THE ARBITRARY LAGRANGIAN–EULERIAN METHOD ON GRIDS OF TRIANGULAR ELEMENTS: A NEW CONVECTION SCHEME

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Abstract: This study investigates the Eulerian step of a split ALE finite element method for quadratic triangular elements. To solve the convection equation for integration point values, a new method is constructed, directly based on integration point values without using intermediate nodal values. The Molenkamp test and a so-called block test were executed to check the performance and stability of the convection scheme. From these tests it is concluded that the new convection scheme shows accurate results.

Keywords: convection schemes, unstructured grids, forming processes

1 Introduction

In forming process simulations the Arbitrary Lagrangian–Eulerian finite element method can be used to keep a mesh regular. In this research we use the split formulation. First the material increment of the variables is determined with a Lagrangian step and then the Eulerian step replaces the mesh. Since the topology of the mesh does not change, the Eulerian step can be represented by the convection equation. The purpose of this research is to investigate the Eulerian part of the calculation for an unstructured mesh of quadratic triangular elements with three integration points.

Most methods for solving the convection equation use a finite volume formulation with a cell-centered approach, see for example Tamamidis [1] for triangular elements. A cell-centered approach may lead to loss of accuracy. If no variation of the convected quantity over an element is taken into account, the method is first-order accurate. For second-order accuracy the gradient of the convected quantity has to be dealt with. Inspired by Van Leer [2], we introduce an integration point based convection scheme, which is second-order accurate, that can be applied to unstructured grids. In order to check the performance and stability of the new scheme, the Molenkamp test and a more severe block test were performed.

2 Convection

In an incremental finite element analysis, the state after an increment is calculated for a finite number of grid points: the nodal points (for e.g. displacements and temperature) and the integration points (for e.g. stresses and strains).

In the ALE-method the material displacement increments are uncoupled from the grid point displacement increments. This implies that in addition to the incremental calculation in the Lagrangian step, convection in the Eulerian step must be taken into account. This can be represented by the convection equation:

$$\begin{cases} \frac{\partial q}{\partial t} + \mathbf{v}^c \cdot \frac{\partial q}{\partial \mathbf{x}} = 0 & \mathbf{x} \in \Omega,\ t \in \mathbb{R}^+ \\ q(\mathbf{x}, 0) = q_0(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases} \tag{1}$$

The convected quantity, e.g. the equivalent plastic strain, is denoted by q. The relative velocity between the material and grid, the convective velocity, is given by $\mathbf{v}^c = \mathbf{v}^g - \mathbf{v}^m$ where $\mathbf{v}^g$ and $\mathbf{v}^m$ represent the grid velocity and the material velocity respectively. Prior to the Eulerian step the convective velocity is fully known.

The initial distribution q_0 represent variables known in the integration points of the element. Hence, in the convection step the integration point values have to be updated. Since quadratic displacement elements can describe linearly varying strains the gradient of the convected quantity over an element should be taken into account to preserve the accuracy of the Lagrangian step. It is most obvious to calculate this gradient from integration point values. Van Leer [2] describes a method to solve the convection equation on one-dimensional grids. This method starts from a piece-wise linear approximation of the initial-value distribution by simple basic functions. Overall a discontinuous distribution results, which is convected explicitly and then remapped piece-wise in terms of the same basic functions. Below, this method is applied on two-dimensional grids of triangles.

Step 1 Construct a linear distribution $\overline{q}_i(x,y,t_0)$ per element i.

