates, where $u = \sqrt{\theta^2+\tau^2}$ and ψ represents the angle with the frequency-shift axis as shown in Fig. 2. The u-axis of the AF making an angle ψ with the θ axis can be expressed as the multiplication of the fractional Fourier transformed signals $f(t)$ and $g(t)$ [24]:

$$A_{f,g}(u\cos(\psi), u\sin(\psi)) = \mathcal{F}^{-\pi/2}\{\sqrt{2\pi}\cdot\mathcal{F}^{\psi}\{f(t)\}\cdot\overline{\mathcal{F}^{\psi}\{g(t)\}}\}. \tag{23}$$

The cross-AF for Hermite functions $h_n(t)$ and $h_m(t)$ along the u-axis making an angle ψ with the frequency-lag axis θ leads to:

$$A_{h_n,h_m}(u\cos(\psi), u\sin(\psi)) = \mathcal{F}^{-\pi/2}\{\sqrt{2\pi}\mathcal{F}^{\psi}\{h_n(t)\}\cdot\overline{\mathcal{F}^{\psi}\{h_m(t)\}}\}. \tag{24}$$

In a similar fashion, as we calculated the WDF, we can simplify this expression such that the AF becomes

$$A_{h_n,h_m}(u\cos(\psi), u\sin(\psi)) = e^{j(n-m)\psi - j(n+m)\frac{\pi}{2}} \cdot\sqrt{\pi} \sum_{k=0}^{\lfloor \frac{m+n}{2} \rfloor} c_{k,n,m}\cdot h_{n+m-2k}\left(\frac{1}{\sqrt{2}}u\right), \tag{25}$$

with $c_{k,n,m}$ as given by (16). In Cartesian coordinates the cross-AF for Hermite functions is provided in

$$A_{h_n(t),h_m(t)}(\theta,\tau) = \sqrt{\pi}e^{j\psi(n-m)} \cdot (-j)^{n+m} \sum_{k=0}^{\lfloor \frac{m+n}{2} \rfloor} c_{k,n,m} h_{n+m-2k} \cdot \left(\frac{1}{\sqrt{2}}\cdot\sqrt{\theta^2+\tau^2}\right) \tag{26}$$

Apart from a constant, the AF reduces to the cross-correlation operation in time or frequency when $\theta = 0$ or $\tau = 0$, respectively. It is noteworthy that the closed-form expressions are remarkably similar for the one- and two-dimensional operations which can be seen in (16), (21), and (26)respectively.

V. Generalizations

A. One Generalized Closed-Form Expression for Operations Involving Hermite Functions

Given the similarity among the closed-form expressions derived for mathematical operations involving Hermite functions, a generalized form can be deduced. The 1-dimensional operations—product, convolution and correlation—as well as the 2-dimensional operations—WDF and AF—can be described by the following generalized form:

$$r_i = \mathcal{A}_i \cdot \sum_{k=0}^{\lfloor \frac{m+n}{2} \rfloor} p_i \cdot c_{k,n,m} \cdot h_{n+m-2k}(u_i), \tag{27}$$

where r_i represents the outcome of a product, convolution, auto/cross-correlation, auto/cross-WDF or auto/cross-AF operation in accordance with the definitions in Table I. For each of the discussed operations on two Hermite functions of order n and m, the normalization $\mathcal{A}$, phase term p and argument u are as given in Table I. A remarkable outcome—although it is a logical consequence of the FrFT eigenfunction property of Hermite functions—is that the difference between all these operations (involving Hermite functions) is primarily determined by the phase term p. Once we have the outcome of a cross-correlation operation we can easily interchange the result to a product, or a product to the WDF and so forth, just by projecting again the result on the orthogonal Hermite basis and changing the phase and argument of the individual Hermite functions. This is particularly useful when Hermite functions are the basis of choice, like in many quantum-mechanical applications or when one wants to perform multiple operations in cascade (e.g., first a correlation in time and then a correlation in frequency) like in Hermite-based communication systems [11].

A Hermite function of order n oscillates between the turning points $[-\sqrt{2n+1}, +\sqrt{2n+1}]$ and has exponential decay outside this interval [25]. As the maximum order of a Hermite function for the operations listed in Table I is at most $m+n$, the oscillatory behavior for argument u is bounded by the interval $[-\phi\sqrt{2(m+n)+1}, +\phi\sqrt{2(m+n)+1}]$ and exponentially decreases outside this region. The parameter ϕ is either $1/\sqrt{2}$ or $\sqrt{2}$ in correspondence with the dilation of the arguments u listed in Table I. The correlation-type of operations (convolution, correlation and the AF) have an argument dilated by a factor two compared to the product and the WDF as shown by Table I.

