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PLANAR AND SPATIAL GRAVITY BALANCING WITH NORMAL SPRINGS

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ABSTRACT

Very often, spring-to-gravity-balancing mechanisms are conceived with ideal (zero-free-length $l_0=0$) springs. However, the use of ideal springs in the conception phase tends to lead to more complex mechanisms because the ideal spring functionality has to be approximated with normal springs. To facilitate construction of (gravity) balancers, employing normal springs ($l_0\neq 0$) directly mounted between the link attachment points of the mechanism in the conception phase therefore seems beneficiary. This paper discusses spring mechanisms that enable perfect balancing of gravity acting on an inverted pendulum while employing normal springs between the spring-attachment points: The design synthesis of such mechanisms will be explained and balancing conditions will be derived, using a potential energy consideration.

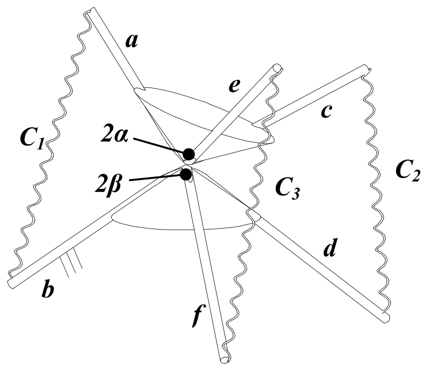
Keywords: static balance, gravity equilibrator, spatial, rolling link mechanisms, rehabilitation technology.

INTRODUCTION

In a companion paper, several spatial mechanisms are presented that enable perfect static gravity balance for rotations about a fixed pivot provided that the determined balancing conditions are satisfied [16]. That paper shows that not satisfying the geometric conditions for perfect spatial static spring-to-spring balance (Fig. 1a) causes a residual torque that can be statically balanced by coupling a mass rotating about a fixed pivot to the spring mechanism (Fig. 1b).

As is the case for many of the quasi-planar systems in the static balancing theory, the conception of these mechanisms was based on the employment of ideal or zero-free-length springs ($l_0=0$) between their attachment points [1, 3, 6]. However, to achieve ideal spring functionality with normal or non-zero-free-length springs ($l_0\neq 0$), special constructions or springs are required [5-7, 9, 11, 15, 16]. Other methods of approximating static balance with normal springs include application of wrapping cams or non-linear springs using special coiling techniques [4, 6, 17]. However, although achieving ideal spring functionality with normal springs is possible, it tends to complicate mechanism construction, especially with large spring pre-stresses in combination with relative large mechanism movements (causing large spring strains). To facilitate construction of (gravity) balancers, employing normal springs directly mounted between the link attachment points of the mechanism in the conception phase therefore seems beneficiary.

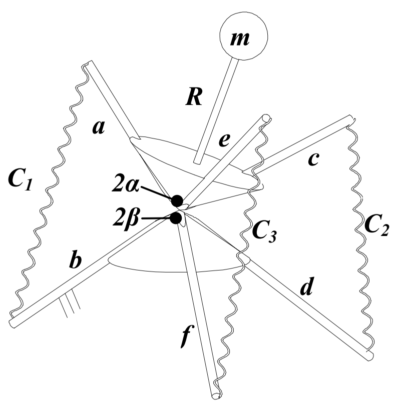
This paper discusses planar (revolute joint) and spatial (spherical joint) spring-mechanisms that enable perfect balancing of gravity acting on an inverted pendulum while employing normal springs between the spring-attachment points. It will be shown that not satisfying the geometric conditions required for perfect static balance *and* replacing the ideal springs with normal springs in an initially perfect ideal spring-to-spring balancer, causes a residual torque that can be statically balanced by coupling a mass rotating about a fixed pivot, to the spring mechanism. This will be done by means of potential energy considerations: First the total potential energy expression of the resulting spring-mechanism is determined as a function of a rotation of the top triangle or cone relative to the bottom one. Subsequently possible solutions are presented that (re)enable a constant potential energy of the mechanism.



Balancing conditions

$$C_1 \cdot ab = C_2 \cdot cd = C_3 \cdot ef$$
$$\tan\alpha \tan\beta = 2$$
$$\sin\alpha \cdot \sin\beta \geq 0 \quad \textit{(Torsion stability)}$$

(a)



Balancing conditions

$$C_1 \cdot ab = C_2 \cdot cd = C_3 \cdot ef$$
$$mgR = -\frac{3}{2} C_1 ab \cos\alpha \cos\beta \left(2 - \tan\alpha \cdot \tan\beta\right)$$
$$\sin\alpha \cdot \sin\beta \geq 0 \quad \textit{(Torsion stability)}$$

(b)

Figure 1. Overview of spatial static balancing mechanisms with ideal linear tension springs between the spring attachment points, with: 2α and 2β the apical angles of respectively the top –and bottom cone, on which the levers a to f are mounted; R the distance from a fixed pivot to the center of gravity of the mass; C_i the spring stiffness.

NOMENCLATURE

a,b,c	=	distance from pivot to spring attachment
a*,b*,c*	=	distance from pivot to spring attachment when projected on the yz-plane
C_i	=	spring stiffness
l_{0i}	=	free length (length of spring when not preloaded or loaded externally)
l_i	=	Actual spring-length (l_{0i}+(pre)strain of springs)
m_i	=	mass
r_r, r_s	=	radius of rollers/ spheres
R_i	=	distance from fixed pivot to center of gravity of a mass
V	=	potential energy
2α, 2β	=	angles between lines from pivot to spring attachment points (apical angle triangle/ cone)
ρ_1, ρ_2, ρ_3	=	angle between the axis of revolution of a cone and the line from pivot to spring attachment when projected on the yz-plane
θ, φ	=	rotation angles
ψ_1, ψ_2, ψ_3	=	angles between symmetry line and a to c respectively.

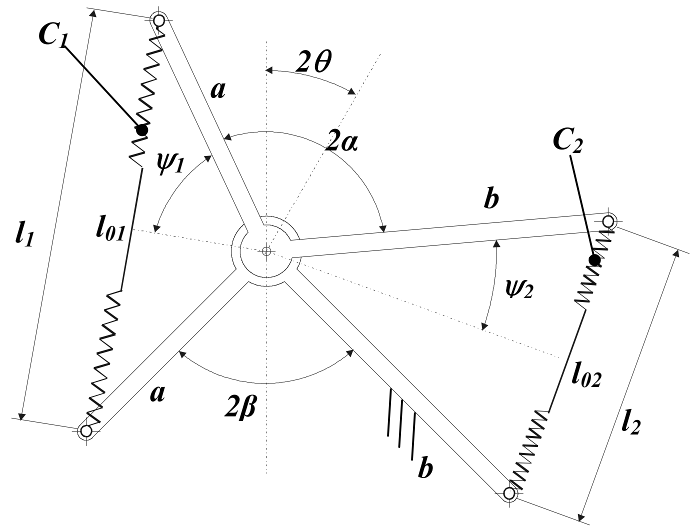


Figure 2. 1dof mechanism with normal linear springs employed between the spring attachment points of levers a and b. The springs employed have a stiffness C_1 and C_2 with a zero length l_{01} and l_{02} respectively.

