Entailment Relations and Distributive Lattices

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Abstract. To any *entailment relation* [Sco74] we associate a distributive lattice. We use this to give a construction of the product of lattices over an arbitrary index set, of the Vietoris construction, of the embedding of a distributive lattice in a boolean algebra, and to give a logical description of some spaces associated to mathematical structures.

1 Introduction

Most spaces associated to mathematical structures: spectrum of a ring, space of valuations of a field, space of bounded linear functionals, ... can be represented as distributive lattices. The key to have a natural definition in these cases is to use the notion of *entailment relation* due to Dana Scott. This note explains the connection between entailment relations and distributive lattices. An entailment relation may be seen as a logical description of a distributive lattice. Furthermore, most operations on distributive lattices are simpler when formulated as operations on entailment relations.

A special kind of distributive lattices (and hence entailment relations) is then used to represent compact regular spaces. We use this to give an alternative construction of the product of a family of compact regular spaces, and of the Vietoris power locale of a compact regular space [Joh82].

2 Entailment Relation

Let S be a set, we think of its elements as abstract "statements" or propositions. We denote by $X, Y, Z, \dots$ arbitrary finite subsets of S. We write X, Y for $X \cup Y$ and X, s for $X \cup \{s\}$.

Definition 1. An entailment relation $\vdash$ on the set S is a relation between finite subsets of S satisfying the following conditions of reflexivity, monotonicity and transitivity:

$$X \vdash Y \textit{ if } X \cap Y \textit{ is inhabited} \tag{R}$$

$$\frac{X \vdash Y}{X, X' \vdash Y, Y'} \tag{M}$$

$$\frac{X \vdash s, Z \quad X, s \vdash Z}{X \vdash Z} \tag{T}$$

The first condition (R) can be replaced by the condition $x \vdash x$ using the second condition (M).

Notice that this definition is "symmetric": the converse of an entailment relation is also an entailment relation.

As emphasised by Scott [Sco71, Sco73, Sco74], this notion of entailment relation can be seen as an abstract generalisation of Gentzen's multi-conclusion sequent calculus. Gentzen was inspired by the notion of consequence relation, due to Hertz, see [Gen69], and was the first to formulate the rule (T) in this setting.

The basic idea of this note is that entailment relations provide a general way of presenting distributive lattices. The reason is as follows. First, relations as equations $e = f$ can be replaced by relations as inequations $e \leq f$ and $f \leq e$. Next, if e is expressed in disjunctive normal form and f in conjunctive normal form, then the inequation $e \leq f$ can be replaced by a set of inequations *disjunct of* $e \leq$ *conjunct of* f, which is the same form as an entailment.

Here is a general lemma about entailment relations that will be needed in one example. We suppose given an entailment relation $\vdash$ on a set S. Let $A \subseteq S$ be a subset of S. We let $X \vdash_A Y$ mean that there exists a finite subset $A_0 \subseteq A$ such that $X, A_0 \vdash Y$.

Lemma 2. *$\vdash_A$ is an entailment relation. It is the least entailment relation $\vdash'$ containing $\vdash$ such that $\vdash' a$ for all $a \in A$.*

Proof. The rules (R) and (M) clearly hold for $\vdash_A$. Let us check the rule (T). If we have $X \vdash_A s, Y$ and $X, s \vdash_A Y$ then there exist $A_1, A_2 \subseteq A$ finite such that $X, A_1 \vdash s, Y$ and $X, A_2, s \vdash Y$. By (T) and (M) for $\vdash$ it follows that we have $X, A_1, A_2 \vdash Y$. Since $A_1 \cup A_2 \subseteq A$ is finite, this implies $X \vdash_A Y$ as desired.

It is clear that we have $\vdash_A a$ for all $a \in A$. Let $\vdash'$ be an entailment relation containing $\vdash$ such that $\vdash' a$ for all $a \in A$. If we have $X, A_0 \vdash Y$ with $A_0 \subseteq A$ finite then by using (T) we get $X \vdash' Y$. This shows that $\vdash'$ contains $\vdash_A$.