TABLE I
GENERALIZED DESCRIPTIONS FOR TIME-FREQUENCY OPERATIONS INVOLVING HERMITE FUNCTIONS

$$\text{Generalized description: } r_i = \mathcal{A}_i \cdot \sum_{k=0}^{\lfloor \frac{m+n}{2} \rfloor} p_i \cdot c_{k,n,m} \cdot h_{n+m-2k}(u_i)$$

Operation	Description	Index	Condition	Scaling $\mathcal{A}$	Phase p	Argument u
Product	$(h_n(t) \cdot h_m(t))(t)$	i=1		1	$(-1)^k$	$\sqrt{2}t$
Convolution	$(h_n(t) * h_m(t))(\tau)$	i=2		$\sqrt{\pi}$	1	$\frac{1}{\sqrt{2}}\tau$
Auto-correlation	$(h_n(t) \star h_m(t))(\tau)$	i=3	$m = n$	$\sqrt{\pi}$	$(-1)^m$	$\frac{1}{\sqrt{2}}\tau$
Cross-correlation	$(h_n(t) \star h_m(t))(\tau)$	i=4	$m \neq n$	$\sqrt{\pi}$	$(-1)^m$	$\frac{1}{\sqrt{2}}\tau$
Auto-WDF	$W_{h_n(t),h_m(t)}(t,\omega)$	i=5	$m = n$	$\sqrt{\pi}$	1	$\sqrt{2} \cdot \sqrt{t^2+\omega^2}$
Cross-WDF	$W_{h_n(t),h_m(t)}(t,\omega)$	i=6	$m \neq n$	$\sqrt{\pi}$	$e^{-j\alpha'(n-m)} \cdot (+j)^{n-m}$	$\sqrt{2} \cdot \sqrt{t^2+\omega^2}$
Auto-AF	$A_{h_n(t),h_m(t)}(\theta,\tau)$	i=7	$m = n$	$\sqrt{\pi}$	$(-1)^n$	$\frac{1}{\sqrt{2}} \cdot \sqrt{\theta^2+\tau^2}$
Cross-AF	$A_{h_n(t),h_m(t)}(\theta,\tau)$	i=8	$m \neq n$	$\sqrt{\pi}$	$e^{j\psi'(n-m)} \cdot (-j)^{n+m}$	$\frac{1}{\sqrt{2}} \cdot \sqrt{\theta^2+\tau^2}$

with $\alpha' = \mathrm{Arg}(t + j \cdot \omega)$, $\psi' = \mathrm{Arg}(\theta + j \cdot \tau)$, $c_{k,n,m} = \frac{\sqrt{n!m!}}{2^k \sqrt{2^{(n+m)}(n+m-2k)!\sqrt{\pi}}} \sum_{u=0}^{\min(k,n,m)} \frac{(-4)^u (n+m-2u)!}{u!(m-u)!(n-u)!(k-u)!}$

Thanks to the closed-form expressions, one can evaluate the product, convolution, correlation, AF and WDF at a single point simply by evaluating the (bounded) sums. The computational complexity of evaluating a single point becomes proportional to N, where N represents the number of Hermite functions in the end-result. Depending on the size of N, this can mean a reduction in computational complexity compared to the straightforward calculations, especially in case of the AF and WDF.

In brief, using the generalized closed-form expression, a strong similarity between one-dimensional and two-dimensional time-frequency operations involving Hermite functions can be observed as shown in Table I. Given that the result of these operations results in a bounded sum of Hermite functions, some well-known characteristics of Hermite functions apply, e.g., the resulting functions exhibit exponential decay both in the time-domain as well as in the frequency-domain.

B. Generalizations to Square-Integrable Functions

As Hermite functions are a complete orthonormal basis for $L^2(\mathbb{R})$, every square-integrable function $f(t)$ can be written as a sum of Hermite functions $f(t) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n \cdot h_n(t)$. This is especially useful for time-frequency localized signals which are well-described by a limited set of Hermite functions; how large N needs to be for a given approximation error depends on the characteristics of the function $f(t)$ [25]. Applying the results of previous sections, we can give the cross-correlation functions for square-integrable functions $f(t)$ and $g(t)$ in terms of normalized Hermite functions:

$$f(t) \star g(t) = \sum_{n=0}^{\infty} \overline{a}_n \sum_{m=0}^{\infty} b_m \sqrt{\pi} \cdot (-1)^m \sum_{k=0}^{\lfloor \frac{m+n}{2} \rfloor} c_{k,n,m} \cdot h_{n+m-2k}\left(\frac{1}{\sqrt{2}}\tau\right), \quad (28)$$

where $c_{k,n,m}$ is as defined before in (16). a_n and b_n are the coefficients from the Hermite expansions of $f(t)$ and $g(t)$, respectively:

$$a_n \triangleq \int_{-\infty}^{\infty} f(t) \cdot h_n(t) dt$$
$$b_m \triangleq \int_{-\infty}^{\infty} g(t) \cdot h_m(t) dt. \quad (29)$$