PERFECT 1DOF STATIC GRAVITY BALANCING

Consider the spring-mechanism given in Fig. 2. The mechanism consists of two triangles with apical angles of *2α* and *2β*. The triangles are interconnected with a revolute joint at their apexes. The lever-arm lengths are *a* and *b*. Furthermore two normal linear extension springs are employed with stiffness C_1 and C_2 and free-length l_{01} and l_{02} respectively.

Note that the mechanism's architecture is the same as the basic ideal spring-to-spring balancer, amply discussed in literature [6, 8, 9, 15], but with the lever arms $a=b\neq c=d$. Furthermore the two employed ideal linear springs have been replaced with normal springs. In the course of this section the consequences of these actions are discussed, using a potential energy consideration.

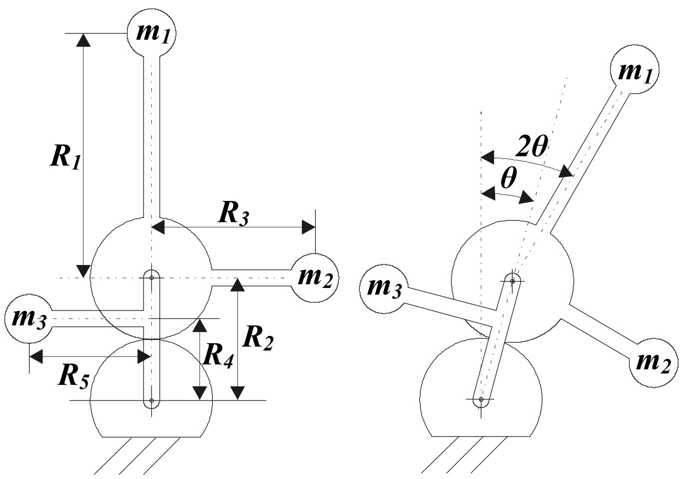


Figure 3. An unstable mechanism consisting of three masses and two rollers. Mass m_1 and m_2 are connected to the top roller. Mass m_3 is connected to the connection link between the centres of the rollers.

Using the symbols shown in Fig. 2, where it is especially noted that the upper half of the mechanism is rotated through an angle of 2θ, the following geometrical relations can be identified:

$$\psi_1 = \tfrac{1}{2}\pi - \tfrac{1}{2}(\alpha+\beta) + \theta; \;\; \psi_2 = \tfrac{1}{2}\pi - \tfrac{1}{2}(\alpha+\beta) - \theta; \\ \ell_1 = 2a\sin\psi_1; \;\; \ell_2 = 2a\sin\psi_2 \tag{1}$$

Writing out eq. (1) gives:

$$\begin{aligned} l_1 &= 2a\sin(\psi_1) = 2a\cos(\gamma-\theta) = 2a(\cos\gamma\cos\theta + \sin\gamma\sin\theta) \\ l_2 &= 2b\sin(\psi_2) = 2b\cos(\gamma+\theta) = 2b(\cos\gamma\cos\theta - \sin\gamma\sin\theta) \\ &\textit{Thus}: \\ l_1^2 &= 4a^2(\cos^2\gamma\cos^2\theta + \sin^2\gamma\sin^2\theta + 2\cos\gamma\cos\theta\sin\gamma\sin\theta) \\ l_2^2 &= 4b^2(\cos^2\gamma\cos^2\theta + \sin^2\gamma\sin^2\theta - 2\cos\gamma\cos\theta\sin\gamma\sin\theta) \end{aligned} \tag{2}$$

where $\gamma = \tfrac{1}{2}(\alpha+\beta)$

The total potential energy of this mechanism yields:

$$V = \tfrac{1}{2}C_1\left(l_1^2 - 2l_{01}l_1 + \ell_{01}^2\right) + \tfrac{1}{2}C_2\left(l_2^2 - 2l_{02}l_2 + \ell_{02}^2\right) \tag{3}$$

Substitution of eq. (2) in eq. (3) and rewriting gives:

$$\begin{aligned} V &= \cos 2\theta\cos 2\gamma\left(C_1a^2 + C_2b^2\right) + \sin 2\theta\sin 2\gamma\left(C_1a^2 - C_2b^2\right) \\ &- 2\cos\theta\cos\gamma\left(C_1l_{01}a + C_2l_{02}b\right) - 2\sin\theta\sin\gamma\left(C_1l_{01}a - C_2l_{02}b\right) + K \end{aligned} \tag{4}$$

where $\gamma = \tfrac{1}{2}(\alpha+\beta)$ and K represents the constant terms in the expression.

From eq. (4) it is readily seen that the potential energy of the mechanism is independent of the rotation 2θ of the top triangle if $C_1a^2 = C_2b^2$, $\alpha+\beta=0.5\pi$ and $l_{01}=l_{02}=0$: The resulting mechanism then satisfies the balancing conditions derived for the basic ideal spring-to-spring balancer [6, 8, 9, 15]. If these static spring-to-spring balancing conditions are not satisfied, it is seen that a $\cos 2\theta$; $\sin 2\theta$; $\cos\theta$; and a $\sin\theta$ term remain in the potential energy expression. These terms can be eliminated from the expression by adding rotating masses, using a similar approach as described in the introduction [15, 16]. A possible embodiment of these rotating masses is given in Fig. 3.

Figure 3 shows a mechanism consisting of three masses and two rollers with radius r_r. The rollers are connected such that only pure rolling of the top roller is possible relative to the bottom roller. This causes the connection link between the centers of the rollers to rotate through an angle θ for a rotation 2θ of the top roller. With the masses m_1 and m_2 mounted on the top roller and the mass m_3 on the extension of the connection link between the centers of the two rollers, the mass rotations are coupled and the total potential energy expression of these masses for a rotation 2θ of the top roller can be written as:

$$\begin{aligned} V_{mass} &= \cos 2\theta\cdot g\cdot m_1R_1 - \sin 2\theta\cdot g\cdot m_2R_3 \\ &+ \cos\theta\cdot g\cdot\left(m_1R_2 + m_2R_2 + m_3R_4\right) + \sin\theta\cdot g\cdot m_3R_5 + K \end{aligned} \tag{5}$$

where K represents the constant terms in the expression and R_2 equals twice the roller radius r_r.

