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The Construction of Frobenius Manifolds from KP tau-Functions

J.W. van de Leur[⋆,⋆⋆]**, R. Martini**

Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

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Abstract: Frobenius manifolds (solutions of WDVV equations) in canonical coordinates are determined by the system of Darboux–Egoroff equations. This system of partial differential equations appears as a specific subset of the n-component KP hierarchy. KP representation theory and the related Sato infinite Grassmannian are used to construct solutions of this Darboux–Egoroff system and the related Frobenius manifolds. Finally we show that for these solutions Dubrovin's isomonodromy tau-function can be expressed in the KP tau-function.

1. Introduction

In the beginning of the 90's in the physics literature on two-dimensional field theory a remarkable and amazingly rich system of partial differential equations emerged. Roughly speaking, this system describes the conditions for a function $F = F(t)$ of the variable $t = (t^1, t^2, \dots, t^n)$ such that the third-order derivatives define structure constants of an associative algebra. These equations are commonly known as the Witten–Dijkgraaf–E. Verlinde–H. Verlinde (WDVV) equations [22,5]. From the geometric point of view the WDVV equations describe the conditions defining a Frobenius manifold. This concept of Frobenius manifold was introduced and extensively studied by Dubrovin, whose lecture notes [3] constitute the primary reference for Frobenius manifolds and many of their applications. The lecture notes of Manin [17] are also a very good general reference. Frobenius manifolds have appeared in a wide range of settings, including quantum cohomology [15], Gromov–Witten invariants, unfolding of singularities, reflection groups and integrable systems. Thus Frobenius manifolds (WDVV equations) are relevant in describing some deep geometrical phenomena. So it is expected that these Frobenius manifold equations are rather difficult to solve. Surprisingly some exact explicit solutions of this system of nonlinear equations do exist.

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⋆⋆ Present address: Mathematical Institute, P.O. Box 80.000, 3508 TA Utrecht, The Netherlands

The WDVV equations first appeared in 2D topological field theory. It was derived as a system of equations for so-called primary free energy. According to an idea of Witten the procedure of coupling to gravity should be described in terms of an integrable hierarchy of partial differential equations. In this context Witten–Kontsevich [23,14] proved that the partition function is a particular tau-function of the KdV hierarchy. For general 2D topological field theories the corresponding integrable hierarchies are not known.

The connection of Frobenius manifolds with integrable systems has been the subject of many investigations. For instance Dubrovin (see e.g. [3], §6) made extensive study of Frobenius manifolds in relation to semi-classical approximations (dispersionless limit, Witham averaging) of integrable hierarchies of partial differential equations. Here also tau-functions emerge, but their representation theoretical meaning remains unclear and under-exposed. Recently tau-functions also reappear in studying one-loop approximations [6,8].

The particular class of semisimple Frobenius manifolds may be effectively studied in the so-called canonical coordinates. In these coordinates Frobenius manifolds are determined by the classical Darboux–Egoroff equations, a system of differential equations, playing a major part in many investigations in classical differential geometry. In terms of the Riemann theta function of auxiliary algebraic curves Krichever constructed in [16] solutions of this system.

It is observed that these Darboux–Egoroff equations are a special case of the n-component KP hierarchy. This observation enables us to study Frobenius manifolds in the context of the KP hierarchy. In particular this implies that we have the machinery from the representation theory for the KP hierarchy at our disposal and may take advantage of it to produce solutions. This is the subject of the present paper.

The paper is devoted to the construction of Frobenius manifolds by considering the WDVV equations in the context of the KP hierarchy and to construct solutions in terms of appropriate classes of tau-functions emerging in the representation theory of the KP hierarchy.

We summarize the contents of the paper. In Sect. 2 we explain the construction of the semi-infinite wedge representation of the group GL_∞ and write down the condition for the GL_∞-orbit $\mathcal{O}_m$ of the highest weight vector $|m\rangle$. The resulting equation is called the KP hierarchy in the fermionic picture. Moreover we briefly discuss the formulation within Sato's Grassmannian. Section 3 is devoted to bosonization of the fermionic picture. We express the fermionic fields in terms of bosonic fields and determine the conditions for elements of orbits $\mathcal{O}_m$ in bosonic terms. Using the so-called boson-fermion correspondence we reformulate in Sect. 4 the KP hierarchy in the bosonic setting. Introducing formal pseudodifferential operators we obtain Sato's equation, another reformulation of the KP hierarchy. In Sect. 5, the central part of the paper, we construct solutions of the Darboux–Egoroff system by considering this system as a special case of the Sato equation and applying the results described in the previous sections and furthermore by introducing appropriate well-chosen tau-functions. The relevance of the orthogonal group is briefly explained. Using the KP wave function corresponding to all solutions of Sect. 5, we construct in Sect. 6 specific eigenfunctions that determine the Frobenius manifold. We find an expression for the flat coordinates and express Dubrovin's isomonodromy tau-function in terms of the KP tau-function. Finally in Sect. 7 as an illustration we describe the simplest example in full detail.

For notations and general background we refer to Dubrovin [3] and Kac and van de Leur [11].

2. The Semi-Infinite Wedge Representation of the Group GL_∞ and Sato's Grassmannian

Consider the infinite complex matrix group

$$GL_\infty=\{A=(a_{ij})_{i,j\in\mathbb{Z}+\frac{1}{2}}|A \text{ is invertible and all but a finite number of } a_{ij}-\delta_{ij} \text{ are } 0\},$$

and its Lie algebra

$$gl_\infty = \{a = (a_{ij})_{i,j\in\mathbb{Z}+\frac{1}{2}}| \text{ all but a finite number of } a_{ij} \text{ are } 0\}$$

with bracket $[a, b] = ab - ba$. The Lie algebra gl_∞ has a basis consisting of matrices $E_{ij},\ i, j \in \mathbb{Z} + \frac{1}{2}$, where E_{ij} is the matrix with a 1 on the $(i, j)^{\text{th}}$ entry and zeros elsewhere. Let $\mathbb{C}^\infty = \bigoplus_{j\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}v_j$ be an infinite dimensional complex vector space with fixed basis $\{v_j\}_{j\in\mathbb{Z}+\frac{1}{2}}$. Both the group GL_∞ and its Lie algebra gl_∞ act linearly on $\mathbb{C}^\infty$ via the usual formula:

$$E_{ij}(v_k) = \delta_{jk}v_i.$$

The well-known semi–infinite wedge representation is constructed as follows [12] (see also [13] and [11]). The semi-infinite wedge space $F = \Lambda^{\frac{1}{2}\infty}\mathbb{C}^\infty$ is the vector space with a basis consisting of all semi-infinite monomials of the form $v_{i_1}\wedge v_{i_2}\wedge v_{i_3}\ldots$, where $i_1 > i_2 > i_3 > \ldots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell >> 0$. We can now define representations R of GL_∞ and r of gl_∞ on F by

$$\begin{aligned} R(A)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) &= Av_{i_1} \wedge Av_{i_2} \wedge Av_{i_3} \wedge \cdots ,\\ r(a)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) &= \sum_k v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{k-1}} \wedge av_{i_k} \wedge v_{i_{k+1}} \wedge \cdots . \end{aligned} \tag{2.1}$$

These equations are related by the usual formula:

$$\exp(r(a)) = R(\exp a) \text{ for } a \in gl_\infty.$$

In order to perform calculations later on, it is convenient to introduce a larger group

$$\overline{GL_\infty} = \{A = (a_{ij})_{i,j\in\mathbb{Z}+\frac{1}{2}}|A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ with } i \geq j \text{ are } 0\},$$

and its Lie algebra

$$\overline{gl_\infty} = \{a = (a_{ij})_{i,j\in\mathbb{Z}+\frac{1}{2}}| \text{ all but a finite number of } a_{ij} \text{ with } i \geq j \text{ are } 0\}.$$

Both $\overline{GL_\infty}$ and $\overline{gl_\infty}$ act on a completion $\overline{\mathbb{C}^\infty}$ of the space $\mathbb{C}^\infty$, where

$$\overline{\mathbb{C}^\infty} = \{\sum_j c_jv_j|c_j = 0 \text{ for } j >> 0\}.$$

It is easy to see that the representations R and r extend to representations of $\overline{GL_\infty}$ and $\overline{gl_\infty}$ on the space F.

The representation r of gl_∞ and $\overline{gl_\infty}$ can be described in terms of wedging and contracting operators in F (see e.g. [12,13]). Let v_j^* be the linear functional on $\mathbb{C}^\infty$

defined by $\langle v_i^*, v_j\rangle := v_i^*(v_j) = \delta_{ij}$ and let $\mathbb{C}^{\infty *} = \bigoplus_{j\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}v_j^*$ be the restricted dual of $\mathbb{C}^\infty$, then for any $w \in \mathbb{C}^\infty$, we define a wedging operator $\psi^+[w]$ on F by

$$\psi^+[w](v_{i_1} \wedge v_{i_2} \wedge \cdots) = w \wedge v_{i_1} \wedge v_{i_2} \cdots . \tag{2.2}$$

Let $w^* \in \mathbb{C}^{\infty *}$, we define a contracting operator

$$\psi^-[w^*](v_{i_1} \wedge v_{i_2} \wedge \cdots) = \sum_{s=1}^{\infty} (-1)^{s+1} \langle w^*, v_{i_s}\rangle v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \cdots . \tag{2.3}$$

For simplicity we write

$$\psi_j^+ = \psi^+[v_{-j}], \quad \psi_j^- = \psi^-[v_j^*] \quad \text{for } j \in \mathbb{Z} + \frac{1}{2}. \tag{2.4}$$

These operators satisfy the following relations ($i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -$):

$$\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda,-\mu}\delta_{i,-j},$$

hence they generate a Clifford algebra, which we denote by $\mathcal{C}\ell$.

Introduce the following elements of F $(m \in \mathbb{Z})$:

$$|m\rangle = v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge v_{m-\frac{5}{2}} \wedge \cdots .$$

It is clear that F is an irreducible $\mathcal{C}\ell$-module generated by the vacuum $|0\rangle$ such that

$$\psi_j^\pm|0\rangle = 0 \text{ for } j > 0.$$

It is straightforward that the representation r is given by the following formula:

$$r(E_{ij}) = \psi_{-i}^+ \psi_j^- . \tag{2.5}$$

Define the *charge decomposition*

$$F = \bigoplus_{m\in\mathbb{Z}} F^{(m)} \tag{2.6}$$

by letting

$$\text{charge } |0\rangle = 0 \text{ and charge } \psi_j^\pm = \pm 1. \tag{2.7}$$

It is clear that the charge decomposition is invariant with respect to $r(g\ell_\infty)$ (and hence with respect to $R(GL_\infty)$). Moreover, it is easy to see that each $F^{(m)}$ is irreducible with respect to $g\ell_\infty$ (and GL_∞). Note that $|m\rangle$ is its highest weight vector, i.e.

$$r(E_{ij})|m\rangle = 0 \text{ for } i < j,$$
$$r(E_{ii})|m\rangle = 0 \text{ (resp. } = |m\rangle) \text{ if } i > m \text{ (resp. if } i < m).$$

Let $w \in F$, we define the Annihilator space $Ann(w)$ of w as follows:

$$Ann(w) = \{v \in \mathbb{C}^\infty | v \wedge w = 0\}. \tag{2.8}$$

Notice that $Ann(w) \neq 0$, since $v_j \in Ann(w)$ for $j << 0$. This Annihilator space for perfect (semi–infinite) wedges $w \in F^{(m)}$ is related to the GL_∞-orbit

$$\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}$$

of the highest weight vector $|m\rangle$ as follows. Let $A = (A_{ij})_{i,j\in\mathbb{Z}} \in GL_\infty$, denote by $A_j = \sum_{i\in\mathbb{Z}} A_{ij} v_i$, then by (2.8),

$$\tau_m = R(A)|m\rangle = A_{m-\frac{1}{2}} \wedge A_{m-\frac{3}{2}} \wedge A_{m-\frac{5}{2}} \wedge \cdots , \tag{2.9}$$

with $A_{-j} = v_{-j}$ for $j >> 0$. Notice that since τ_m is a perfect (semi-infinite) wedge

$$Ann(\tau_m) = \sum_{j<m} \mathbb{C} A_j \subset \mathbb{C}^\infty.$$

The following theorem also characterizes the group orbit. For a proof, see [12,13]:

Theorem 2.1. *Let $\tau_m \in F^{(m)}$, then $\tau_m \in \mathcal{O}_m$ if and only if τ_m satisfies the (fermionic) KP hierarchy:*

$$\sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k^+ \tau_m \otimes \psi_{-k}^- \tau_m = 0. \tag{2.10}$$

It is obvious from the construction that if $w \in \mathbb{C}^\infty$ and $\tau_m \in \mathcal{O}_m$ that $w \wedge \tau_m \in \mathcal{O}_{m+1}$. In fact one has the following useful lemma.