Given two square-integrable functions $f(t)$ and $g(t)$, the cross-WDF and cross-AF can be written by Hermite expansions as well:

$$W_{f,g}(t,\omega) = \sum_{n,m} a_n \overline{b}_m W_{h_n(t),h_m(t)}(t,\omega) \quad (30)$$
$$A_{f,g}(\theta,\tau) = \sum_{n,m} a_n \overline{b}_m A_{h_n(t),h_m(t)}(\theta,\tau), \quad (31)$$

where a_n and b_m are as defined in (29). These expressions prove useful in Section VI.

VI. Application of the Rotational Symmetric AF for the Design of Communication Signals

This section applies the theoretical results derived in last section to design orthogonal, time-frequency localized and spectrally efficient waveforms for communications.

It was already stated in [26] that signals constructed by Hermite functions of order $n = 0, 4, \ldots 4N$ are characterized by a time representation which is equal in magnitude to the frequency representation. We are able to generalize this notice to a class of signals characterized by a rotationally symmetric AF based on (26):

$$A_{h_n,h_m}(u\cos(\psi), u\sin(\psi)) = e^{j(n-m)\psi - j(n+m)\frac{\pi}{2}} \cdot \sqrt{\pi} \sum_{k=0}^{\lfloor \frac{m+n}{2} \rfloor} c_{k,n,m} \cdot h_{n+m-2k}\left(\frac{1}{\sqrt{2}}u\right).$$

The AF of two Hermite functions is isotropic, i.e., the magnitude of the AF is the same regardless of the angle ψ. However, when the AF is calculated for signals in $L^2(\mathbb{R})$ which are constructed by sums of Hermite functions, the AF is not automatically isotropic anymore. The order-dependent phase terms $e^{j(n-m)\psi}$ cause constructive and destructive summations of Hermite functions such that the magnitude of the AF is varying for different angles ψ. The key insight is that by summing Hermite functions of specific orders, we can design (communication) signals which are time-frequency localized, orthogonal and achieve nearly the critical symbol density of 1 s·Hz per signal.

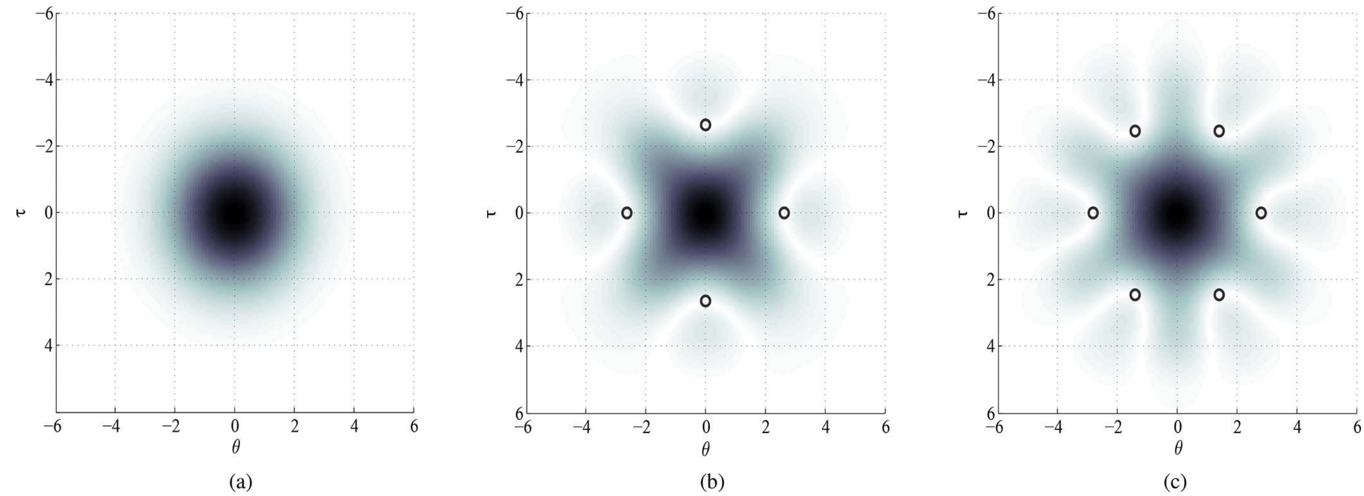

Fig. 3. The magnitude of the ambiguity function (AF) of (a) the zeroth-order Hermite function (Gaussian signal) (b) a highly time-frequency localized signal for a rectangular grid and (c) a highly time-frequency localized signal for a hexagonal grid. The signals in (a), (b) and (c) are based on Hermite functions of order 0, $\{0,4\}\{0,6\}$ and are characterized by rotational symmetry of order 1, 4 and 6, respectively. The zeros of the AFs are indicated by circles.