With the potential energy expression of the mass mechanism containing the same products of rotation-dependent goniometric terms and constants as for the spring mechanism, it should be possible to achieve perfect static balance by mounting the mass-mechanism to the spring-mechanism as shown in Fig. 4. This is possible because the potential energy characteristic of the spring-mechanism given in Fig. 2 is not

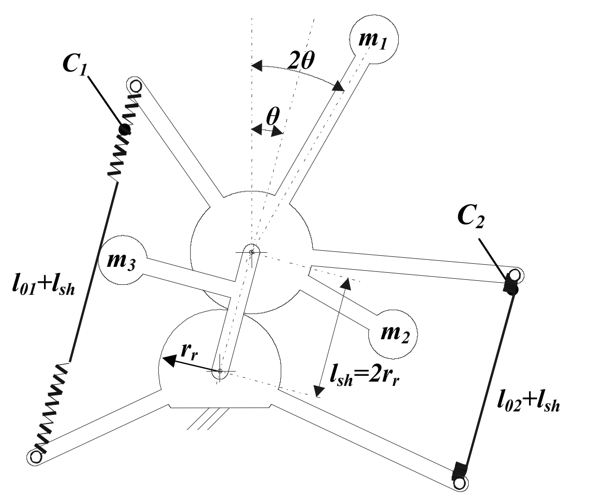
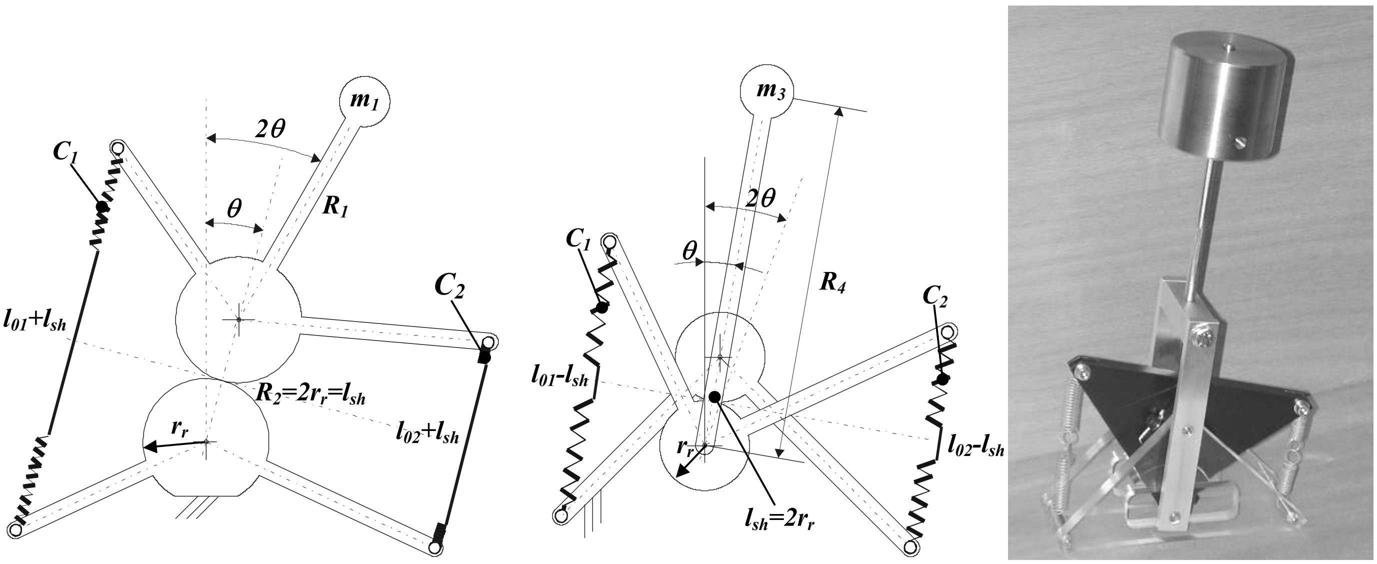


Figure 4. The mechanisms given in Figs. 2 and 3 combined. The triangle-spring mechanism is as described in Fig. 2 only the apexes of the triangles are shifted apart with a distance $l_{sh}=2r_r$ and $\alpha=\beta$.



(a) (b) (c)

Figure 5. Mechanisms as given in Fig. 4, but with: a) m_2=0 and m_3=0. b) m_1=0; m_2=0; α=0.25π; and R_5=0, with triangles overlapped. c) Working prototype of b), where flexible bands are used to prevent the rollers from slipping [13-15].

influenced by shifting apart or overlapping the apexes with a distance $l_{sh}=2r_r$ provided that the shifting distance is compensated for in the free-length of the normal springs (=> free-length=$l_{0i}+l_{sh}$) and $\alpha=\beta$ [1, 6, 12-14, 16].

In combining the mass and spring mechanism, the potential energy terms of both mechanisms can be summed and the following conditions for perfect static balance can be derived:

$$\begin{aligned}
&\cos 2\alpha\left(C_1 a^2 + C_2 b^2\right) = -m_1 g R_1 \\
&\sin 2\alpha\left(C_1 a^2 - C_2 b^2\right) = m_2 g R_3 \\
&2\cos\alpha\left(C_1 l_{01} a + C_2 l_{02} b\right) = g\left(m_1 R_2 + m_2 R_2 + m_3 R_4\right) \\
&2\sin\alpha\left(C_1 l_{01} a - C_2 l_{02} b\right) = m_3 g R_5
\end{aligned} \tag{6}$$

Note that $\alpha=\beta$ thus $\gamma=\alpha=\beta$ and $R_2=2r_r$.

Figure 5 shows two possible 'single mass'-spring mechanisms, directly derived from the described basic gravity-spring balancer (Fig. 4). It is noted that for eq. (6) to remain valid, springs with free-length $l_{01}-l_{sh}$ and $l_{02}-l_{sh}$ have to be employed with overlapping the triangles (Fig. 5b and c) whereas springs with free-length $l_{01}+l_{sh}$ and $l_{02}+l_{sh}$ have to be employed with shifting them apart (Fig. 5a). The working model [13-15] (Fig.5c) functions quite satisfactorily. Provided it is placed on a horizontal base, the mass is in equilibrium throughout its range of motion while hysteresis is negligible.