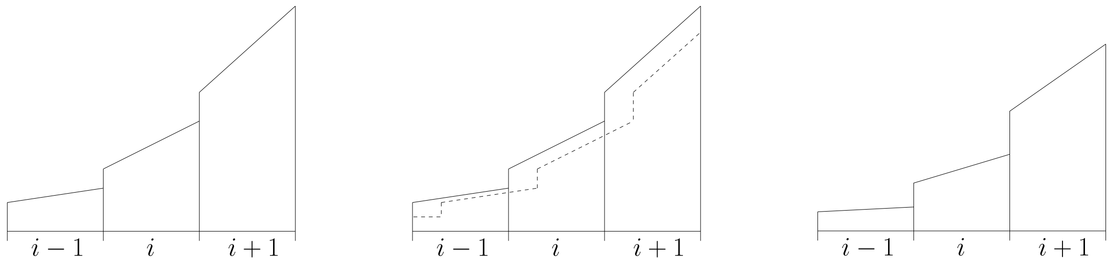
$$\overline{q}_i(x,y,t_0) = \overline{q}_i^0 + \frac{\overline{\partial q_i}^0}{\partial x}(x-\bar{x}_i) + \frac{\overline{\partial q_i}^0}{\partial y}(x-\bar{y}_i) \qquad (x,y) \in \text{element } i \tag{2}$$

where $(\bar{x}_i, \bar{y}_i)$ is the central point. The functions $\overline{q}_i(x,y,t_0)$ for all elements together form the total approximate distribution $\overline{q}(x,y,t_0)$ which is discontinuous over the element boundaries. Figure 1a shows $\overline{q}(x,y,t_0)$ for a one-dimensional situation which is easier to interpret than a two-dimensional situation.

Step 2 Starting from the approximate distribution $\overline{q}(x,y,t_0)$, Equation (1) is integrated over a finite time step Δt. This is achieved by shifting $\overline{q}(x,y,t_0)$ over a distance $\mathbf{v}^c \Delta t$:

$$\overline{q}(x,y,t_1) = \overline{q}(x-u\Delta t, y - v\Delta t, t_0) \tag{3}$$

where $t_1 = t_0 + \Delta t$ and u and v are the components of the convective velocity $\mathbf{v}^c$. This leads to a shifted piece-wise linear distribution where discontinuities appear inside the elements as demonstrated in Figure 1b. For each element i an updated linear distribution $\overline{q}_i(x,y,t_1)$ as shown in Figure 1c must be



(a) (b) (c)

Figure 1. Shifting the piece-wise linear distribution for a one-dimensional situation. (a) The approximate distribution $\overline{q}(x,t_0)$. (b) The approximate distribution $\overline{q}(x,t_0)$ (solid line) and the shifted distribution $\overline{q}(x-u\Delta t, t_0)$ (dashed line). (c) The updated approximate distribution $\overline{q}(x,t_1)$.

constructed. Therefore, it is needed to determine updated mean values $\overline{q}_i^1$, and updated gradients $\frac{\overline{\partial q_i}^1}{\partial x}$ and $\frac{\overline{\partial q_i}^1}{\partial y}$ from the shifted distribution at $t = t_1$.

Step 3 New integration point values are calculated from the new approximate distribution $\overline{q}(x,y,t_1)$ based on $\overline{q}_i^1$, $\frac{\overline{\partial q_i}^1}{\partial x}$ and $\frac{\overline{\partial q_i}^1}{\partial y}$.

The mean value is updated on the basis of the following equation:

$$\int_{A_i} \overline{q}(x,y,t_1)\,\mathrm{d}\Omega = \int_{A_i} \overline{q}(x-u\Delta t, y-v\Delta t, t_0)\,\mathrm{d}\Omega \tag{4}$$

where A_i is the area of element i. Since the linear distribution is constructed from the central point of the triangular element (2), the linear terms of $\overline{q}(x,y,t_1)$ in the left hand side vanish. The right hand side is an integral over a discontinuous distribution (see Figure 1b), but can be approximated quite simply. Equation (4) yields:

$$A_i \overline{q}_i^1 = A_i \overline{q}_i^0 - \sum_{k=1}^{3} F_{i,k}^0 \tag{5}$$

where $F_{i,k}^0$ is the flux 'through' side k of element i, calculated as the volume under the shaded part of Figure 2a or 2b dependent on the direction of the normal velocity $v_{n,k}$. On the basis of the normal velocity the upwind element is determined. Here, the upwind element is defined as the element from which the information, needed to calculate $F_{i,k}^0$, comes. After that, $F_{i,k}^0$ is given by the following integral:

$$F_{i,k}^0 = \operatorname{sgn}(v_{n,k}) \int_{A_{i,k}} \left\{ \overline{q}_{\text{up}}^0 + \frac{\overline{\partial q}_{\text{up}}^0}{\partial x}(x-\bar{x}_{\text{up}}) + \frac{\overline{\partial q}_{\text{up}}^0}{\partial y}(y-\bar{y}_{\text{up}}) \right\} \mathrm{d}\Omega \tag{6}$$