3 Distributive Lattices

Given a set S with a binary relation R on finite subsets of S we say that a map $f : S \to D$ from S to a distributive lattice D *preserves* R iff $X \; R \; Y$ implies $\wedge_{x \in X} f(x) \leq \vee_{y \in Y} f(y)$. We are interested in the following universal problem: a distributive lattice D together with a map $i : S \to D$ preserving R such that for any other map $f : S \to L$ preserving R there is a unique lattice map $f' : D \to L$ such that $f'i = f$. We say that $D, i : S \to D$ is *generated* by S, R. Since the theory of distributive lattice is equational, there is a solution to this universal problem. The goal of this section is to prove the following result.

Theorem 3. *Let S be a set with an entailment relation $\vdash$. If $D, i : S \to D$ is the distributive lattice generated by $S, \vdash$ then $X \vdash Y$ iff $\wedge_{x \in X} i(x) \leq \vee_{y \in Y} i(y)$.*

Corollary 4. *Let S be a set with a binary relation R on finite subsets of S. If $D, i : S \to D$ is the distributive lattice generated by S, R then the relation $X\ R^+\ Y$ defined by $\wedge_{x \in X} i(x) \leq \vee_{y \in Y} i(y)$ is the least entailment relation containing R.*

We shall prove this theorem by building explicitly a distributive lattice $D, i : S \to D$ generated by a given entailment relation $S, \vdash$. Notice that, for any solution, using distributivity, any element of D is equal to one element $\vee_j \wedge_{x \in Y_j} i(x)$ for some finite set $\{Y_0, \ldots, Y_{m-1}\}$ of finite subsets of S. This suggests the following construction.

Let D be the set of finite sets of finite subsets of S. Intuitively $A \in D$ is thought of as $\vee_{X \in A} \wedge_{x \in X} x$. If $A, B \in D$ let $A \wedge B$ be the finite set of all unions $X \cup Y$, $X \in A, Y \in B$ and $A \vee B$ be the union of A and B. To each $A \in D$ we can associate $A^* \in D$ such that Z meets all elements of A iff Z contains one element of A^* : we take A^* to be the set $\{\{x\} \mid x \in X\}$ if A is a singleton $\{X\}$ and $(A \cup B)^* = A^* \wedge B^*$. We define then $A \leq B$ to mean $X \vdash Y$ for all $X \in A$ and $Y \in B^*$. Finally, we let $i : S \to D$ be the map $i(a) = \{\{a\}\}$.

Lemma 5. *If $X \vdash y, Z$ for all $y \in Y$ and $Y \vdash Z$ then $X \vdash Z$.*

Proof. This is a direct consequence of the rules (M) and (T).

Lemma 6. *Let B be an element of D. If $Y \vdash Z$ for all $Y \in B$ and $X \vdash Y, Z$ for all $Y \in B^*$ then $X \vdash Z$.*

Proof. We write $B = \{Y_0, \ldots, Y_{m-1}\}$ and reason by induction on m. The base case is trivial. If $m > 0$ then for any $y \in Y_{m-1}$ and any $Y' \in \{Y_0, \ldots, Y_{m-2}\}^*$, we have $Y', y \in \{Y_0, \ldots, Y_{m-1}\}^*$ and hence $X \vdash Y', y$. By hypothesis $Y_0 \vdash Z, \ldots, Y_{m-2} \vdash Z$ and hence $Y_0 \vdash y, Z, \ldots, Y_{m-2} \vdash y, Z$. By induction hypothesis $X \vdash y, Z$, then by the previous lemma $X \vdash Z$.