The basic mechanism given in Fig. 4 allows for several other configurations of mass-spring mechanisms that enable perfect static balance. The thought is to zero out the cosine and sine terms in the potential energy expression of the (normal) spring-mechanism (eq. (4)) with a combination of rotating masses (or ideal springs [16]). Since the potential energy expression of the normal spring-mechanism discussed, contains goniometric terms with both 2θ and θ dependencies, a coupling of two rotating masses is demanded to achieve this. This can be achieved with two rollers as described, but numerous different embodiments including gears, belts, or linkage type mechanisms can also be used. However, the use of rollers allows for a relative simple mechanism with little hysteresis during operation [10].

PERFECT 2DOF STATIC GRAVITY BALANCING

Consider the cone-spring mechanism given in Fig. 6. The mechanism consists of two cones with equal apical angles of 2α. The cones are interconnected with a spherical joint at their apexes. On the surfaces of the cones, six lever arms (*a to c*) are mounted such that their projections on the *xy*-plane have in-between angles of 120° (with mechanism in neutral position). Three normal linear springs with stiffness C_1, C_2, C_3 and free lengths l_{01}, l_{02} *and* l_{03} respectively, are connected to the ends of respectively the lever arms $a \neq b \neq c$. The bottom cone and the *xyz*-coordinate system are considered fixed to the ground with the apex of the bottom cone and the origin of the coordinate system coinciding. The coordinate system is drawn outside of the apex for clarity reasons.

Note that the architecture of this mechanism is basically the same as the one shown in Fig. 1a but with mutual equal lever-arms $a \neq b \neq c$, and two equal apical angles 2α of the cones. Furthermore the three employed ideal springs have been

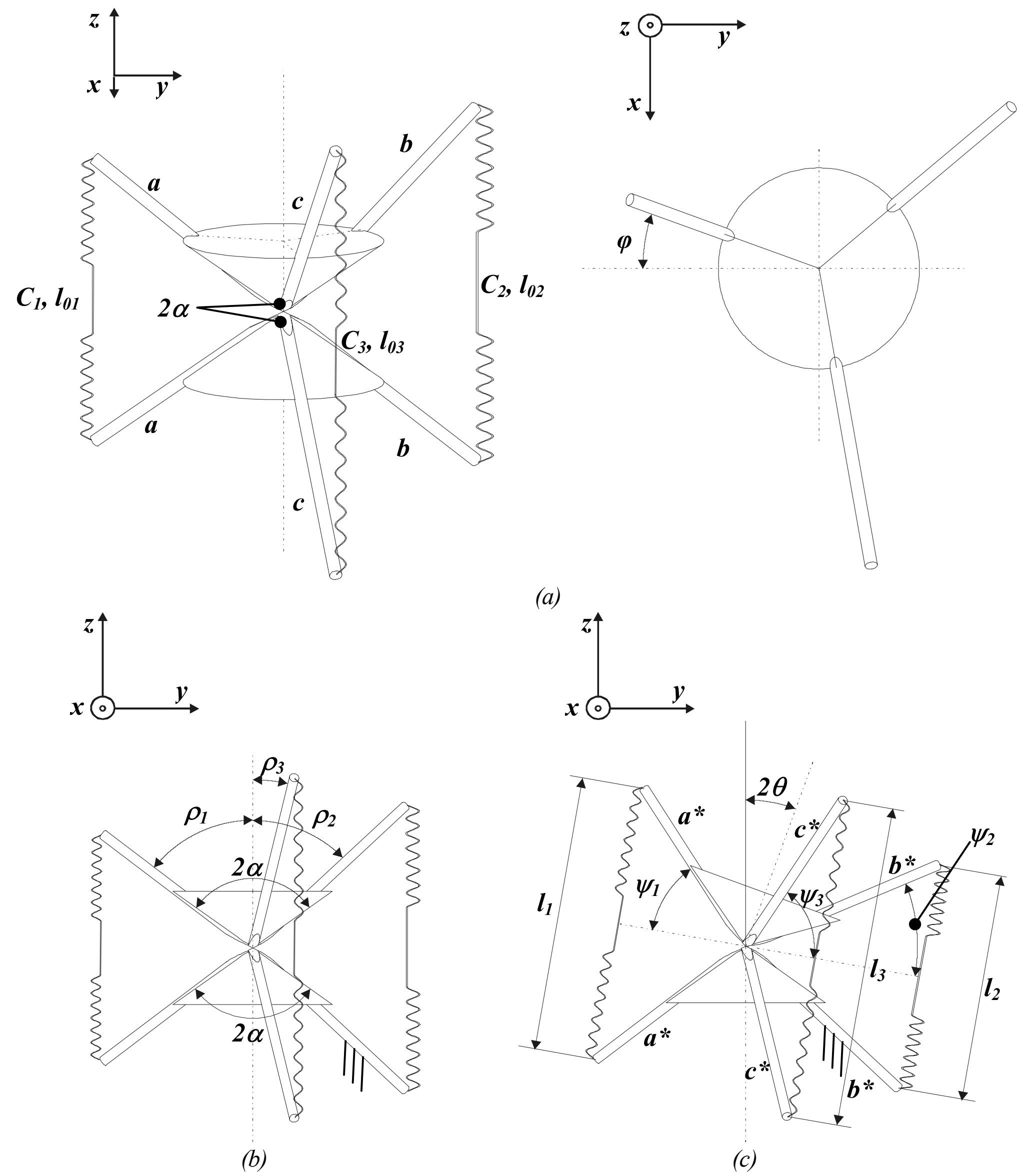


Figure 6. Spatial spring-mechanism consisting of two cones, interconnected with a spherical joint in the apexes and three normal springs employed between the ends of levers a, b, and c. a) 3D and top view of the mechanism in neutral position. b)/ c) Projections of the spring mechanism on the yz-plane with: b) The mechanism in neutral position. c) The top cone of mechanism rotated through an angle 2θ about the x-axis. a, b* and c* are the projections of the levers a, b and c on the yz-plane respectively.*

replaced with technical springs. In the course of this section the consequences of these actions are discussed, using a potential energy consideration.

Every position of the top cone relative to the bottom cone can be described by a rotation of the top cone about an arbitrary chosen axis in the *xy*-plane (passing through the spherical joint) followed by an axial rotation about its centerline (torsion rotation). In this section the potential energy expression of this three-spring mechanism will be derived for rotations of the top cone about an arbitrary chosen axis lying in the *xy*-plane. The mechanism is considered as follows: Instead of rotating the top cone about an arbitrary axis lying in the *xy*-plane, the mechanism –as a whole- is rotated through an (arbitrary chosen) angle φ about the *z*-axis (Fig. 6a). Subsequently the bottom cone is again considered fixed to the *xyz*-coordinate system and a rotation of 2θ is enforced on the top cone about the *x*-axis (Fig. 6b→6c).