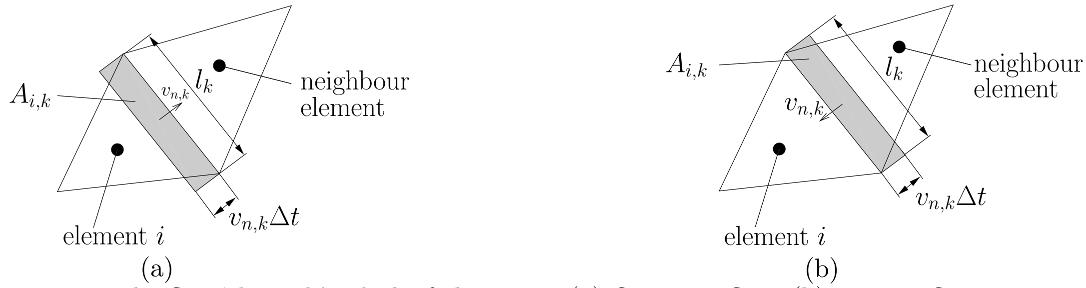


Figure 2. The flux 'through' side k of element i. (a) Outgoing flux. (b) Ingoing flux.

where $A_{i,k}$ is the area of the shaded part in the upwind element.

The flux $F^0_{i,k}$ gives a positive contribution if the normal velocity $v_{n,k}$ is outward pointing ($v_{n,k} \geq 0$) and a negative contribution if the normal velocity is inward pointing ($v_{n,k} < 0$). Of course, if the flux is an *outgoing* flux for one element itself, it will be an *ingoing* flux for its neighbour element, hence conservativity is guaranteed.

To update the gradients $\frac{\overline{\partial q_i}^0}{\partial x}$ and $\frac{\overline{\partial q_i}^0}{\partial y}$ the first moment of $\overline{q}$ along the x- and y-axis are kept constant:

$$\int_{A_i} \overline{q}(x, y, t_1)(x - \bar{x}_i)\,\mathrm{d}\Omega = \int_{A_i} \overline{q}(x - u\Delta t, y - v\Delta t, t_0)(x - \bar{x}_i)\,\mathrm{d}\Omega \tag{7}$$

and analogous for the y-direction. This method coincides with a minimization of the least square error for the new distribution.

3 Results

3.1 Molenkamp tests

To check the performance of the convection method the two-dimensional Molenkamp test [3] is used. The Molenkamp test is a test problem where a Gaussian profile is rotated. The velocity field is described in the nodal points and represents a pure rigid-body rotation:

$$u(x, y) = -2\pi y \qquad v(x, y) = 2\pi x \tag{8}$$

in the domain: $-1 \leq x \leq 1 \quad -1 \leq y \leq 1$. The initial distribution of the scalar field q is a Gaussian distribution and is defined in the integration points:

$$q(x, y, 0) = 0.01^{4r^2} \tag{9}$$

with r the distance from the point $(-\frac{1}{2}, 0)$.

At $t = 1$ one full revolution of the profile is done. At that moment the exact solution q_{exact} equals the initial situation (9), so that the following error norms can be determined:

$$\|\Delta q\|_n = \sqrt[n]{\frac{1}{A}\sum_{i=1}^{N_{\text{el}}} \int_{A_i} |q - q_{\text{exact}}|^n \,\mathrm{d}\Omega} \tag{10}$$

where N_{el} represents the number of elements in the domain and A is the total area of the domain.

The Molenkamp problem is solved on unstructured grids of triangles. The notation $n_x \times n_y$ refers to the number of elements on the boundary of the domain. The grids are refined with a factor two and four, approximately resulting in four times and sixteen times the number of elements.

The calculations were done with a *First Order Upwind* method (FOU) where a constant distribution per element is assumed and with the described method.