Lemma 2.1. *Let $\tau_m \in \mathcal{O}_m$, $w \in \mathbb{C}^\infty$ and $w^* \in \mathbb{C}^{\infty *}$. If $\psi^+[w]\tau_m \neq 0$ (resp. $\psi^-[w^*]\tau_m \neq 0$), then $\psi^+[w]\tau_m \in \mathcal{O}_{m+1}$ (resp. $\psi^-[w^*]\tau_m \in \mathcal{O}_{m-1}$).*

Proof. We only have to prove the statement for $\psi^-[w^*]\tau_m$. Let $\psi^-[w^*] \otimes \psi^-[w^*]$ act on (2.10), then we obtain

$$\begin{aligned} 0 &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k^+ \psi^-[w^*]\tau_m \otimes \psi_{-k}^- \psi^-[w^*]\tau_m - \sum_{k\in\mathbb{Z}+\frac{1}{2}} \tau_m \otimes \langle w^*, v_{-k}\rangle \psi_{-k}^- \psi^-[w^*]\tau_m \\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k^+ \psi^-[w^*]\tau_m \otimes \psi_{-k}^- \psi^-[w^*]\tau_m - \tau_m \otimes \psi^-[w^*]\psi^-[w^*]\tau_m. \end{aligned}$$

Since the last term is clearly zero we obtain the desired result. □

Choose a positive integer n and relabel the basis vectors v_i as follows. Define for $j \in \mathbb{Z},\ 1 \le j \le n,\ k \in \mathbb{Z} + \frac{1}{2}$:

$$v_k^{(j)} = v_{nk-\frac{1}{2}(n-2j+1)}, \tag{2.11}$$

and identify

$$v_k^{(j)} = t^{-k-\frac{1}{2}} e_j, \tag{2.12}$$

where e_j, $1 \le j \le n$, is a basis of $\mathbb{C}^n$. We can thus write the vectors A_ℓ in (2.9) as

$$A_\ell = A_\ell(t) = \sum_{j=1}^n \left(\sum_{i\in\mathbb{Z}+\frac{1}{2}} A_{ni-\frac{1}{2}(n-2j+1),\ell} t^{-i-\frac{1}{2}} \right) e_j, \tag{2.13}$$

hence as a vector in $H = (\mathbb{C}[t, t^{-1}])^n$. In this way we can identify $Ann(\tau_m)$ with a subspace $W_{\tau_m} = \sum_{j<m} \mathbb{C}A_j(t)$ of the space H and hence with a point in an infinite (polynomial) Grassmannian Gr. A point of Gr is a linear subspace of H which contains

$$H_\ell := \sum_{j=1}^{n} \sum_{i=\ell}^{\infty} \mathbb{C}t^i e_j$$

for $\ell >> 0$. Now $Gr = \cup_{m\in\mathbb{Z}} Gr_m$ (disjoint union) with

$$Gr_m = \{W \in Gr | H_\ell \subset W \text{ and } \dim W/H_\ell = \ell n + m \text{ for } \ell >> 0\},$$

and we can construct a canonical map

$$\phi : \mathcal{O}_m \to Gr_m, \quad \phi(\tau_m) = W_{\tau_m} := \sum_{i<m} \mathbb{C}A_i(t).$$

It is clear that $\phi(|mn\rangle) = H_{-m}$ and that ϕ is surjective with fibers $\mathbb{C}^\times$. This construction is due to Sato [S].

3. The Boson-Fermion Correspondence

The relabeling of the v_i's given by (2.11) induces a relabeling of the $\psi_j^\pm$'s, viz.,

$$\psi_k^{\pm(j)} = \psi^\pm_{nk\pm\frac{1}{2}(n-2j+1)}.$$

Notice that with this relabeling we have:

$$\psi_k^{\pm(j)}|0\rangle = 0 \text{ for } k > 0.$$

Besides the charge decomposition, we also introduce an *energy decomposition* defined by

$$\text{energy } |0\rangle = 0, \text{ energy } \psi_k^{\pm(j)} = -k. \tag{3.1}$$

Note that energy on F is never negative. Introduce the fermionic fields ($z \in \mathbb{C}^\times$) by

$$\psi^{\pm(j)}(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k^{\pm(j)} z^{-k-\frac{1}{2}}, \tag{3.2}$$

and bosonic fields ($1 \le i, j \le n$) by

$$\alpha^{(ij)}(z) = \sum_{k\in\mathbb{Z}} \alpha_k^{(ij)} z^{-k-1} =: \psi^{+(i)}(z)\psi^{-(j)}(z) :, \tag{3.3}$$

where : : stands for the *normal ordered product* defined in the usual way ($\lambda, \mu = +$ or $-$):

$$: \psi_k^{\lambda(i)}\psi_\ell^{\mu(j)} := \begin{cases} \psi_k^{\lambda(i)}\psi_\ell^{\mu(j)} & \text{if } \ell \ge k, \\ -\psi_\ell^{\mu(j)}\psi_k^{\lambda(i)} & \text{if } \ell < k. \end{cases} \tag{3.4}$$

One checks (using e.g. the Wick formula) that the operators $\alpha_k^{(ij)}$ satisfy the commutation relations of the affine algebra $gl_n(\mathbb{C})^\wedge$ with central charge 1, i.e.:

$$[\alpha_p^{(ij)}, \alpha_q^{(k\ell)}] = \delta_{jk}\alpha_{p+q}^{(i\ell)} - \delta_{i\ell}\alpha_{p+q}^{(kj)} + p\delta_{i\ell}\delta_{jk}\delta_{p,-q}, \tag{3.5}$$

and that

$$\alpha_k^{(ij)}|m\rangle = 0 \text{ if } k > 0 \text{ or } k = 0 \text{ and } i < j. \tag{3.6}$$

The operators $\alpha_k^{(i)} \equiv \alpha_k^{(ii)}$ satisfy the canonical commutation relation of the associative oscillator algebra, which we denote by α:

$$[\alpha_k^{(i)}, \alpha_\ell^{(j)}] = k\delta_{ij}\delta_{k,-\ell}, \tag{3.7}$$

and one has

$$\alpha_k^{(i)}|m\rangle = 0 \text{ for } k > 0. \tag{3.8}$$

It is easy to see that restricted to $g\ell_n(\mathbb{C})^\wedge$, $F^{(0)}$ is its basic highest weight representation (see [10]).

In order to express the fermionic fields $\psi^{\pm(i)}(z)$ in terms of the bosonic fields $\alpha^{(i)}(z)$, we need some additional operators Q_i, $i = 1, \ldots, n$, on F. These operators are uniquely defined by the following conditions:

$$Q_i|0\rangle = \psi_{-\frac{1}{2}}^{+(i)}|0\rangle, \;\; Q_i\psi_k^{\pm(j)} = (-1)^{\delta_{ij}+1}\psi_{k\mp\delta_{ij}}^{\pm(j)}Q_i. \tag{3.9}$$

They satisfy the following commutation relations:

$$Q_iQ_j = -Q_jQ_i \text{ if } i \neq j, \;\; [\alpha_k^{(i)}, Q_j] = \delta_{ij}\delta_{k0}Q_j. \tag{3.10}$$

Theorem 3.1 ([1,9]**).**

$$\psi^{\pm(i)}(z) = Q_i^{\pm 1}z^{\pm\alpha_0^{(i)}} \exp(\mp\sum_{k<0}\frac{1}{k}\alpha_k^{(i)}z^{-k}) \exp(\mp\sum_{k>0}\frac{1}{k}\alpha_k^{(i)}z^{-k}). \tag{3.11}$$

Proof. See [18].

The operators on the right-hand side of (3.11) are called vertex operators. They made their first appearance in string theory (cf. [7]).

We can describe now the n-component boson-fermion correspondence. Let $\mathbb{C}[x]$ be the space of polynomials in indeterminates $x = \{x_k^{(i)}\}$, $k = 1, 2, \ldots$, $i = 1, 2, \ldots, n$. Let L be a lattice with a basis $\delta_1, \ldots, \delta_n$ over $\mathbb{Z}$ and the symmetric bilinear form $(\delta_i|\delta_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Let

$$\varepsilon_{ij} = \begin{cases} -1 & \text{if } i > j, \\ 1 & \text{if } i \leq j. \end{cases} \tag{3.12}$$

Define a bimultiplicative function $\varepsilon : L \times L \to \{\pm 1\}$ by letting

$$\varepsilon(\delta_i, \delta_j) = \varepsilon_{ij}. \tag{3.13}$$

Let $\delta = \delta_1 + \ldots + \delta_n$, $M = \{\gamma \in L| (\delta|\gamma) = 0\}$, $\Delta = \{\alpha_{ij} := \delta_i - \delta_j | i, j = 1, \ldots, n, \; i \neq j\}$. Of course M is the root lattice of $s\ell_n(\mathbb{C})$, the set Δ being the root system.

Consider the vector space $\mathbb{C}[L]$ with basis e^γ, $\gamma \in L$, and the following twisted group algebra product:

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}. \tag{3.14}$$

Let $B = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[L]$ be the tensor product of algebras. Then the n-component boson-fermion correspondence is the vector space isomorphism

$$\sigma : F \to B, \tag{3.15}$$

given by

$$\sigma(\alpha_{-m_1}^{(i_1)} \ldots \alpha_{-m_s}^{(i_s)} Q_1^{k_1} \ldots Q_n^{k_n} |0\rangle) = m_1 \ldots m_s x_{m_1}^{(i_1)} \ldots x_{m_s}^{(i_s)} \otimes e^{k_1\delta_1 + \ldots + k_n \delta_n}. \tag{3.16}$$

The transported charge and energy then will be as follows:

$$\begin{aligned} &\text{charge } p(x) \otimes e^\gamma = (\delta|\gamma), \\ &\text{energy } x_{m_1}^{(i_1)} \ldots x_{m_s}^{(i_s)} \otimes e^\gamma = m_1 + \ldots + m_s + \frac{1}{2}(\gamma|\gamma). \end{aligned} \tag{3.17}$$

We denote the transported charge decomposition by

$$B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}.$$

The transported action of the operators $\alpha_m^{(i)}$ and Q_j looks as follows:

$$\begin{cases} \sigma\alpha_{-m}^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) = m x_m^{(j)} p(x) \otimes e^\gamma, \text{ if } m > 0, \\ \sigma\alpha_m^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) = \frac{\partial p(x)}{\partial x_m} \otimes e^\gamma, \text{ if } m > 0, \\ \sigma\alpha_0^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) = (\delta_j|\gamma) p(x) \otimes e^\gamma, \\ \sigma Q_j \sigma^{-1}(p(x) \otimes e^\gamma) = \varepsilon(\delta_j, \gamma) p(x) \otimes e^{\gamma+\delta_j}. \end{cases} \tag{3.18}$$