With rotations 2θ of the top cone, it is seen that two mutual spring attachment points always move in one plane that is perpendicular to the axis of rotation or, for this consideration, parallel to the *yz*-plane. Therefore the distance between two mutual spring attachment points can be determined directly from the projection of the mechanism on the *yz*-plane. Considering that linear technical springs are employed, the potential energy of the mechanism is therefore given by:

$$\begin{aligned} V &= \tfrac{1}{2}C_1\cdot\left(l_1^{\,2}+l_{01}^{\,2}-2\cdot l_{01}\cdot l_1\right)+\tfrac{1}{2}C_2\cdot\left(l_2^{\,2}+l_{02}^{\,2}-2\cdot l_{02}\cdot l_2\right) \\ &\quad+\tfrac{1}{2}C_3\cdot\left(l_3^{\,2}+l_{03}^{\,2}-2\cdot l_{03}\cdot l_3\right) \end{aligned} \tag{7}$$

with l_1 to l_3 the distance between a set of two spring attachment points.

Figure 6 yields the following geometric relations:

$$l_1 = 2\cdot a^*\cdot\sin(\psi_1);\; l_2 = 2\cdot b^*\cdot\sin(\psi_2);\; l_3 = 2\cdot c^*\cdot\sin(\psi_3) \tag{8}$$

With $\psi_1 = 0.5\pi-\rho_1+\theta;\;\; \psi_2 = 0.5\pi-\rho_2-\theta;\;\; \psi_3 = 0.5\pi-\rho_3-\theta$

The sine terms in the equations can be written as:

$$\begin{aligned} \sin(\psi_1) &= \sin(0.5\pi-\rho_1+\theta) = \cos(\rho_1)\cdot\cos(\theta)+\sin(\rho_1)\cdot\sin(\theta) \\ \sin(\psi_2) &= \sin(0.5\pi-\rho_2-\theta) = \cos(\rho_2)\cdot\cos(\theta)-\sin(\rho_2)\cdot\sin(\theta) \\ \sin(\psi_3) &= \sin(0.5\pi-\rho_3-\theta) = \cos(\rho_3)\cdot\cos(\theta)-\sin(\rho_3)\cdot\sin(\theta) \end{aligned} \tag{9}$$

Finally the angles ρ_i can be derived from:

$$\begin{aligned} \sin(\rho_1) &= \frac{a\cdot\sin(\alpha)\cdot\cos(\varphi)}{a^*} & \cos(\rho_1) &= \frac{a\cdot\cos(\alpha)}{a^*} \\ \sin(\rho_2) &= \frac{b\cdot\sin(\alpha)\cdot\cos(\tfrac{1}{3}\pi-\varphi)}{b^*}; & \cos(\rho_2) &= \frac{b\cdot\cos(\alpha)}{b^*} \\ \sin(\rho_3) &= \frac{c\cdot\sin(\alpha)\cdot\cos(\tfrac{1}{3}\pi+\varphi)}{c^*} & \cos(\rho_3) &= \frac{c\cdot\cos(\alpha)}{c^*} \end{aligned} \tag{10}$$

Substituting eq. (9) and eq. (10) in eq. (8) gives:

$$\begin{aligned} l_1 &= 2\cdot a\cdot\cos\alpha\cos\theta+2\cdot a\cdot\sin\alpha\sin\theta\cos\varphi \\ l_2 &= 2\cdot b\cdot\cos\alpha\cos\theta-2\cdot b\cdot\sin\alpha\sin\theta\cos(\tfrac{1}{3}\pi-\varphi) \\ l_3 &= 2\cdot c\cdot\cos\alpha\cos\theta-2\cdot c\cdot\sin\alpha\sin\theta\cos(\tfrac{1}{3}\pi+\varphi) \end{aligned} \tag{11}$$

Thus:

$$\begin{aligned} l_1^{\,2} &= 4\cdot a^2\cdot\cos^2\alpha\cos^2\theta+4\cdot a^2\cdot\sin^2\alpha\sin^2\theta\cos^2\varphi \\ &\quad+8\cdot a^2\cdot\cos\alpha\cos\theta\sin\alpha\sin\theta\cos\varphi \\ l_2^{\,2} &= 4\cdot b^2\cdot\cos^2\alpha\cos^2\theta+4\cdot b^2\cdot\sin^2\alpha\sin^2\theta\cos^2(\tfrac{1}{3}\pi-\varphi) \\ &\quad-8\cdot b^2\cdot\cos\alpha\cos\theta\sin\alpha\sin\theta\cos(\tfrac{1}{3}\pi-\varphi) \\ l_3^{\,2} &= 4\cdot c^2\cdot\cos^2\alpha\cos^2\theta+4\cdot c^2\cdot\sin^2\alpha\sin^2\theta\cos^2(\tfrac{1}{3}\pi+\varphi) \\ &\quad-8\cdot c^2\cdot\cos\alpha\cos\theta\sin\alpha\sin\theta\cos(\tfrac{1}{3}\pi+\varphi) \end{aligned}$$

With these geometric relations determined, the total potential energy of this mechanism can be determined from to eq. (7). Substituting eq. (11) into eq. (7) and rearranging gives:

$$\begin{aligned} V &= \tfrac{1}{2}\left(C_1\cdot l_{01}^2+C_2\cdot l_{02}^2+C_3\cdot l_{03}^2\right) \\ &\quad+2\cdot\sin^2\alpha\cdot\sin^2\theta\cdot\begin{pmatrix} C_1\cdot a^2\cdot\cos^2\varphi+C_2\cdot b^2\cdot\cos^2(\tfrac{1}{3}\pi-\varphi) \\ +C_3\cdot c^2\cdot\cos^2(\tfrac{1}{3}\pi+\varphi)\end{pmatrix} \\ &\quad+4\cdot\cos\alpha\cdot\sin\alpha\cdot\cos\theta\cdot\sin\theta\cdot\begin{pmatrix} C_1\cdot a^2\cdot\cos\varphi-C_2\cdot b^2\cdot\cos(\tfrac{1}{3}\pi-\varphi) \\ -C_3\cdot c^2\cdot\cos(\tfrac{1}{3}\pi+\varphi)\end{pmatrix} \\ &\quad+2\cdot\cos^2\alpha\cdot\cos^2\theta\cdot\left(C_1\cdot a^2+C_2\cdot b^2+C_3\cdot c^2\right) \\ &\quad-2\cdot\cos(\alpha)\cdot\cos(\theta)\cdot\begin{pmatrix} C_1\cdot l_{01}\cdot a+C_2\cdot l_{02}\cdot b \\ +C_3\cdot l_{03}\cdot c\end{pmatrix} \\ &\quad-2\cdot\sin(\alpha)\cdot\sin(\theta)\cdot\begin{pmatrix} C_1\cdot l_{01}\cdot a\cdot\cos(\varphi)-C_2\cdot l_{02}\cdot b\cdot\cos(\tfrac{1}{3}\pi-\varphi) \\ -C_3\cdot l_{03}\cdot c\cdot\cos(\tfrac{1}{3}\pi+\varphi)\end{pmatrix} \end{aligned} \tag{12}$$