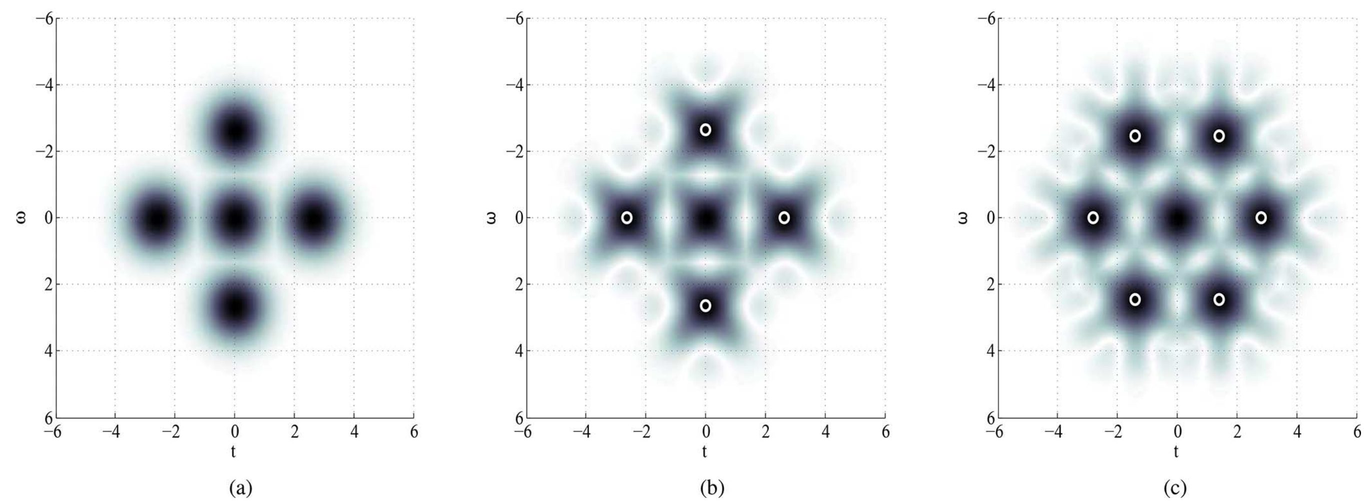

Fig. 4. Surface plot of superimposed Wigner distribution function (WDFs) of (a) five zeroth-order Hermite functions (Gaussian signals) (b) five highly time-frequency localized signals for a rectangular grid and (c) seven highly time-frequency localized signals for a hexagonal grid. The signals in (a), (b) and (c) are based on Hermite functions of order 0, $\{0,4\}\{0,6\}$ and are characterized by rotational symmetry of order 1, 4 and 6, respectively. By virtue of the orthogonalization procedure the signal in (b) and (c) in the center is orthogonal to its direct neighbors.

A weighted sum of Hermite functions of order $n \in \{0, k, 2k, \ldots, kN\}$ leads to a phase term $e^{j\psi k}$ which is periodic with $\psi = 2\pi/k$. E.g., a summation of Hermite functions of order $n \in \{0, 4, 8, \ldots, 4N\}$ leads to an AF which is constructed by Hermite functions of order h_n with $n = 0, 4, 8, \ldots, 4N$, which are all rotationally symmetric with $\psi = 2\pi/4$. An example would be a signal $s(t) = h_0(t) + \rho_4 h_4(t)$. Choosing ρ_4 such that $s(t) = 0$ at $t = C$ will lead—due to the rotational symmetry—automatically to four zeros of the AF at the coordinates $[\theta_4, \tau_4] \in \{[0, C], [C, 0], [0, -C], [-C, 0]\}$. This greatly reduces the computational complexity. Instead of solving a second-order equation in four variables, we can now force a zero at each of these coordinates by solving a simple quadratic equation (assuming ρ_4 is real), i.e.,

$$A_{h_0,h_0}(\theta_4, \tau_4) + 2\rho_4 A_{h_0,h_4}(\theta_4, \tau_4) + \rho_4^2 A_{h_4,h_4}(\theta_4, \tau_4) = 0. \tag{32}$$