The transported action of the fermionic fields is as follows:

$$\sigma\psi^{\pm(j)}(z)\sigma^{-1} = e^{\pm\delta_j} z^{\pm\delta_j} \exp(\pm \sum_{k=1}^{\infty} x_k^{(j)}). \exp(\mp \sum_{k=1}^{\infty} \frac{\partial}{\partial x_k^{(j)}} \frac{z^{-k}}{k}). \tag{3.19}$$

We will now determine the second part of the boson–fermion correspondence, i.e., we want to determine $\sigma(\tau_m)$, where τ_m is given by (2.9). Since all spaces $F^{(m)}$ give a similar representation of gl_∞, we will restrict our attention to the case that $m = 0$ and we write τ instead of τ_0. We will generalize the proof of Theorem 6.1 of [13]. For this purpose we have to introduce elements $\Lambda_\ell^{(j)} \in \overline{gl_\infty}$, $1 \leq j \leq n$, $\ell \in \mathbb{N}$, by

$$\Lambda_\ell^{(j)} = \sum_{k \in \mathbb{Z}+\frac{1}{2}} E_{nk-\frac{1}{2}(n-2j+1), nk+\ell-\frac{1}{2}(n-2j+1)}. \tag{3.20}$$

Notice that $\Lambda_\ell^{(j)} = (\Lambda_1^{(j)})^\ell, r(\Lambda_\ell^{(j)}) = \alpha_\ell^{(j)}$ and that $\exp \Lambda_\ell^{(j)} \in \overline{Gl_\infty}$. With the relabeling $|0\rangle$ becomes

$$|0\rangle = v_{-\frac{1}{2}}^{(n)} \wedge v_{-\frac{1}{2}}^{(n-1)} \wedge \cdots \wedge v_{-\frac{1}{2}}^{(1)} \wedge v_{-\frac{3}{2}}^{(n)} \wedge v_{-\frac{3}{2}}^{(n-1)} \wedge \cdots ,$$

and

$$Q_i Q_j^{-1}|0\rangle = (-)^{n-j} v_{\frac{1}{2}}^{(i)} v_{-\frac{1}{2}}^{(n)} \wedge v_{-\frac{1}{2}}^{(n-1)} \wedge \cdots \wedge v_{-\frac{1}{2}}^{(j+1)} \wedge v_{-\frac{1}{2}}^{(j-1)} \wedge \cdots \wedge v_{-\frac{1}{2}}^{(1)} \wedge v_{-\frac{3}{2}}^{(n)} \wedge v_{-\frac{3}{2}}^{(n-1)} \wedge \cdots .$$

We now want to determine $\sigma(\tau)$, where

$$\tau = R(A)|0\rangle = A_{-\frac{1}{2}} \wedge A_{-\frac{3}{2}} \wedge A_{-\frac{5}{2}} \wedge \cdots , \text{ with } A_{-p} = v_{-p} \text{ for all } p > P >> 0. \tag{3.21}$$

Let $\sigma(\tau) = \sum_{\alpha\in M} \tau_\alpha(x) e^\alpha$; we want to compute

$$\sigma\left(R\left(\exp\left(\sum_{j=1}^n\sum_{k=1}^\infty y_k^{(j)}\Lambda_k^{(j)}\right)\right)\tau\right) = \exp\left(\sum_{j=1}^n\sum_{k=1}^\infty y_k^{(j)}\frac{\partial}{\partial x_k^{(j)}}\right)\sum_{\alpha\in M}\tau_\alpha(x)e^\alpha.$$

Now let $F_\alpha(y)$ denote the coefficient of $1\otimes e^\alpha$ in this expression, then

$$F_\alpha(y) = \exp\left(\sum_{j=1}^n\sum_{k=1}^\infty y_k^{(j)}\frac{\partial}{\partial x_k^{(j)}}\right)\tau_\alpha(x)|_{x=0} = \tau_\alpha(x+y)|_{x=0} = \tau_\alpha(y).$$

So $\tau_\alpha(y)$ is the coefficient of $1\otimes e^\alpha$ in

$$\sigma\left(R\left(\exp\left(\sum_{j=1}^n\sum_{k=1}^\infty y_k^{(j)}\Lambda_k^{(j)}\right)A\right)|0\rangle\right).$$

Now let $\alpha = \sum_{j=1}^n k_j\delta_j$; then

$$1\otimes e^\alpha = \sigma(Q_1^{k_1}Q_2^{k_2}\cdots Q_n^{k_n}|0\rangle),$$

hence $\tau_\alpha(y)$ is the coefficient of

$$R\left(\exp\left(\sum_{j=1}^n\sum_{k=1}^\infty y_k^{(j)}\Lambda_k^{(j)}\right)A\right)|0\rangle = R\left(\left(\sum_{j=1}^n\sum_{k=0}^\infty S_k(y^{(j)})\Lambda_k^{(j)}\right)A\right)|0\rangle$$
$$= R\left(\sum_{\ell<0}\sum_{j=1}^n\sum_{q\in\mathbb{Z}+\frac{1}{2}}\left(\sum_{k=0}^\infty A_{n(q+k)-\frac{1}{2}(n-2j+1),n(q)-\frac{1}{2}(n-2j+1)}S_k(y^{(j)})\right) E_{nq-\frac{1}{2}(n-2j+1),\ell}\right)|0\rangle,$$

where $S_k(y)$ are the elementary Schur functions defined by

$$\sum_{k\in\mathbb{Z}} S_k(y)z^k = \exp\,(\sum_{k=1}^{\infty} y_k z^k).$$

Using formula (4.48) of [13], i.e.,

$$R(A)|0\rangle = \sum_{j_{-\frac12}>j_{-\frac32}>j_{-\frac52}>\cdots} \det\left(A^{-\frac12,-\frac32,-\frac52,\cdots}_{j_{-\frac12},j_{-\frac32},j_{-\frac52},\cdots}\right) v_{j_{-\frac12}}\wedge v_{j_{-\frac32}}\wedge v_{j_{-\frac52}}\wedge\cdots,$$

where $A^{-\frac12,-\frac32,-\frac52,\cdots}_{j_{-\frac12},j_{-\frac32},j_{-\frac52},\cdots}$ denotes the matrix located at the intersection of the rows $j_{-\frac12}$, $j_{-\frac32}, j_{-\frac52},\cdots$ and the columns $-\frac12,-\frac32,-\frac52,\cdots$ of the matrix A, we can calculate $\tau_\alpha(y)$ if we can determine $Q_1^{k_1}Q_2^{k_2}\cdots Q_n^{k_n}|0\rangle$ as a perfect simple wedge. This is in general quite complicated, so we assume for the moment that

$$Q_1^{k_1}Q_2^{k_2}\cdots Q_n^{k_n}|0\rangle = \lambda_\alpha v_{j_{-\frac12}}\wedge v_{j_{-\frac32}}\wedge v_{j_{-\frac52}}\wedge\cdots,$$

with $j_{-q} = -q$ for all $q > Q >> 0$ and $\lambda_\alpha = \pm 1$, then

$$\tau_\alpha(y) = \lambda_\alpha \det\left(\sum_{\ell<0}\sum_{r=j_{-\frac12},j_{-\frac32},j_{-\frac52},\cdots}\sum_{\substack{1\le j\le n, q\in\mathbb{Z}+\frac12\\ nq-\frac12(n-2j+1)=r}}\left(\sum_{k=0}^{\infty} A_{r+nk,\ell}S_k(y^{(j)})\right)E_{r,\ell}\right).$$

Finally notice that this is in fact only a finite determinant of size $R = \max(P, Q)$, hence we have determined

Proposition 3.1. *Let $A = (A_{i,j})_{i,j\in\mathbb{Z}+\frac12} \in GL_\infty$ be such that $A_{ij} = \delta_{ij}$ for $j < -P$, then $\sigma(R(A)|0\rangle = \sum_{\alpha\in M}\tau_\alpha(x)e^\alpha$. Assume that $\alpha = \sum_{j=1}^n k_j\delta_j$ and suppose that*

$$Q_1^{k_1}Q_2^{k_2}\cdots Q_n^{k_n}|0\rangle = \lambda_\alpha v_{j_{-\frac12}}\wedge v_{j_{-\frac32}}\wedge v_{j_{-\frac52}}\wedge\cdots,$$

with $j_{-\frac12} > j_{-\frac32} > j_{-\frac52}\cdots$ and $j_{-q} = -q$ for all $q > Q >> 0$ and $\lambda_\alpha = \pm 1$, then

$$\tau_\alpha(x) =$$

$$= \lambda_\alpha \det\left(\sum_{-R<\ell<0}\sum_{r=j_{-\frac12},j_{-\frac32},\cdots,j_{-R+\frac12}}\sum_{\substack{1\le j\le n, q\in\mathbb{Z}+\frac12\\ nq-\frac12(n-2j+1)=r}}\left(\sum_{k=0}^{\infty} A_{r+nk,\ell}S_k(x^{(j)})\right)E_{r,\ell}\right),$$

where $R = \max(P, Q)$. In particular if $1 \le i < j \le n$ and $\alpha = 0$, $\delta_i - \delta_j$, $\delta_j - \delta_i$, respectively, then $\lambda_0 = 1$, $\lambda_{\delta_i-\delta_j} = (-1)^{n-j}$, $\lambda_{\delta_j-\delta_i} = (-1)^{n-i+1}$ and $(j_{-\frac12}, j_{-\frac32},\cdots) = (-\frac12,-\frac32,-\frac52,\ldots), = (i-\frac12,-\frac12,-\frac32,\ldots, j-n+\frac12, j-n+\frac32\ldots)$, $= (j-\frac12,-\frac12,-\frac32,\ldots, i-n+\frac12, i-n+\frac32\ldots)$, respectively.

4. The KP Hierarchy as a Dynamical System

Using the isomorphism σ we can reformulate the KP hierarchy (2.10) in the bosonic picture. We start by observing that (2.10) can be rewritten as follows:

$$\mathrm{Res}_{z=0}\sum_{j=1}^{n}\psi^{+(j)}(z)\tau\otimes\psi^{-(j)}(z)\tau=0,\ \tau\in F^{(0)}. \tag{4.1}$$

Here and further $\mathrm{Res}_{z=0}\sum_j f_j z^j$ (where f_j are independent of z) stands for f_{-1}. Notice that for $\tau\in F^{(0)}$, $\sigma(\tau)=\sum_{\gamma\in M}\tau_\gamma(x)e^\gamma$. Here and further we write $\tau_\gamma(x)e^\gamma$ for $\tau_\gamma\otimes e^\gamma$. Using Theorem 3.1, Eq. (4.1) turns under $\sigma\otimes\sigma:\ F\otimes F\longrightarrow\mathbb{C}[x',x'']\otimes(\mathbb{C}[L']\otimes\mathbb{C}[L''])$ into the following equations, which we call the n-component KP hierarchy. Let $1\le a,b\le n,\alpha,\ \beta\in M$:

$$\begin{aligned}&\mathrm{Res}_{z=0}(\sum_{j=1}^{n}\varepsilon(\delta_j,\alpha+\delta_a-\beta+\delta_b)z^{(\delta_j|\alpha+\delta_a-\beta+\delta_b-2\delta_j)}\\&\times\exp(\sum_{k=1}^{\infty}(x_k^{(j)'}-x_k^{(j)''})z^k)\exp(-\sum_{k=1}^{\infty}(\frac{\partial}{\partial x_k^{(j)'}}-\frac{\partial}{\partial x_k^{(j)''}})\frac{z^{-k}}{k})\\&\tau_{\alpha+\alpha_{aj}}(x')\tau_{\beta-\alpha_{bj}}(x''))=0\quad(\alpha,\beta\in M).\end{aligned} \tag{4.2}$$