with $C_1\cdot a^2 = C_2\cdot b^2 = C_3\cdot c^2$ and $C_1\cdot a\cdot l_{01} = C_2\cdot b\cdot l_{02} = C_3\cdot c\cdot l_{03}$ eq. (12) can be simplified to:

$$\begin{aligned} V &= 2\sin^2\alpha\sin^2\theta\cdot C_1a^2\begin{pmatrix}\cos^2\varphi+\cos^2(\tfrac{1}{3}\pi-\varphi) \\ +\cos^2(\tfrac{1}{3}\pi+\varphi)\end{pmatrix} \\ &\quad+4\cos\alpha\sin\alpha\cos\theta\sin\theta\cdot C_1a^2\begin{pmatrix}\cos\varphi-\cos(\tfrac{1}{3}\pi-\varphi) \\ -\cos(\tfrac{1}{3}\pi+\varphi)\end{pmatrix} \\ &\quad+6\cos^2\alpha\cos^2\theta\cdot C_1\cdot a^2-6C_1\cdot a\cdot l_{01}\cdot\cos\alpha\cdot\cos\theta \\ &\quad-2C_1\cdot a\cdot l_{01}\cdot\sin\alpha\sin\theta\cdot\begin{pmatrix}\cos\varphi-\cos(\tfrac{1}{3}\pi-\varphi) \\ -\cos(\tfrac{1}{3}\pi+\varphi)\end{pmatrix}+K \end{aligned} \tag{13}$$

where K represents the constant terms in the equation.

Knowing that:

$$\begin{aligned} &\cos^2(\tfrac{1}{3}\pi-\varphi)+\cos^2(\tfrac{1}{3}\pi+\varphi) = \tfrac{1}{2}\cos^2\varphi+1.5\sin^2\varphi \\ &\cos(\tfrac{1}{3}\pi-\varphi)+\cos(\tfrac{1}{3}\pi+\varphi) = \cos\varphi \end{aligned}$$

it is seen that the φ-dependency can be eliminated from eq. (13) (condition for rotation symmetric mechanism). Rewriting of eq. (13) therefore gives:

$$\begin{aligned} V &= 1.5C_1a^2\cos^2\alpha\left(2-\tan^2\alpha\right)\cos(2\theta) \\ &\quad-6C_1al_{01}\cdot\cos\alpha\cdot\cos\theta+K \end{aligned} \tag{14}$$

With $C_1\cdot a^2 = C_2\cdot b^2 = C_3\cdot c^2;\; C_1\cdot a\cdot l_{01} = C_2\cdot b\cdot l_{02} = C_3\cdot c\cdot l_{03}$ and where K represents the constant terms.

From eq. (14) it is readily seen that the potential energy of the mechanism is independent of the rotation 2θ of the top triangle if $C_1\cdot a^2 = C_2\cdot b^2 = C_3\cdot c^2;\; \tan^2\alpha = 2$ and $l_{0i}=0$ (note that the resulting mechanism then is basically the three-spring balancer shown in Fig. 1). If these static spring-to-spring balancing conditions are not satisfied, a $\cos 2\theta$ and a $\cos\theta$ term remain in the expression. As was the case for the 1dof mechanism, these terms can be eliminated from the expression through the

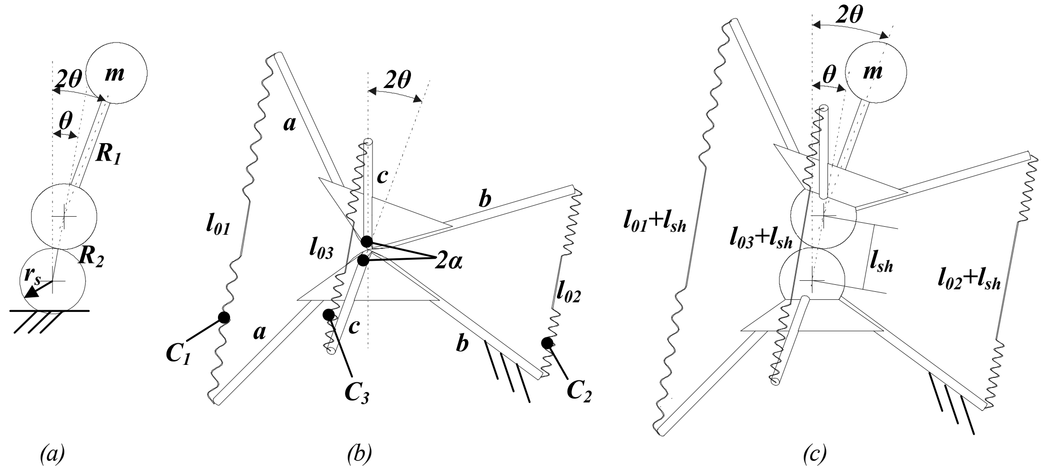


Figure 7. Conception of 2dof gravity balancer consisting of 6 levers, three normal springs and two cones, interconnected with a spherical joint in the apexes. The three-spring mechanism is as described in Fig. 6.

addition of rotating masses. This can be achieved by coupling two inversed pendulums such that a rotation 2θ of one pendulum causes a rotation θ of the other pendulum. A possible embodiment of such a mechanism is given in Fig. 7a.

Figure 7a shows a mechanism consisting of a mass and two spheres with radius r_s. The spheres are connected such that only rolling of the top sphere is possible relative to the bottom sphere. This causes the connection line between the centers of the spheres to rotate through an angle θ for a rotation 2θ of the top sphere. With the mass m mounted on the top sphere, it rotates about two pivots, such that a rotation 2θ of link R_1 about the centre of the top sphere causes link R_2 to rotate through an angle θ about the center of the bottom sphere. The total potential energy expression of these masses for a rotation 2θ of the top sphere therefore becomes:

$$V_{mass} = mgR_1 \cdot \cos 2\theta + mgR_2 \cdot \cos\theta + K \tag{15}$$

where K represents the constant terms and where $R_2 = 2r_s$.