For every performed calculation q_{max} and q_{min} are the maximum and minimum value of the scalar function q evaluated in the integration points.

3.2 Block test

The Molenkamp tests in the preceding section have shown us that the new method gives very accurate results. In order to perform an extra check on stability, we did a more severe test with the unstructured 28×28 mesh. This test can be compared with the Molenkamp tests, but instead of a Gaussian profile a block profile is rotated.

Table 1. Results for the Molenkamp test.

	Grid	$q_{\max}$	$q_{\min}$	$\|\Delta q\|_1$ [$\cdot 10^{-4}$]	$\|\Delta q\|_2$ [$\cdot 10^{-4}$]
FOU	14×14	0.184	0.0	418.5	1116.
	28×28	0.315	0.0	323.8	888.3
	56×56	0.472	0.0	227.5	656.9
New method	14×14	0.843	$-9.15 \cdot 10^{-3}$	65.44	197.5
	28×28	0.969	$-1.54 \cdot 10^{-4}$	18.19	60.88
	56×56	1.02	$-5.50 \cdot 10^{-5}$	10.52	39.07

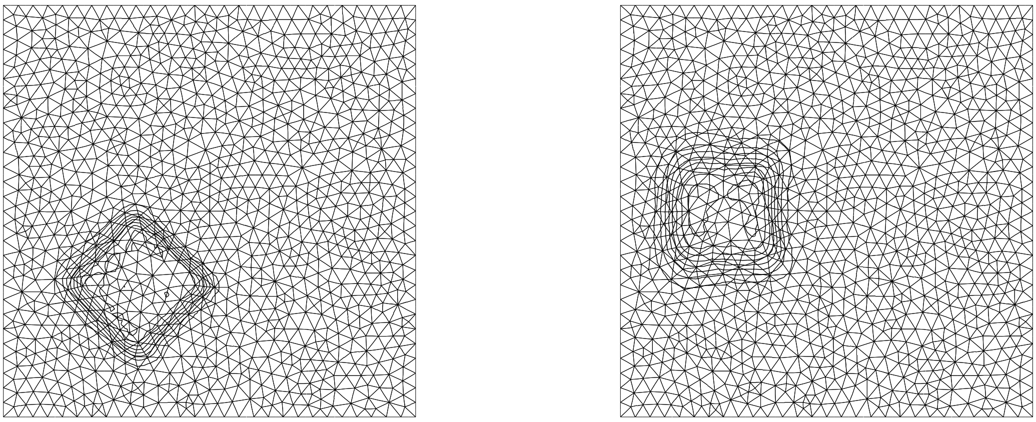
The velocity field is described in the nodal points and is given by (8). The domain is also equal to the domain used for the Molenkamp test. The initial distribution of the scalar field q however is defined as:

$$q(x,y,0) = \begin{cases} 1 & \text{if } -0.75 \leq x \leq -0.25,\ -0.25 \leq y \leq 0.25 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

The block is revolved in 1600 time steps of 0.000625 sec. At that moment ($t = 1$ sec) the calculated distribution can be compared with the initial situation.

After an overshoot of the maximum and an undershoot of the minimum, the maximum and minimum values remain oscillating a little around steady values. It is concluded that the new method leads to stable results for this severe test.

In Figure 3a and 3b the scalar field distributions are shown after one eighth of the revolution and after a full revolution respectively. Comparing these two figures we see that the block profile is present



(a) (b)

Figure 3. Scalar field distribution for the unstructured 28×28 mesh. (a) After one eighth of the revolution. (b) After a full revolution. Values belonging to the isolines from the outer to the inner one: 0.022; 0.167; 0.311; 0.455; 0.600; 0.744; 0.888; 1.033.

well after one eighth of the revolution and reasonably well after a full revolution.

4 Conclusions

The presented convection method shows very accurate results for the Molenkamp tests as well as the block test. In this scheme no direct time discretization is done. The application of the integration point based convection scheme as discussed in this article is not limited to quadratic triangular elements. The principle can be applied to other types of 2- and 3-dimensional elements.

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