Define the support of τ by $\mathrm{supp}\,\tau=\{\alpha\in M|\tau_\alpha\neq 0\}$, then for each $\alpha\in\ \mathrm{supp}\,\tau$ we define the (matrix valued) wave functions

$$V^{\pm}(\alpha,x,z)=(V_{ij}^{\pm}(\alpha,x,z))_{i,j=1}^{n} \tag{4.3}$$

as follows:

$$\begin{aligned}&V_{ij}^{\pm}(\alpha,x,z):=\varepsilon(\delta_j,\alpha+\delta_i)z^{(\delta_j|\pm\alpha+\alpha_{ij})}\\&\times\exp(\pm\sum_{k=1}^{\infty}x_k^{(j)}z^k)\exp(\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}\frac{z^{-k}}{k})\tau_{\alpha\pm\alpha_{ij}}(x)/\tau_\alpha(x).\end{aligned} \tag{4.4}$$

It is easy to see that Eq. (4.2) is equivalent to the following bilinear identity:

$$\mathrm{Res}_{z=0}V^{+}(\alpha,x,z)\,{}^tV^{-}(\beta,x',z)=0\text{ for all }\alpha,\beta\in M, \tag{4.5}$$

where ${}^tV$ stands for the transposed of the matrix V. Define $n\times n$ matrices $W^{\pm(m)}(\alpha,x)$ by the following generating series (cf. (4.4)):

$$\sum_{m=0}^{\infty}W_{ij}^{\pm(m)}(\alpha,x)(\pm z)^{-m}=\varepsilon_{ji}z^{\delta_{ij}-1}(\exp\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}\frac{z^{-k}}{k})\tau_{\alpha\pm\alpha_{ij}}(x))/\tau_\alpha(x). \tag{4.6}$$

Note that

$$W^{\pm(0)}(\alpha,x)=I_n, \tag{4.7}$$

$$W_{ij}^{\pm(1)}(\alpha,x)=\begin{cases}\varepsilon_{ji}\tau_{\alpha\pm\alpha_{ij}}/\tau_\alpha & \text{if } i\neq j,\\ -\tau_\alpha^{-1}\frac{\partial\tau_\alpha}{\partial x_1^{(i)}} & \text{if } i=j.\end{cases} \tag{4.8}$$

We see from (4.4) that $V^{\pm}(\alpha, x, z)$ can be written in the following form:

$$V^{\pm}(\alpha, x, z) = \sum_{m=0}^{\infty} W^{\pm(m)}(\alpha, x)(\pm z)^{-m} R^{\pm}(\alpha, \pm z) S^{\pm}(x, z), \tag{4.9}$$

where

$$\begin{aligned} R^{\pm}(\alpha, z) &= \sum_{i=1}^{n} \varepsilon(\delta_i, \alpha) E_{ii} (\pm z)^{\pm(\delta_i|\alpha)}, \\ S^{\pm}(x, z) &= \sum_{i=1}^{n} e^{\pm \sum_{j=1}^{\infty} x_j^{(i)} z^j} E_{ii}. \end{aligned} \tag{4.10}$$

Here E_{ij} stands for the $n \times n$ matrix whose (i, j) entry is 1 and all other entries are zero. Now let $\partial = \sum_{j=1}^{n} \frac{\partial}{\partial x_1^{(j)}}$, then $V^{\pm}(\alpha, x, z)$ can be written in terms of formal pseudo-differential operators (see [11] for more details).

$$P^{\pm}(\alpha) \equiv P^{\pm}(\alpha, x, \partial) = I_n + \sum_{m=1}^{\infty} W^{\pm(m)}(\alpha, x)\partial^{-m}, \;\; R^{\pm}(\alpha) = R^{\pm}(\alpha, \partial) \tag{4.11}$$

as follows:

$$V^{\pm}(\alpha, x, z) = P^{\pm}(\alpha) R^{\pm}(\alpha) S^{\pm}(x, z). \tag{4.12}$$

Since obviously

$$R^{-}(\alpha, \partial)^{-1} = R^{+}(\alpha, \partial)^{*}, \tag{4.13}$$

where $P^* = \sum_k (-\partial)^{k\,t} P^{(k)}$ stands for the formal adjoint of $P = \sum_k P^{(k)} \partial^k$. Moreover one can deduce (see [11]) from the bilinear identity (4.5):

$$(P^{+}(\alpha, x, \partial) R^{+}(\alpha - \beta, \partial) P^{-}(\beta, x'\partial)^{*})_{-} = 0 \tag{4.14}$$

for any $\alpha, \beta \in$ supp τ. Here $Q_- = Q - Q_+$, where Q_+ stands for the differential operator part of Q.

Furthermore, put $x = x'$, then one deduces from (4.14) with $\alpha = \beta$ that

$$P^{-}(\alpha) = (P^{+}(\alpha)^{*})^{-1}, \tag{4.15}$$

since $R^{\pm}(0) = I_n$ and $P^{\pm}(\alpha) \in I_n +$ lower order terms. With all these ingredients one can prove the following lemma:

Proposition 4.1. *Let $\alpha, \beta \in$ supp τ, then $P^{+}(\alpha)$ satisfies the Sato equations:*

$$\frac{\partial P^{+}(\alpha)}{\partial x_k^{(j)}} = -(P^{+}(\alpha) E_{jj} \partial^k P^{+}(\alpha)^{-1})_{-} P^{+}(\alpha) \tag{4.16}$$

and $P^{+}(\alpha),\;\; P^{+}(\beta)$ satisfy

$$(P^{+}(\alpha) R^{+}(\alpha - \beta) P^{+}(\beta)^{-1})_{-} = 0 \textit{ for all } \alpha, \beta \in \textit{supp } \tau. \tag{4.17}$$

This is another formulation of the n-component KP hierarchy (see [11]). Introduce the following formal pseudo-differential operators $L(\alpha),\ C^{(j)}(\alpha)$:

$$\begin{aligned} L(\alpha) &\equiv L(\alpha, x, \partial) = P^+(\alpha)\partial P^+(\alpha)^{-1}, \\ C^{(j)}(\alpha) &\equiv C^{(j)}(\alpha, x, \partial) = P^+(\alpha)E_{jj}P^+(\alpha)^{-1}, \end{aligned} \tag{4.18}$$

then related to the Sato equation is the following linear system:

$$\begin{aligned} L(\alpha)V^+(\alpha, x, z) &= zV^+(\alpha, x, z), \\ C^{(i)}(\alpha)V^+(\alpha, x, z) &= V^+(\alpha, x, z)E_{ii}, \\ \frac{\partial V^+(\alpha, x, z)}{\partial x_k^{(i)}} &= (L(\alpha)^k C^{(i)}(\alpha))_+ V^+(\alpha, x, z). \end{aligned} \tag{4.19}$$

To end this section we write down explicitly some of the Sato equations (4.16) on the matrix elements $W_{ij}^{(s)}$ of the coefficients $W^{(s)}(x)$ of the pseudo-differential operator

$$P = P^+(\alpha) = I_n + \sum_{m=1}^{\infty} W^{(m)}(x)\partial^{-m}.$$

We shall write $W = W^{(1)}$ and W_{ij} for $W_{ij}^{(1)}$ to simplify notation, then the simplest Sato equation is

$$\frac{\partial P}{\partial x_1^{(k)}} = [\partial E_{kk}, P] + [W, E_{kk}]P. \tag{4.20}$$

In particular we have for $i \neq k$:

$$\frac{\partial W_{ij}}{\partial x_1^{(k)}} = W_{ik}W_{kj} - \delta_{jk}W_{ij}^{(2)}. \tag{4.21}$$

Equation (4.20) is equivalent to the following equation for $V = V^+(\alpha)$:

$$\frac{\partial V}{\partial x_1^{(k)}} = (E_{kk}\partial + [W, E_{kk}])V. \tag{4.22}$$

5. Solutions of the Darboux–Egoroff System

Define

$$\gamma_{ij}(x) = W_{ij}^{(1)}(0, x)|_{x_k^{(i)} = c_k^{(i)} \text{ for } k>1}, \tag{5.1}$$

where the $x_k^{(i)}$ for $k > 0$ are chosen to be certain specific but at the moment still unknown constants. From (4.21) we already know that

$$\frac{\partial \gamma_{ij}(x)}{\partial x_1^{(k)}} = \gamma_{ik}(x)\gamma_{kj}(x) \quad i \neq k \neq j. \tag{5.2}$$

This is the n-wave equation if i, j, k are distinct. The aim of this section is to construct specific γ_{ij}'s which satisfy

$$\sum_{k=1}^{n} \frac{\partial \gamma_{ij}(x)}{\partial x_1^{(k)}} = 0, \tag{5.3}$$

and

$$\gamma_{ij}(x) = \gamma_{ji}(x). \tag{5.4}$$

In other words we want to find the rotation coefficients γ_{ij} for the Darboux–Egoroff system (5.2)-(5.4). Sometimes we will assume an additional equation, viz.

$$\sum_{k=1}^{n} x_1^{(k)} \frac{\partial \gamma_{ij}(x)}{\partial x_1^{(k)}} = -\gamma_{ij}(x), \tag{5.5}$$

which means that γ_{ij} has degree -1. This equation holds for the so-called semisimple conformal invariant Frobenius manifolds, see [2].

The restriction

$$\sum_{k=1}^{n} \frac{\partial W_{ij}^{(1)}(0, x)}{\partial x_1^{(k)}} = 0, \tag{5.6}$$

is a very natural restriction. If we assume that

$$\sum_{k=1}^{n} \frac{\partial \tau(x)}{\partial x_1^{(k)}} = 0, \tag{5.7}$$

then this clearly implies (5.6). Notice that one may even assume that $\sum_{k=1}^{n} \frac{\partial \tau(x)}{\partial x_1^{(k)}} = \lambda \tau(x)$, but since we are in the polynomial case λ must be 0. Equation (5.7) means that τ (in the fermionic picture) belongs to the $GL_n(\mathbb{C}[t, t^{-1}])$-loop group orbit or even the $SL_n(\mathbb{C}[t, t^{-1}])$-loop group orbit of $|0\rangle$ (see [11] for more details). The homogeneous space for this group is in fact the restricted Grassmannian

$$\overline{Gr} = \{W \in Gr_0 | \sum_{k=1}^{n} t E_{kk} W \subset W\}.$$

In fact τ satisfies (5.7) if and only if

$$\sum_{k=1}^{n} t E_{kk} W_\tau \subset W_\tau. \tag{5.8}$$

Since Eq. (5.7) holds for τ we do not only find Eq. (5.6) for $W^{(1)}(0, x)$, but we find that this equation holds for all $W^{(s)}(\alpha, x)$'s and hence

$$\sum_{k=1}^{n} \frac{\partial P^{+}(\alpha, x)}{\partial x_1^{(k)}} = 0. \tag{5.9}$$

This means that we do not really have formal pseudo-differential operators, but rather formal matrix-valued Laurent series in z^{-1}. The Sato equation takes the following simple form. Let $P(z) = P^+(\alpha, x, z)$, then

$$\frac{\partial P(z)}{\partial x_k^{(j)}} = -(P(z)E_{jj}P(z)^{-1}z^k)_- P(z)$$

and the simplest Sato equation becomes

$$\frac{\partial P(z)}{\partial x_1^{(k)}} = z[E_{kk}, P(z)] + [W, E_{kk}]P(z).$$

Equation (4.22) turns into

$$\frac{\partial V(z)}{\partial x_1^{(k)}} = (zE_{kk} + [W, E_{kk}])V(z), \tag{5.10}$$

where $V(z) = V^+(\alpha, x, z)$. Define $X = \sum_{j=1}^n x_1^{(j)} E_{jj}$, then

$$\sum_{j=1}^n x_1^{(j)} \frac{\partial}{\partial x_1^{(j)}} V(z) = (zX + [W, X])V(z). \tag{5.11}$$