To arrive at an effective time-bandwidth product of 1.10 s·Hz per signal we choose $C = \sqrt{1.1 \cdot 2\pi} \approx 2.63$. Solving the quadratic equations for $[\theta_4, \tau_4] = [0, C]$ gives $\rho_4 \approx -0.219$, such that $s(t) = h_0(t) - 0.219 \cdot h_4(t)$. The ambiguity plot is shown in Fig. 3(b) and shows the four zeros of the AF at $\tau = \pm 2.63$ s and $\theta = \pm 2.63$ rad. This means that the signal designed is orthogonal to shifted signals in time-frequency by $\tau = \pm 2.63$ s or $\theta = \pm 2.63$ rad/s. The WDF of the designed signal—and four time-frequency shifted ones at the coordinates of the zeros of the AF—are shown in Fig. 4(b). For comparison, Fig. 4(a) shows five non-orthogonal Gaussians (zeroth-order Hermite functions). Given the resemblance between the closed-form equation for the AF and the WDF (Table I), the WDF is easily and very efficiently calculated once the AF is known.

Similarly, we can design highly time-frequency localized pulses suitable for efficient hexagonal packing of signals in time-frequency. The zeros are placed at

$[\theta_6, \tau_6] \in \{[\pm C, 0], [\pm 1/2C, \pm\sqrt{3}/2C]\}$ and the second-order equation to be solved is (assuming ρ_6 is real),

$$A_{h_0,h_0}(\theta_6, \tau_6) + 2\rho_6 A_{h_0,h_6}(\theta_6, \tau_6) + \rho_6^2 A_{h_6,h_6}(\theta_6, \tau_6) = 0. \tag{33}$$

Without the rotational symmetry of (26) the orthogonalization procedure would require solving a second-order equation in six variables. For an effective time-bandwidth product of 1.10 s·Hz per signal for the hexagonal lattice C needs to be approximately 2.83. Solving now (33) leads to $\rho_6 \approx -0.196$ and the AF is thereby characterized by six zeros at the coordinates of the hexagon as shown in Fig. 3(c). The designed signal is orthogonal to six time-frequency displaced variants (corresponding to the coordinates of the hexagon). The WDF of the seven signals in a hexagonal setup are shown in Fig. 4(c).

In brief, the design of orthogonal, rotationally symmetric WDFs and AFs becomes possible by appropriately selecting and weighting Hermite functions of order $n \in \{0, k, 2k, \ldots, kN\}$. For the rectangular grid, the orthogonalized signal consists of only a 0th and 4th order Hermite function, whereas the hexagonal grid leads to an orthogonalized signal consisting of a 0th and 6th order Hermite function. These signals are highly time-frequency localized and exhibit, as stated in Section V, exponential tails. The main aim of this application section is illustrating how the closed-form nature of our expressions can be useful to significantly simplify an orthogonalization procedure in communication signal design. A comprehensive treatment of the properties and the use of these insights for rectangular and hexagonal lattices attaining high spectral efficiencies is outside the scope of this paper and subject of ongoing work.

VII. Conclusion

Closed-form expressions for the product, convolution, correlation, the WDF and AF of two Hermite functions have been derived. We investigated the proposition posed by Abreu [15] that the (cross-)correlation operation of two Hermite functions leads to a new Hermite function. It is prior knowledge that the product, convolution, correlation, WDF and AF of two Gaussian functions (the zeroth-order Hermite functions) lead to one- and two-dimensional Gaussian functions. We generalized this for Hermite functions of arbitrary order and have shown that the results of all mentioned operations on two Hermite functions of order n and m can be expressed as a sum of $\lfloor (n+m)/2 \rfloor$ Hermite functions with an order up to $n+m$.

An interesting resemblance among all mentioned operations has been found. The differences between the product, convolution, correlation, WDF and AF of two Hermite functions are primarily determined by distinct phase changes of the individual Hermite functions of which the resulting function consists.

As an illustration, we have shown the application of the closed-form expressions for the design of orthogonal, time-frequency localized communication signals. By constructing signals based on Hermite functions of order $n \in \{n = 0, k, 2k, \ldots, kN\}$ the AF of the signals becomes rotationally symmetric with order k. Application of this key insight allows for an orthogonalization procedure which leads to orthogonal, time-frequency localized signals which attain a high spectral efficiency. Given the usage of Hermite functions in many scientific fields like quantum mechanics, optics, image processing and spectrum estimation, many applications—beyond communications—may benefit from the contributed closed-form expressions.

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