With the potential energy expression of the mass mechanism containing the same products of rotation-dependent goniometric terms and constants as for the spring mechanism, it should be possible to achieve perfect static balance by mounting the mass-mechanism to the spring-mechanism as shown in Fig. 6b-c. This is possible because the potential energy characteristic of the spring-mechanism given in Fig. 6 is not influenced by shifting apart or overlapping the apexes with a distance $l_{sh}=2r_s$ provided the shifting/ overlapping distance is compensated for in the free-length of the springs (=> in this case the normal spring free-length therefore becomes: $l_{0i}+l_{sh}$) [1, 6, 12-14, 16].

The potential energy expression for the total mechanism (Fig. 7c) then becomes:

$$\begin{aligned} V_{tot} &= V_{spring} + V_{mass} = 1.5 \cdot C_1 a^2 \cos^2\alpha \left(2 - \tan^2\alpha\right)\cos 2\theta \\ &- 6 \cdot C_1 \cdot l_{01} \cdot a \cdot \cos\alpha \cdot \cos\theta + mgR_1 \cdot \cos 2\theta \\ &+ mgR_2 \cdot \cos\theta + K \end{aligned} \tag{16}$$

With $C_1 \cdot a^2 = C_2 \cdot b^2 = C_3 \cdot c^2$; $C_1 \cdot a \cdot l_{01} = C_2 \cdot b \cdot l_{02} = C_3 \cdot c \cdot l_{03}$; $R_2=2r_s$ and where K represents the constant terms.

Equation (16) yields the following conditions for perfect gravity balance of the mechanism given in Fig. 7c:

$$\begin{aligned} mgR_1 &= -1\tfrac{1}{2} C_1 a^2 \cos^2\alpha \left(2 - \tan^2\alpha\right) \\ mgR_2 &= 6 l_{01} C_1 a \cos\alpha \end{aligned} \tag{17}$$

With $C_1 \cdot a^2 = C_2 \cdot b^2 = C_3 \cdot c^2$; $C_1 \cdot a \cdot l_{01} = C_2 \cdot b \cdot l_{02} = C_3 \cdot c \cdot l_{03}$ and $R_2=2r_s$.

Figure 8 shows a working model of the mechanism displayed in Fig. 7c. It should be noted that the spheres rely only on friction and spring tension forces to ensure a 2dof movement. This is because flexible bands could not be used, as was the case for the 1dof mechanism. The prototype incorporates six equal lever arms and three equal linear normal springs. Although the model worked quite satisfactorily when the correct balancing conditions were set, it is observed during experiments that a correct alignment is as necessary as it is difficult to obtain and sustain. This is of course because of the lack of hysteresis with the current rolling link embodiment and the non-holonomic nature of the contact.

The torsion stability, or rather torsion stiffness, of the prototype has been determined numerically, using a similar method as set out in the companion paper [16]. No single stability condition or stiffness expression could be derived (symbolically), therefore it is only remarked here that the prototype proved torsion stable in theory and, more importantly, in practice. If the described mechanism is actually going to be employed and torsion stability of the mechanism is an issue, this should be looked into further.

CONCLUSION

The use of normal springs in static balancing generally results in approximate balance or requires more complex adjustment mechanisms. In this paper several mechanisms that statically balance gravity acting on an inverted pendulum, with normal (non-zero free length) springs employed directly between the spring attachment points, have been described and working models were presented. This has been done for mechanisms employed with both a 1dof pivot (revolute joint) as well as for a 3dof pivot (spherical joint) of which the basic configurations are summarized in Fig. 9 together with the derived balancing conditions.

It has been shown that not satisfying the balancing conditions *and* replacing the ideal springs with normal springs of an initially static spring-to-spring balancer results in a potential energy expression with rotation dependent terms which can be perfectly zeroed out by means of rotating masses. However, to obtain perfect gravity balance, the two rotating segments always have to be coupled such that a rotation of 2θ of one segment causes a rotation θ of the other segment. This kinematic demand can be satisfied by stacking two rollers (1dof) -or spheres (2dof) in which a rotation 2θ of one roller - or sphere relative to the other, causes a rotation θ of the connection line between the centers of the rollers (assuming pure rolling). Using this kinematic coupling of two rotating segments results in mechanisms that are relative easy to construct (compared to some other mechanisms employing technical springs), with minimal hysteresis during operation.

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REFERENCES

[1] Carwardine G. (1932) Improvements in elastic force mechanisms, UK Patent 377.251, *Specifications of Inventions*, Vol. 2773, Patent Office Sale Branch, London.

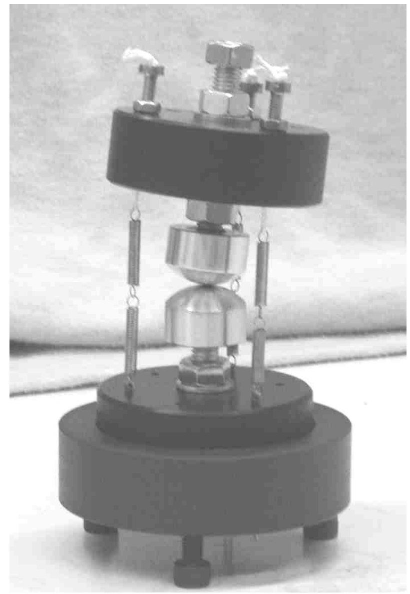


Figure 8. Working prototype of the mechanism given in Fig. 7c. All lever arms for spring attachment points and the springs are equal.

[2] Grootenboer H.J. (1995) *Stabiliteit*, Lecture Notes, Twente University, Mechanical Engineering dept., Enschede, The Netherlands.

[3] Hain K. (1952) Der Federausgleich von Lasten, *Grundlagen der Landtechnik*, (3)38/50.

[4] Hain K. (1953) Gelenkarme Bandgetriebe für den Kraftausbleich durch Federn, *Grundlagen der Landtechnik*, (4)100/9.

[5] Haupt G., Grewolls J. (1963) Über das Gleichgewicht zwischen Federkräften und konstanten Kräften, *Maschinen bautechnik* 8(12)423/8.

[6] Herder J.L. (2001) *Energy Free Systems; theory, conception and design of statically balanced spring mechanisms*, Ponsen en Looijen BV, Delft, ISBN 90-370-0192-0

[7] Hilpert H. (1968) Weight balancing of precision mechanical instruments, *Jnl Mechanisms,* (3)289/302.

[8] Hoek W. van der (1986) *A mechanical designer's case book: DDP*, Philips CFT Report 59/85, Compiled by the Mechanical Engineering Dept of the Eindhoven University of Technology.

[9] Koster M.P. (1996) *Construction principles for accurate motion and positioning* (in Dutch), Uitgeverij Universiteit Twente, Enschede, ISBN 90-365-0832-0

[10] Kuntz J.P. (1995) *Rolling Link Mechanisms*, PhD thesis, Delft University of technology, Delft, The Netherlands.