From now on we will only consider tau-functions that are homogeneous with respect to the energy. Notice that if energy $\tau = N$, then energy $\tau_\alpha = N - \frac{1}{2}(\alpha|\alpha)$, in particular energy $\tau_{\delta_i - \delta_j}$ = energy $\tau_0 - 1$. Since the energy $x_k^{(j)} = k$, it is straightforward to check that for $\alpha = 0$,

$$\begin{aligned} L_0 V(z) &= z\frac{\partial V(z)}{\partial z}, \quad \text{where} \\ L_0 &= \sum_{j=1}^n \sum_{k=1}^\infty k x_k^{(j)} \frac{\partial}{\partial x_k^{(j)}}. \end{aligned} \tag{5.12}$$

We will now describe a class of homogeneous tau-functions, in the fermionic picture that satisfy (5.7). First choose two positive integers m_1 and m_2 such that $m_1 + m_2 \le n$. Next choose m_1 positive integers k_i, $1 \le i \le m_1$ and m_2 positive integers ℓ_j, $1 \le j \le m_2$, such that $\sum_{i=1}^{m_1} k_i - \sum_{j=1}^{m_2} \ell_j = 0$. Next choose m_1 linearly independent vectors $a_i = (a_{i1}, a_{i2}, \dots, a_{in})$ and m_2 linearly independent vectors $b_j = (b_{j1}, b_{j2}, \dots, b_{jn})$ in $\mathbb{C}^n$ such that

$$(a_i, b_j) = \sum_{k=1}^n a_{ik} b_{jk} = 0 \text{ for all } 1 \le i \le m_1 \text{ and } 1 \le j \le m_2. \tag{5.13}$$

Using Lemma 2.1 we construct a $\tau \in \mathcal{O}_0$ as follows:

$$\begin{aligned}\tau =&(\sum_p a_{1p}\psi^{+(p)}_{-k_1+\frac12})(\sum_p a_{1p}\psi^{+(p)}_{-k_1+\frac32})\cdots\\ &\cdots(\sum_p a_{1p}\psi^{+(p)}_{-\frac12})(\sum_p a_{2p}\psi^{+(p)}_{-k_2+\frac12})(\sum_p a_{2p}\psi^{+(p)}_{-k_2+\frac32})\cdots\\ &\cdots(\sum_p a_{2p}\psi^{+(p)}_{-\frac12})(\sum_p a_{3p}\psi^{+(p)}_{-k_3+\frac12})\cdots\\ &\cdots(\sum_p a_{m_1,p}\psi^{+(p)}_{-\frac12})(\sum_p b_{1p}\psi^{-(p)}_{-\ell_1+\frac12})(\sum_p b_{1p}\psi^{-(p)}_{-\ell_1+\frac32})\cdots\\ &\cdots(\sum_p b_{1p}\psi^{-(p)}_{-\frac12})(\sum_p b_{2p}\psi^{-(p)}_{-\ell_2+\frac12})\cdots(\sum_p b_{m_2,p}\psi^{-(p)}_{-\frac12})|0\rangle.\end{aligned} \tag{5.14}$$

The point of the Grassmannian W_τ corresponding to this τ satisfies (5.8).

The symmetry conditions (5.4) of the γ_{ij}'s are not so natural. Using (4.8), it is equivalent to

$$\tau_{\delta_i-\delta_j}(x_1^{(\ell)}, c_2^{(\ell)}, c_3^{(\ell)}, \dots) = -\tau_{\delta_j-\delta_i}(x_1^{(\ell)}, c_2^{(\ell)}, c_3^{(\ell)}, \dots). \tag{5.15}$$

To achieve this result, we define an automorphism ω on F as follows:

$$\begin{aligned}\omega(|0\rangle) &= |0\rangle,\\ \omega(\psi_k^{\pm(i)}) &= c_i^{\pm1}\psi_k^{\mp(i)}, \text{ with } 1 \le i \le n \text{ and } c_i \in \mathbb{C}^\times.\end{aligned} \tag{5.16}$$

We will fix the c_i later all to be equal to 1, but for the moment we keep them arbitrary. This gives

$$\omega(\alpha_k^{(i)}) = -\alpha_k^{(i)} \text{ and } \omega(Q_i^{\pm1}) = c_i^{\pm1}Q_i^{\mp1}. \tag{5.17}$$

Using the boson-fermion correspondence this induces an automorphism on B, which we will also denote by ω,

$$\omega(x_k^{(i)}) = -x_k^{(i)},\ \omega(\frac{\partial}{\partial x_k^{(i)}}) = -\frac{\partial}{\partial x_k^{(i)}},\ \omega(\delta_i) = -\delta_i \text{ and } \omega(e^{\pm\delta_i}) = c_i^{\pm1}e^{\mp\delta_i}. \tag{5.18}$$

Define for $\alpha = \sum_{j=1}^n p_i\delta_i \in M$, $c_\alpha = \prod_{j=1}^n c_i^{p_i}$, then

$$\omega\left(\sum_{\alpha\in M}\tau_\alpha(x)e^\alpha\right) = \sum_{\alpha\in M} c_\alpha\tau_\alpha(-x)e^{-\alpha}. \tag{5.19}$$

We now want to find homogeneous tau-functions that satisfy $\omega(\tau(x)) = \lambda\tau(x)$ for some $\lambda \in \mathbb{C}^\times$. Since $\omega^2(\tau_0(x)) = \tau_0(x)$, $\lambda = 1$ or -1. From (5.19) we deduce that

$$\tau_\alpha(x) = \lambda c_\alpha\tau_{-\alpha}(-x), \tag{5.20}$$

and we want this for $\alpha \in \Delta$, of course after a specific choice of constants $x_k^{(i)}$'s for $k \ge 2$, to be equal to $-\tau_\alpha(x)$. Since we have assumed that τ is homogeneous (in the energy), say that it has energy N, then we can get rid of the $-x$ in the right-hand side of (5.20) if we put all $x_{2k}^{(i)}$'s equal to zero. So define

$$\overline{\tau}(x) = \tau(x)|_{x_{2k}^{(i)}=0}, \tag{5.21}$$

then clearly (5.20) turns into

$$\overline{\tau}_\alpha(x) = \lambda c_\alpha (-)^{N-\frac{1}{2}(\alpha|\alpha)}\overline{\tau}_{-\alpha}(x).$$

Because this also has to hold for $\alpha = 0$, we obtain that $\lambda = (-1)^N$ and hence $c_\alpha = 1$ for all $\alpha \in \Delta$. Thus $c_i = 1$ for all $1 \le i \le n$ or $c_i = -1$ for all i, we may choose either of these two cases, for simplicity we choose

$$c_i = 1 \text{ for all } 1 \le i \le n.$$

With all these choices, we have finally that

$$\omega(\overline{\tau}_\alpha(x)) = (-)^{N-\frac{1}{2}(\alpha|\alpha)}\overline{\tau}_{-\alpha}(x). \tag{5.22}$$

Return to the tau-functions of the form (5.14). If such a τ satisfies (5.22) and it contains a factor $\sum_i a_{\ell i}\psi_k^{+(i)}$ for a certain ℓ, then it must also contain a factor $\sum_j b_{mj}\psi_k^{-(j)}$. Since

$$\text{energy}\left(\sum_i a_{\ell i}\psi_k^{+(i)}\right)\left(\sum_j b_{mj}\psi_k^{-(j)}\right) = -2k \in 2\mathbb{Z}+1,$$

we must assume that there exists an m such that

$$\begin{aligned}\omega\left(\left(\sum_i a_{\ell i}\psi_k^{+(i)}\right)\left(\sum_j b_{mj}\psi_k^{-(j)}\right)\right) &= -\left(\sum_j b_{mj}\psi_k^{+(j)}\right)\left(\sum_i a_{\ell i}\psi_k^{-(i)}\right)\\ &= -\left(\sum_i a_{\ell i}\psi_k^{+(i)}\right)\left(\sum_j b_{mj}\psi_k^{-(j)}\right).\end{aligned}$$

So

$$a_{\ell i}b_{mj} = a_{\ell j}b_{mi} \text{ for all } 1 \le i, j \le n$$

and b_m must be a multiple of a_ℓ. Since the length of such a vector does not matter much (only a scalar multiple of the whole tau-function), we may assume that $a_\ell = b_m$ and since also $(a_\ell, b_\ell) = 0$ (see (5.13)), we obtain that a_ℓ is an isotropic vector in $\mathbb{C}^n$.

Finally we conclude the following

Proposition 5.1. *Let m be the integer part of $\frac{n}{2}$. Choose m linearly independent vectors $a_i = (a_{i1}, a_{i2}, \dots, a_{in})$ in $\mathbb{C}^n$ which span a maximal isotropic subspace of $\mathbb{C}^n$, i.e.*

$$(a_i, a_j) = \sum_{k=1}^n a_{ik}a_{jk} = 0 \textit{ for all } 1 \le i, j \le m.$$

Choose m non-negative integers k_i, $1 \le i \le m$ such that

$$k_1 \ge k_2 \ge \ldots \ge k_m \ge 0, \tag{5.23}$$

then $\sigma(\tau) = \sum_{\alpha\in M} \tau_\alpha(x)e^\alpha$, with

$$\begin{aligned}\tau = &(\sum_p a_{1p}\psi^{+(p)}_{-k_1+\frac{1}{2}})(\sum_p a_{1p}\psi^{+(p)}_{-k_1+\frac{3}{2}})\cdots\\ &\cdots(\sum_p a_{1p}\psi^{+(p)}_{-\frac{1}{2}})(\sum_p a_{2p}\psi^{+(p)}_{-k_2+\frac{1}{2}})(\sum_p a_{2p}\psi^{+(p)}_{-k_2+\frac{3}{2}})\cdots\\ &\cdots(\sum_p a_{2p}\psi^{+(p)}_{-\frac{1}{2}})(\sum_p a_{3p}\psi^{+(p)}_{-k_3+\frac{1}{2}})\cdots\\ &\cdots(\sum_p a_{mp}\psi^{+(p)}_{-\frac{1}{2}})(\sum_p a_{1p}\psi^{-(p)}_{-k_1+\frac{1}{2}})(\sum_p a_{1p}\psi^{-(p)}_{-k_1+\frac{3}{2}})\cdots\\ &\cdots(\sum_p a_{1p}\psi^{-(p)}_{-\frac{1}{2}})(\sum_p a_{2p}\psi^{-(p)}_{-k_2+\frac{1}{2}})\cdots\\ &\cdots(\sum_p a_{mp}\psi^{-(p)}_{-\frac{1}{2}})|0\rangle,\end{aligned} \tag{5.24}$$

satisfies the n-component KP hierarchy (4.2) and

$$\omega(\tau) = (-)^{k_1+k_2+\cdots+k_m}\tau.$$

Moreover

$$\text{energy } \tau_\alpha(x) = k_1^2 + k_2^2 + \cdots + k_m^2 - \frac{1}{2}(\alpha|\alpha), \tag{5.25}$$

$$\sum_{j=1}^{n} \frac{\partial\tau_\alpha(x)}{\partial x_1^{(j)}} = 0$$

and

$$\overline{\tau}_\alpha(x) = (-)^{\frac{1}{2}(\alpha|\alpha)}\overline{\tau}_{-\alpha}(x),$$

where $\overline{\tau}$ is defined by (5.21).

Notice that the restriction (5.23) is not essential, but we may assume it without loss of generality. Since the energy is nowhere negative, formula (5.25) gives a restriction for supp τ.

It is not difficult to prove that the perfect wedge τ, given by (5.24), is also a highest weight vector for the $W_{1+\infty}$-algebra generated by

$$J^{(\ell+1)}(z) = \sum_{k\in\mathbb{Z}} J_k^{(\ell+1)} z^{-k-\ell-1} = \sum_{j=1}^{n} :\psi^{+(j)}(z)\frac{\partial^\ell\psi^{-(j)}(z)}{\partial z^\ell}: \quad \ell = 0, 1, 2, \ldots,$$

i.e.,

$$J_k^{(\ell+1)}\tau = \delta_{k0}c_\ell(k_1, k_2, \ldots, k_m)\tau \quad \text{for } k \ge 0.$$

Here $c_\ell \in \mathbb{C}$ only depend on the integers $k_1, k_2, \dots, k_m$. This induces the following restriction on $W_\tau \in Gr_0$:

$$\sum_{j=1}^{n} t^{k+\ell} \left(\frac{\partial}{\partial t}\right)^{\ell} E_{jj} W_\tau \subset W_\tau \quad \text{for all } k, \ell = 0, 1, 2, \dots .$$

If we now rewrite the element (5.24) as a perfect wedge, we can use Proposition 3.1 to determine τ_α for $\alpha = 0$ or $\alpha \in \Delta$. Add to the vectors a_i, $1 \le i \le m$ vectors a_j, $m+1 \le j \le n$ such that they form a basis of $\mathbb{C}^n$, which satisfies

$$(a_\ell, a_k) = \delta_{k+\ell, 2m+1} + \delta_{k+\ell, 4m+2} \text{ for all } 1 \le k, \ell \le n. \tag{5.26}$$

Define

$$k_{2m+1-i} = -k_i \text{ for } 1 \le i \le m. \tag{5.27}$$

Then the τ given by (5.24) is up to a scalar multiple equal to the following perfect wedge:

$$A_{-\frac{1}{2}} \wedge A_{-\frac{3}{2}} \wedge A_{-\frac{5}{2}} \wedge \cdots ,$$

with

$$\begin{aligned}
A_{-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell} &= \sum_{j=1}^{n} a_{qj} v_\ell^{(j)} \\
&\qquad \text{with } 1 \le q \le 2m-1 \text{ and } -k_1 + \frac{1}{2} \le \ell \le k_q - \frac{1}{2}, \\
A_{-(2m)-(k_1+k_2+\cdots k_{2m-1})-\ell} &= \sum_{j=1}^{n} a_{n,j} v_\ell^{(j)} \\
&\qquad \text{with } -k_1 + \frac{1}{2} \le \ell \le -\frac{1}{2} \text{ this only if } n = 2m+1, \\
A_q = v_q &\text{ for } q < -nk_1 - k_2 - \cdots - k_{2m-1}.
\end{aligned} \tag{5.28}$$

Now using (2.11), this is equal to

$$\begin{aligned}
A_{-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell} &= \sum_{j=1}^{n} a_{qj} v_{n\ell-\frac{1}{2}(n-2j+1)} \\
&\qquad \text{with } 1 \le q \le 2m-1 \text{ and } -k_1 + \frac{1}{2} \le \ell \le k_q - \frac{1}{2}, \\
A_{-(2m)-(k_1+k_2+\cdots k_{2m-1})-\ell} &= \sum_{j=1}^{n} a_{n,j} v_{n\ell-\frac{1}{2}(n-2j+1)} \\
&\qquad \text{with } -k_1 + \frac{1}{2} \le \ell \le -\frac{1}{2} \text{ this only if } n = 2m+1, \\
A_q = v_q &\text{ for } q < -nk_1 - k_2 - \cdots - k_{2m-1}.
\end{aligned} \tag{5.29}$$

Using Proposition 3.1, one easily deduces that (5.24) corresponding to τ_0 is given by

$$\tau_0 = \det\Big(\sum_{q=1}^{2m-1} \sum_{j=1}^{n} \sum_{i=1}^{k_1} \sum_{\ell=-i}^{k_q-1} a_{qj} S_{\ell+i}(x^{(j)}) E_{j-in-\frac{1}{2},-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell-\frac{1}{2}} + \delta_{(-1)^n,-1} \sum_{j=1}^{n} \sum_{i=1}^{k_1} \sum_{\ell=-i}^{-1} a_{n,j} S_{\ell+i}(x^{(j)}) E_{j-in-\frac{1}{2},-(2m)-(k_1+k_2+\cdots k_{2m-1})-\ell-\frac{1}{2}} \Big)$$

and $\tau_{\delta_r-\delta_s}$ for $1 \le r, s \le n$ is equal to the determinant of τ_0, but then with the $(s-n-\frac{1}{2})^{\text{th}}$ row replaced by

$$\sum_{q=1}^{2m-1} \sum_{\ell=0}^{k_q-1} a_{qr} S_\ell(x^{(r)}) E_{s-n-\frac{1}{2},-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell-\frac{1}{2}}.$$

Now change the indices and we obtain

Theorem 5.1. *Let τ be given by (5.24), and let $\sigma(\tau) = \sum_{\alpha\in M} \tau_\alpha(x)e^\alpha$, then up to a common scalar factor*

$$\tau_0 = \det\Big(\sum_{q=1}^{2m-1} \sum_{j=1}^{n} \sum_{i=1}^{k_1} \sum_{\ell=1-k_q}^{i} a_{qj} S_{i-\ell}(x^{(j)}) E_{in-j+1,qk_1+(k_1+k_2+\cdots+k_{q-1})-\ell+1} + \delta_{(-1)^n,-1} \sum_{j=1}^{n} \sum_{i=1}^{k_1} \sum_{\ell=1}^{i} a_{n,j} S_{i-\ell}(x^{(j)}) E_{in-j+1,(2m)+(k_1+k_2+\cdots k_{2m-1})-\ell+1} \Big) \tag{5.30}$$

and $\tau_{\delta_r-\delta_s}$ for $1 \le r, s \le n$ is equal to the determinant of τ_0, but then with the $(n+1-s)^{\text{th}}$ row replaced by

$$\sum_{q=1}^{2m-1} \sum_{\ell=0}^{k_q-1} a_{qr} S_\ell(x^{(r)}) E_{n+1-s,qk_1+(k_1+k_2+\cdots+k_{q-1})+\ell+1}, \tag{5.31}$$

where the a_ℓ, $1 \le \ell \le n$, satisfy (5.26) and the k_j, $m+1 \le j \le 2m$ are given by (5.27). Moreover the

$$\overline{\gamma}_{rs}(x) = \begin{cases} \epsilon_{sr} \frac{\overline{\tau}_{\delta_r-\delta_s}(x)}{\overline{\tau}_0(x)}, & \text{if } 1 \le r, s \le n \text{ and } r \ne s, \\ -\frac{\partial \log \overline{\tau}_0(x)}{\partial x_1^{(r)}}, & \text{if } 1 \le r, s \le n \text{ and } r = s, \end{cases} \tag{5.32}$$

satisfy the Darboux–Egoroff system (5.2)–(5.4). If we define

$$\gamma_{rs}(x) = \overline{\gamma}_{rs}(x)|_{x_k^{(i)}=0 \text{ for all } k>1}, \tag{5.33}$$

then these elements satisfy (5.2)–(5.5).

Let $f(t) = \sum_i f_i(t)e_i$ and $g(t) = \sum_i g_i(t)e_i$ be two elements in H. Define the following bilinear form:

$$B(f, g) = \text{Res}_{t=0} \sum_{i=1}^{n} f_i(t)g_i(t). \tag{5.34}$$

Then the orthogonal restricted Grassmannian is

$$\widehat{Gr} = \{W \in \overline{Gr} | B(W, W) = 0\}. \tag{5.35}$$

All $W \in \widehat{Gr}$ are maximal isotropic subspaces with respect to $B(\cdot, \cdot)$. This Grassmannian is the homogeneous space for the $O_n(\mathbb{C}[t, t^{-1}])$-loop group. The $O_n(\mathbb{C}[t, t^{-1}])$-orbit of $|0\rangle$ corresponds exactly to this Grassmannian (see e.g. [19]). Notice that all the W_τ's corresponding to the tau-functions given by (5.24) exactly satisfy this condition. Hence the tau-functions we have constructed to solve the Darboux–Egoroff system are in fact homogeneous tau-functions in the $O_n(\mathbb{C}[t, t^{-1}])$-orbit of $|0\rangle$. If we consider the affine Lie algebra $gl_n(\mathbb{C})^\wedge$ with central charge 1, defined by (3.5), then the special orthogonal Lie algebra $so_n(\mathbb{C})^\wedge$ is given by

$$so_n(\mathbb{C})^\wedge = \{x \in gl_n(\mathbb{C})^\wedge | \omega(x) = x\}.$$

Recall that $\omega(\psi_k^{\pm(i)}) = \psi_k^{\mp(i)}$. The Grassmannian $\widehat{Gr}$ has two connected components, which are distinguished by the parity of the dimension of the kernel of the projection $W \to H_0$. Depending on the energy of our (homogeneous) tau-function, $\omega(\tau) = (-)^{\text{energy}\ \tau}\tau$, the space W_τ belongs to one of these two components.

It is obvious, from the above description and from the construction of the tau-functions given by (5.24), that the orthogonal group O_n acts on these tau-functions and hence on the rotation coefficients. One has

Proposition 5.2. *The orthogonal group O_n acts on the rotation coefficients of Theorem 5.1. Let $X = (X_{ij})_{1\le i,j\le n} \in O_n$, then replacing a_{ij}, $1 \le i, j \le n$, (even if $a_{ij} = 0$) by $\sum_{\ell=1}^{n} X_{j\ell}a_{i\ell}$ in (5.30) and (5.31) gives a new solution of the Darboux–Egoroff system.*

6. Semisimple Frobenius Manifolds

Let $\gamma_{ij}(x)$, $1 \le i, j \le n$, be a solution of the Darboux–Egoroff system. If we can find n linearly independent vector functions $\psi_j = \psi_j(x) = {}^t(\psi_{1j}, \psi_{2j}, \ldots, \psi_{nj})$ such that

$$\begin{aligned} \frac{\partial \psi_{ij}}{\partial x_1^{(k)}} &= \gamma_{ik}\psi_{kj}, \quad k \ne i, \\ \sum_{k=1}^{n} \frac{\partial \psi_{ij}}{\partial x_1^{(k)}} &= 0, \end{aligned} \tag{6.1}$$

then they determine under certain conditions (locally) a semisimple (i.e. massive) Frobenius manifold (see [2,3]).