[11] Nathan R.H. (1985) A constant force generating mechanism, *Journal of Mechanisms, Transmissions, and Automation in Design*, 107(12)508/12.

[12] Riele te F.L.S. (1997) The design of a system that attains neutral equilibrium for rotations about a moving pivot (in Dutch), Masters thesis, Report BW-94, Twente University, Enschede, The Netherlands.

[13] Riele F.L.S. te, Hekman E.E.G., Cool J.C. (2000) The development of a highly efficient energetic locomotion foot*, Proceedings of the Dutch Annual Conference on BioMedical Engineering*, Bante I, Feijen J "eds.", Oct 9-10, Papendal, The Netherlands, pp.112-113.

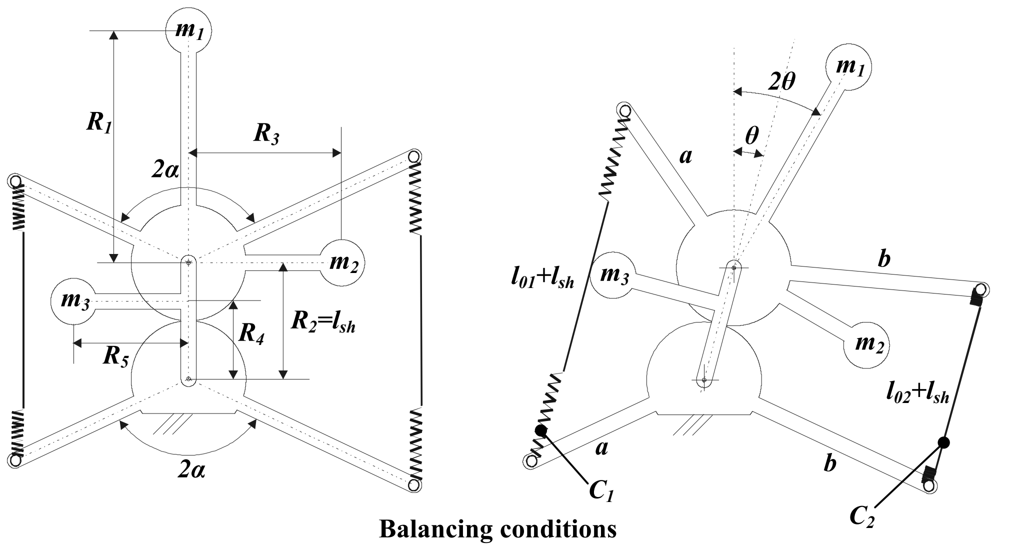
[14] Riele F.L.S., Herder J.L. (2001) Perfect static balance with normal springs*, CDROM proceedings ASME 2001 Design Engineering Technical conference*, Pittsburgh, Pennsylvania, paper number DETC2001/DAC-21069.

[15] Riele te F.L.S. (2003) *The HEELfoot; Design of a plantarflexing prosthetic foot*, Enschede, ISBN 90-365-1974-8.

[16] Riele F.L.S., Herder J.L., E.E.G. Hekman (2004) Spatial static gravity balancing with ideal springs*, CDROM proceedings ASME 2004 Design Engineering Technical conference*, Salt Lake City, Utah, paper number DETC2004-57166.

[17] Streit D.A., Shin E. (1993) Equilibrators for planar linkages, *Journal of Mechanical Design*, 115 (3) 604/11.

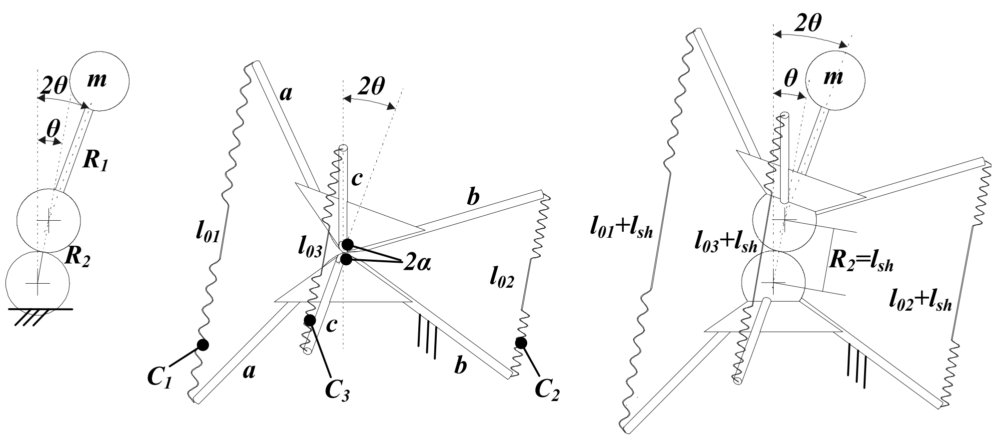
[18] Ulrich N., Kumar V. (1991) Passive mechanical gravity compensation for robot manipulators, *Proceedings of the 1991 IEEE International Conference on Robotics and Automation,* Sacramento, California, April 1991, p1536-1541.



Balancing conditions

$$\cos 2\alpha \left(C_1 a^2 + C_2 b^2\right) = -m_1 g R_1$$
$$\sin 2\alpha \left(C_1 a^2 - C_2 b^2\right) = m_2 g R_3$$
$$2\cos\alpha \left(C_1 l_{01} a + C_2 l_{02} b\right) = g\left(m_1 R_2 + m_2 R_2 + m_3 R_4\right)$$
$$2\sin\alpha \left(C_1 l_{01} a - C_2 l_{02} b\right) = m_3 g R_5$$

(a)



Balancing conditions

$$mgR_1 = -1\tfrac{1}{2} C_1 a^2 \cos^2\alpha \left(2 - \tan^2\alpha\right)$$
$$mgR_2 = 6 l_{01} C_1 a \cos\alpha$$
$$C_1 \cdot a^2 = C_2 \cdot b^2 = C_3 \cdot c^2$$
$$C_1 \cdot a \cdot l_{01} = C_2 \cdot b \cdot l_{02} = C_3 \cdot c \cdot l_{03}$$

Torsion stiffness is mechanism dependent[16]

(b)

Figure 9. Overview of perfect static gravity balancing mechanisms with normal springs employed between the spring attachment points, with: 2α the apical angles of a triangle or cone on which the levers for the spring attachment points a to c are mounted; R_i segment lengths from which the distance from the center of gravity of a mass to a pivot can be determined; l_{0i} the spring free-length; C_i the spring stiffness. a) 1dof gravity balancer. b) 2dof gravity balancer.