Recall from (5.10), that the wave function $V(z) = V^+(0, x, z)$ corresponding to the tau-functions of Proposition 3.1 and Theorem 5.1 satisfy

$$\begin{aligned} \frac{\partial V_{ij}(z)}{\partial x_1^{(k)}} &= W_{ik} V_{kj}(z), \quad k \neq i, \\ \sum_{k=1}^{n} \frac{\partial V_{ij}(z)}{\partial x_1^{(k)}} &= z V_{ij}(z). \end{aligned} \tag{6.2}$$

Comparing (6.1) and (6.2), one would like to take $z = 0$ in (6.2), however this does not make sense. There is a way to use the wave function $V(z)$ to construct the ψ_{ij}'s of (6.1). Suppose that we have a tau-function of the form (5.24), with the corresponding k_q's, $1 \le q \le n$, (in the case that n is odd, we define $k_n = 0$) and a_{qj}'s $1 \le q, j \le n$. Let

$$X_q(t) = \sum_{j=1}^{n} a_{qj} t^{-k_q - 1} e_j \in H, \quad 1 \le q \le n, \tag{6.3}$$

then it easy to check that

$$W_\tau + \mathbb{C} X_q(t) \neq W_\tau \text{ and } W_\tau + \mathbb{C} t X_q(t) = W_\tau.$$

Hence,

$$\sum_{j=1}^{n} a_{qj} \psi^{+(j)}_{-k_q - \frac{1}{2}} \tau \neq 0 \quad \text{and} \quad \sum_{j=1}^{n} a_{qj} \psi^{+(j)}_{-k_q + \frac{1}{2}} \tau = 0. \tag{6.4}$$

We rewrite this as follows:

$$\mathrm{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q - 1} \psi^{+(j)}(z) \tau \neq 0 \quad \text{and} \quad \mathrm{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q} \psi^{+(j)}(z) \tau = 0. \tag{6.5}$$

From this we deduce that

$$\mathrm{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q - 1} z^{1-\delta_{ij}} e^{\sum_{\ell=1}^{\infty} x_\ell^{(j)} z^\ell} e^{-\sum_{\ell=1}^{\infty} \frac{\partial}{\partial x_\ell^{(j)}} \frac{z^{-\ell}}{\ell}} \tau_{\delta_i - \delta_j}(x) \neq 0 \quad \text{and}$$
$$\mathrm{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q} z^{1-\delta_{ij}} e^{\sum_{\ell=1}^{\infty} x_\ell^{(j)} z^\ell} e^{-\sum_{\ell=1}^{\infty} \frac{\partial}{\partial x_\ell^{(j)}} \frac{z^{-\ell}}{\ell}} \tau_{\delta_i - \delta_j}(x) = 0.$$

Dividing this by $\tau_0(x)$ we obtain

$$\mathrm{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q - 1} V_{ij}(z) \neq 0 \quad \text{and} \quad \mathrm{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q} V_{ij}(z) = 0. \tag{6.6}$$

Now define for $1 \le i, q \le n$,

$$\Psi_{iq} = \mathrm{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q - 1} V_{ij}(z), \tag{6.7}$$

then it is straightforward to check, using (6.2) and (6.6) that

$$\begin{aligned} \frac{\partial \Psi_{ij}}{\partial x_1^{(k)}} &= W_{ik}\Psi_{kj}, \quad k \neq i, \\ \sum_{k=1}^{n} \frac{\partial \Psi_{ij}}{\partial x_1^{(k)}} &= 0. \end{aligned} \tag{6.8}$$

Notice that the vector functions $\Psi_q =^t (\Psi_{1q}, \Psi_{2q}, \ldots, \Psi_{nq})$ are "eigenfunctions" of the KP hierarchy which lie in the kernel of L. From all this we finally obtain the following

Theorem 6.1. *Let $V(z) = V^+(0, x, z)$ be the wave function corresponding to the tau-function of (5.24) with a_{qj}, $1 \leq q, j \leq n$ and k_ℓ, $1 \leq \ell \leq 2m$, as given in Theorem 5.1 and $k_n = 0$ if n is odd. Denote by*

$$\begin{aligned} \psi_{iq} &= \mathrm{Res}_{z=0}\lambda_q \sum_{j=1}^{n} a_{qj} z^{-k_q-1} V_{ij}^+(0, x, z)|_{x_k^{(\ell)}=0 \text{ for all } k>1}, \\ \overline{\psi}_{iq} &= \mathrm{Res}_{z=0}\lambda_q \sum_{j=1}^{n} a_{qj} z^{-k_q-1} V_{ij}^+(0, x, z)|_{x_{2k}^{(\ell)}=0 \text{ for all } k}, \end{aligned} \tag{6.9}$$

where $1 \leq q \leq n$ and $\lambda_q \in \mathbb{C}^\times$. Then these ψ_{iq}'s satisfy Eqs. (6.1), with γ_{ij} given by (5.32) and the formulas

$$\begin{aligned} \eta_{ii} &= \psi_{i1}^2, \\ \eta_{\alpha\beta} &= \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i\beta}, \\ \frac{\partial t_\alpha}{\partial x_1^{(i)}} &= \psi_{i1}\psi_{i\alpha}, \\ c_{\alpha\beta\gamma} &= \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}, \end{aligned} \tag{6.10}$$

with $t_\alpha = \sum_{\epsilon=1}^{n} \eta_{\alpha\epsilon} t^\epsilon$, determine (locally) a semisimple Frobenius manifold on the domain $x_1^{(i)} \neq x_1^{(j)}$ and $\psi_{11}\psi_{21}\cdots\psi_{n1} \neq 0$. The $\overline{\psi}_{iq}$'s also satisfy (6.1), but now with the γ_{ij} replaced by $\overline{\gamma}_{ij}$ of (5.33). Equations (6.10) for these $\overline{\psi}_{ij}$'s also determine a semisimple Frobenius manifold.

Proof. Formula (6.10) is a direct consequence of the following proposition, see [4] (cf. [2] and [3]) for more details. □

Proposition 6.1. *Let $X = \sum_{i=1}^{n} x_1^{(i)} E_{ii}$, $\Gamma = (\gamma_{ij})_{1\leq i,j\leq n}$, $\mathcal{V} = [\Gamma, X]$ and $\mathcal{V}_k = [\Gamma, E_{kk}]$, then $\mathcal{V} = (\mathcal{V}_{ij})_{1\leq i,j\leq n}$ is anti-symmetric and satisfies*

$$\frac{\partial \mathcal{V}}{\partial x_1^{(k)}} = [\mathcal{V}_k, \mathcal{V}] \tag{6.11}$$

and also

$$\begin{aligned} \mathcal{V}\psi_q &= \sum_{j=1}^{n} x_1^{(j)} \frac{\partial \psi_q}{\partial x_1^{(j)}} = k_q \psi_q, \\ \frac{\partial \psi_q}{\partial x_1^{(k)}} &= \mathcal{V}_k \psi_q, \end{aligned} \tag{6.12}$$

for $\psi_q = {}^t(\psi_{1q}, \psi_{2q}, \dots, \psi_{nq})$.

Proof. Equation (6.11) follows from (5.2), (5.3) and the fact that Γ is symmetric. We prove (6.12) as follows. Let $\mathcal{V}$ act on ψ_q. Using (5.11) and (6.7) one deduces

$$\mathcal{V}\psi_q = \sum_{j=1}^{n} x_1^{(j)} \frac{\partial \psi_q}{\partial x_1^{(j)}}.$$

Since ψ_q is independent of $x_k^{(j)}$ for all $k > 1$, we can use (5.12), to rewrite this as follows

$$\begin{aligned} &\sum_{j=1}^{n} x_1^{(j)} \frac{\partial \psi_{iq}}{\partial x_1^{(j)}} = \\ &\quad = \mathrm{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{qj} z^{-k_q-1} z \frac{\partial}{\partial z} \left(V_{ij}^{+}(0, x, z)|_{x_k^{(\ell)}=0 \text{ for all } k>1} \right) \\ &\quad = \mathrm{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{qj} \left(\frac{\partial}{\partial z} z^{-k_q} + k_q z^{-k_q-1} \right) \left(V_{ij}^{+}(0, x, z)|_{x_k^{(\ell)}=0 \text{ for all } k>1} \right) \\ &\quad = k_q \mathrm{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{qj} z^{-k_q-1} V_{ij}^{+}(0, x, z)|_{x_k^{(\ell)}=0 \text{ for all } k>1} \\ &\quad = k_q \psi_{iq}. \end{aligned}$$

The second equation of (6.12) can be proved in a similar way, using (5.10). □

From (6.12) we determine the degrees $d_1, d_2, \dots, d_n$ and d (resp. d_F) of the corresponding t^α,

$$d_1 = 1,\ d_\alpha = 1 + k_1 - k_\alpha,\ 2 \le \alpha \le n,\ d = -2k_1 \text{ and } d_F = 3 + 2k_1. \tag{6.13}$$

With our choice of k_α we have

$$d_\alpha + d_{2m+1-\alpha} = 2 - d,\ 1 \le \alpha \le m \text{ and } d_n = 1 + k_1 \text{ if } n = 2m + 1 \text{ is odd.}$$

Notice that if we define

$$\Phi(z) = V(0, x, z)|_{x_k^{(\ell)}=0 \text{ for all } k>1}, \tag{6.14}$$

then $\Phi(z)$ satisfies

$$
\begin{aligned}
z\frac{\partial\Phi(z)}{\partial z} &= \sum_{j=1}^{n} x_1^{(j)}\frac{\partial\Phi(z)}{\partial x_1^{(j)}} = (zX+\mathcal{V})\Phi(z),\\
\frac{\partial\Phi(z)}{\partial x_1^{(k)}} &= (zE_{kk}+\mathcal{V}_k)\Phi(z).
\end{aligned}
\tag{6.15}
$$

Theorem 6.2. *Let* $\Psi = (\psi_{ij})_{1\le i,j\le n}$ *and define* $\xi(z) = {}^t\Psi\Phi(z) = \eta\Psi^{-1}\Phi(z)$, $\mathcal{U} = \eta\Psi^{-1}X\Psi\eta^{-1}$, $\mu = -\eta\Psi^{-1}\mathcal{V}\Psi\eta^{-1} = \sum_{i=1}^{n} k_i E_{ii}$ *and* $\Pi_i = \eta\Psi^{-1}E_{ii}\Psi\eta^{-1}$, *then* $\eta({}^t\mathcal{U}) = \mathcal{U}\eta$, $\mu\eta+\eta\mu = 0$ *and*

$$
\begin{aligned}
z\frac{\partial\xi(z)}{\partial z} &= (z\mathcal{U}-\mu)\xi(z),\\
\sum_{j=1}^{n} x_1^{(j)}\frac{\partial\xi(z)}{\partial x_1^{(j)}} &= z\mathcal{U}\xi(z),\\
\frac{\partial\xi(z)}{\partial x_1^{(k)}} &= z\Pi_k\xi(z),\\
\frac{\partial\xi(z)}{\partial t^\alpha} &= zC_\alpha\xi(z),
\end{aligned}
\tag{6.16}
$$

where $C_\alpha = \sum_{\beta,\gamma=1}^{n} c_{\alpha\beta}^{\gamma}E_{\beta\gamma}$.

Proof. All formulas except the last one of (6.16) follow immediately from (6.11), (6.12), (6.15) and the fact that ${}^t\Psi\Psi = \eta$. Use the last formula of (6.10), $c_{\alpha\beta}^{\gamma} = \sum_{\epsilon=1}^{n} c_{\beta\alpha\epsilon}\eta^{\epsilon\gamma}$ and $\frac{\partial x_1^{(i)}}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}$ to rewrite

$$
\begin{aligned}
\frac{\partial\xi}{\partial t^\alpha} &= \sum_{i=1}^{n}\frac{\partial x_1^{(i)}}{\partial t^\alpha}\frac{\partial\xi}{\partial x_1^{(i)}}\\
&= z\sum_{i=1}^{n}\frac{\partial x_1^{(i)}}{\partial t^\alpha}\eta\Psi^{-1}E_{ii}\Psi\eta^{-1}\xi\\
&= z\,{}^t\Psi\sum_{i=1}^{n}\frac{\psi_{i\alpha}}{\psi_{i1}}E_{ii}\Psi\eta^{-1}\xi\\
&= zC_\alpha\xi.
\end{aligned}
$$

This finishes the proof of the theorem. □

As in [3,4] we can reformulate (6.11) as an $\{x_1^{(i)}\}_{1\le i\le n}$-dependent commuting Hamiltonian system

$$
\frac{\partial\mathcal{V}}{\partial x_1^{(k)}} = \{\mathcal{V}, H_k(\mathcal{V}, X)\},
$$

with quadratic Hamiltonians

$$H_i(\mathcal{V}, X) = \frac{1}{2}\sum_{j\neq i} \frac{\mathcal{V}_{ij}\mathcal{V}_{ji}}{x_1^{(i)} - x_1^{(j)}} = \frac{1}{2}\sum_{j\neq i} \gamma_{ij}\gamma_{ji}(x_1^{(i)} - x_1^{(j)}) \tag{6.17}$$

with respect to the standard Poisson bracket on so_n:

$$\{\mathcal{V}_{ij}, \mathcal{V}_{k\ell}\} = \delta_{jk}\mathcal{V}_{i\ell} - \delta_{ik}\mathcal{V}_{j\ell} + \delta_{i\ell}\mathcal{V}_{jk} - \delta_{j\ell}\mathcal{V}_{ik}.$$

Now consider the 1-form

$$\sum_{i=1}^{n} H_i(\mathcal{V}, X)dx_1^{(i)}. \tag{6.18}$$

Since it is closed for any such $\mathcal{V}$(see [2,3]), there exists a function $\tau_I(X)$, the isomonodromy tau-function, such that

$$d\log\tau_I(X) = \sum_{i=1}^{n} H_i(\mathcal{V}, X)dx_1^{(i)}. \tag{6.19}$$

Using (5.2), we rewrite $H_i(\mathcal{V}, X)$ as follows. Let $\tilde{\tau}_0(X) = \tau_0(x)|_{x_k^{(\ell)}=0 \text{ for all } k>1}$, then

$$\begin{aligned}
H_i(\mathcal{V}, X) =& \frac{1}{2}\sum_{j\neq i}\gamma_{ij}\gamma_{ji}(x_1^{(i)} - x_1^{(j)})\\
=& \frac{1}{2}\sum_{j\neq i}\frac{\partial\gamma_{ii}}{\partial x_1^{(j)}}(x_1^{(i)} - x_1^{(j)})\\
=& \frac{1}{2}\sum_{j=1}^{n} x_1^{(i)}\frac{\partial\gamma_{ii}}{\partial x_1^{(j)}} - \frac{1}{2}\sum_{j=1}^{n} x_1^{(j)}\frac{\partial\gamma_{ii}}{\partial x_1^{(j)}}\\
=& -\frac{1}{2}\sum_{j=1}^{n} x_1^{(j)}\frac{\partial\gamma_{ii}}{\partial x_1^{(j)}}\\
=& \frac{1}{2}\sum_{j=1}^{n} x_1^{(j)}\frac{\partial}{\partial x_1^{(j)}}\frac{\partial}{\partial x_1^{(i)}}(\log\tilde{\tau}_0(X))\\
=& \frac{1}{2}\frac{\partial}{\partial x_1^{(i)}}\left(\sum_{j=1}^{n} x_1^{(j)}\frac{\partial}{\partial x_1^{(j)}}(\log\tilde{\tau}_0(X))\right) - \frac{1}{2}\frac{\partial}{\partial x_1^{(i)}}(\log\tilde{\tau}_0(X))\\
=& -\frac{1}{2}\frac{\partial}{\partial x_1^{(i)}}(\log\tilde{\tau}_0(X)).
\end{aligned}$$

Hence

$$d\log\tau_I(X) = -\frac{1}{2}d\log\tilde{\tau}_0(X). \tag{6.20}$$

Dubrovin and Zhang defined in [6] a Gromov–Witten type G-function of a Frobenius manifold as follows:

$$\begin{aligned} G &= \log\left(\frac{\tau_I}{J^{\frac{1}{24}}}\right), \quad \text{where} \\ J &= \det\left(\frac{\partial t^\alpha}{\partial x_1^{(i)}}\right) = \log\left(\psi_{11}\psi_{21}\cdots\psi_{n1}\right). \end{aligned} \tag{6.21}$$

We can explicitly determine this function in the cases of the Frobenius manifolds corresponding to Theorem 6.1.

Theorem 6.3. *Let τ be given by (5.24) and let ψ_{i1} be defined as in (6.9). Let $\tilde{\tau}_0(X) = \tau_0(x)|_{x_k^{(\ell)}=0 \text{ for all } k>1}$, i.e.,*

$$\begin{aligned} \tilde{\tau}_0(X) = \det\Big(&\sum_{q=1}^{2m-1}\sum_{j=1}^{n}\sum_{i=1}^{k_1}\sum_{\ell=1-k_q}^{i} a_{qj}\frac{(x_1^{(j)})^{i-\ell}}{(i-\ell)!}E_{in-j+1,qk_1+(k_1+k_2+\cdots+k_{q-1})-\ell+1} \\ &+\delta_{(-1)^n,-1}\sum_{j=1}^{n}\sum_{i=1}^{k_1}\sum_{\ell=1}^{i} a_{n,j}\frac{(x_1^{(j)})^{i-\ell}}{(i-\ell)!}E_{in-j+1,(2m)+(k_1+k_2+\cdots k_{2m-1})-\ell+1}\Big). \end{aligned} \tag{6.22}$$

Then up to an additive scalar factor,

$$G = -\frac{1}{2}\log\tilde{\tau}_0(X) - \frac{1}{24}\log\left(\psi_{11}\psi_{21}\cdots\psi_{n1}\right). \tag{6.23}$$

Moreover,

$$\sum_{j=1}^{n} x_1^{(j)}\frac{\partial G}{\partial x_1^{(j)}} = \gamma G,$$

where

$$\gamma = -\frac{1}{4}\sum_{j=1}^{n} k_j^2 - \frac{nk_1}{24} \tag{6.24}$$

and

$$\frac{\partial}{\partial x_1^{(i)}}\frac{\partial}{\partial x_1^{(j)}}\left(\log\tilde{\tau}_0(X)\right) = -\gamma_{ij}^2 \quad i \neq j,$$

where γ_{ij} is defined by formula (5.33).

7. An Example

In this section we describe the simplest example in more detail. Let $n = 2m$, respectively $n = 2m+1$ if n is even respectively odd. Since the choices of the order of $k_1, k_2 \dots k_m \in \mathbb{Z}$ is rather arbitrary, we choose for simplicity of notation and calculation $k_1 = -k_n = -1$ and all other $k_i = 0$. Hence $d_1 = 1,\ d_n = -1,\ d_\alpha = 0,\ \alpha \neq 1, n,\ d = 2$ and $d_F = 1$. Choose vectors $a_i = (a_{i1}, a_{i2}, \dots, a_{in})$, such that

$$(a_i, a_j) = \delta_{i+j,n+1}.$$

Then

$$\tau_0 = \sum_{j=1}^{n} a_{ni}^2 u_i \text{ and } \tau_{\delta_i - \delta_j} = -\tau_{\delta_j - \delta_i} = a_{ni} a_{nj} \text{ for } i < j,$$

where we use the notation $u_i = x_1^{(i)}$. Hence,

$$\gamma_{ij} = -\frac{a_{ni} a_{nj}}{\sum_{j=1}^{n} a_{ni}^2 u_i} \text{ for } 1 \leq i, j \leq n$$

and the wave function is equal to

$$V(z) = \left(I - \frac{1}{\tau_0} \sum_{i,j=1}^{n} a_{ni} a_{nj} E_{ij} z^{-1} \right) \sum_{\ell=1}^{n} \sum_{k=0}^{\infty} S_k(x^{(\ell)}) E_{\ell\ell} z^k.$$

From which we deduce that

$$\psi_{i,1} = -\frac{a_{ni}}{\tau_0},$$
$$\psi_{in} = -a_{ni} \left(u_i - \frac{1}{2\tau_0} \sum_{j=1}^{n} a_{nj}^2 u_j^2 \right),$$
$$\psi_{ik} = a_{ki} - \frac{a_{ni}}{\tau_0} \sum_{j=1}^{n} a_{kj} a_{nj} u_j \text{ for } k \neq 1, n.$$

Then using the formulas (6.10) it is straightforward to check that

$$t_1 = -\frac{1}{\tau_0}, \quad t_n = \frac{\sum_{j=1}^{n} a_{nj}^2 u_j^2}{2\tau_0}, \quad t_k = -\frac{\sum_{j=1}^{n} a_{kj} a_{nj} u_j}{\tau_0},$$

and hence that

$$\psi_{i,1} = a_{ni} t_1, \quad \psi_{in} = a_{ni}(t_n - u_i), \quad \psi_{ik} = a_{ki} + a_{ni} t_k,$$

$\eta_{\alpha,\beta} = \delta_{\alpha+\beta,n+1}$ and $t^\ell = t_{n+1-\ell}$. Assume from now on that all $a_{ni} \neq 0$. Since $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$, the solution $F(t)$ of the WDVV equations is of the form (see [3]):

$$F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n+1-\alpha} + f(t^2, t^3, \dots, t^n).$$

Since $d_n = -1$, $d_\alpha = 0$ for $\alpha \neq 1, n$ and $d_F = 1$, it suffices to determine c_{nnn}, which is

$$c_{nnn} = \sum_{i=1}^{n} \frac{a_{ni}^2 (t^1 - u_i)^3}{t^n}.$$

A straightforward calculation shows that

$$u_i = t^1 - \frac{1}{a_{ni} t^n} \left(a_{1i} - \sum_{\alpha=2}^{n-1} \left(a_{\alpha i} t^\alpha + \frac{a_{ni}}{2} t^\alpha t^{n+1-\alpha} \right) \right).$$

Hence,

$$\frac{\partial^3 f}{\partial u_n^3} = \frac{1}{(t^n)^4} \left(\sum_{i=1}^{n} \frac{a_{1i}}{a_{ni}} - \sum_{\alpha=2}^{n-1} \left(\frac{a_{\alpha i}}{a_{ni}} t^\alpha + \frac{1}{2} t^\alpha t^{n+1-\alpha} \right) \right)^3,$$

and thus

$$F(t) = \frac{1}{2} (t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n+1-\alpha} - \frac{1}{6t^n} \left(\sum_{i=1}^{n} \frac{a_{1i}}{a_{ni}} - \sum_{\alpha=2}^{n-1} \left(\frac{a_{\alpha i}}{a_{ni}} t^\alpha + \frac{1}{2} t^\alpha t^{n+1-\alpha} \right) \right)^3.$$

Next we give the ξ_{ij}'s ($\alpha \neq 1, n$):

$$\begin{aligned} \xi_{1j} &= a_{nj} t^n e^{zu_j}, \\ \xi_{\alpha j} &= \left(a_{\alpha j} + a_{nj} t^{n+1-\alpha} \right) e^{zu_j}, \\ \xi_{nj} &= \left(a_{nj} z^{-1} + \frac{1}{t^n} \left(a_{1j} - \sum_{\alpha=2}^{n-1} \left(a_{\alpha j} t^\alpha + \frac{a_{nj}}{2} t^\alpha t^{n+1-\alpha} \right) \right) \right) e^{zu_j}. \end{aligned}$$

One easily sees that $\xi_{ij} = \frac{\partial h_j}{\partial t^i}$ with

$$\begin{aligned} h_j &= \frac{a_{nj} t^n}{z} e^{zu_j} \\ &= \frac{a_{nj} t^n}{z} e^{z\left(t^1 - \frac{1}{a_{nj} t^n} \left(a_{1j} - \sum_{\alpha=2}^{n-1} \left(a_{\alpha j} t^\alpha + \frac{a_{nj}}{2} t^\alpha t^{n+1-\alpha} \right) \right) \right)}. \end{aligned}$$

To see that these are deformed flat coordinates, we determine

$$\tilde{t}^\alpha = (-)^{\delta_{\alpha 1}} \sum_{j=1}^{n} a_{n+1-\alpha, j} h_j.$$

We find

$$\begin{aligned} \tilde{t}^1 &= 1 + t^1 z + O(z^2), \\ \tilde{t}^\alpha &= t^\alpha + O(z), \quad \alpha \neq 1, n \\ \tilde{t}^n &= t^n z^{-1} + O(z^0). \end{aligned}$$

Finally we calculate the G-function of the Frobenius manifold. Notice that $\tilde{\tau}_0(X) = \tau_0(x) = -\frac{1}{t^n}$ and that

$$\psi_{11}\psi_{21}\cdots\psi_{n1} = \prod_{i=1}^{n}(a_{ni}t^n).$$

So using Theorem 6.3, we obtain that $\gamma = \frac{n-12}{24}$ and that up to an additive constant,

$$G(t) = \frac{12-n}{24}\log(t^n).$$

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