Proposition 7. *The relation $\leq$ is reflexive and transitive on D. Furthermore D is a distributive lattice for the operation $A \wedge B$ and $A \vee B$ with a least element $0 = \emptyset$ and a greatest element $1 = \{\emptyset\}$.*

Proof. Reflexivity of $\leq$ follows from (R). Transitivity is a consequence of the previous lemma. For checking that $\wedge$ is indeed a meet operation, we remark that each element of $(A \wedge B)^*$ either contains an element of A^* or contains an element of B^*. Distributivity holds because $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$.

Proposition 8. *The distributive lattice $D, i : S \to D$ is generated by the entailment relation $\vdash$.*

Proof. Let L be a distributive lattice and $f : S \longrightarrow L$ a map preserving $\vdash$. If $A = \{X_i \mid i \in I\}$ we define $f'(A)$ to be $\wedge_{i \in I} \vee_{x \in X_i} f(x)$. This is a lattice morphism from D to L such that $f'i = f$. Furthermore, it is clear that the values of f' are uniquely determined by the condition $f'i = f$.

Theorem 3 is a direct consequence. In this particular construction we do have $X \vdash Y$ iff $\wedge_{x \in X} i(x) \vdash \vee_{y \in Y} i(y)$ and hence, by unicity of the solution of an universal problem, this holds for any solution. Another way of proving this theorem, closer to the way taken in [JKM97], would be to consider the set S^* of syntactical $\wedge, \vee$-formulae on S, to define a sequent calculus on S^* taking as axiom the sequents on atomic formulae given by the entailment relation. We can then prove a cut-elimination result that gives another proof of theorem 3 [JKM97].

4 Some Universal Constructions

The goal of this section is to show that the notion of entailment relation simplifies the construction of the solution of some universal problems for distributive lattices.

4.1 Product

Let D_j be a family of distributive lattices, indexed over a set J. We consider $S = (\Sigma j \in J) D_j$ and the following relation: $X \vdash Y$ iff there exists $j \in J$ and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}$ in D_j such that $(j, a_k) \in X$ and $(j, b_l) \in Y$ and $a_0 \wedge \ldots \wedge a_{n-1} \leq b_0 \vee \ldots \vee b_{m-1}$ in D_j.

Theorem 9. *The relation $\vdash$ is an entailment relation on S. Let $D, i : S \to D$ be the distributive lattice generated by $S, \vdash$ and $\sigma_j : D_j \to D$ be the map $a \longmapsto i(a, j)$. Then $D, \sigma_j : D_j \to D$ is the coproduct lattice of the family D_j.*

Proof. The fact that $\vdash$ is an entailment relation has a direct proof.

Also, we have $(j, a)(j, b) \vdash (j, ab)$ and $(j, ab) \vdash (j, a)$, $(j, ab) \vdash (j, b)$ for each $a, b \in D_j$ so that $\sigma_j(ab) = \sigma_j(a)\sigma_j(b)$. Similarly we prove $\sigma_j(a \vee b) = \sigma_j(a) \vee \sigma_j(b)$, $\sigma_j(0) = 0$ and $\sigma_j(1) = 1$ so that σ_j is a morphism.

If we have a lattice L with a family of morphisms $f_j : D_j \to L$ we can define

$$f : S \to L,\ (j, a) \longmapsto f_j(a).$$

It is direct that f preserves $\vdash$ because each f_j is a morphism and hence there is a unique lattice map $f' : D \to L$ such that $f' i = f$ which is also the unique lattice map such that $f' \sigma_j = f_j$ for all j.

Remark. If D_j is generated by an entailment relation $S_j, \vdash$ another entailment relation generating the coproduct of the family (D_j) is given by the set $S' = (\Sigma j \in J) S_j$ with the entailment relation: $X \vdash Y$ iff there exists $j \in J$ and $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}$ in S_j such that $(j, x_k) \in X$ and $(j, y_l) \in Y$ and $x_0, \ldots, x_{n-1} \vdash y_0, \ldots, y_{m-1}$ in S_j.

4.2 Vietoris Construction

Let D be a distributive lattice. We take for S the set of elements $\Box x$ and $\Diamond x$ for $x \in D$. We define the relation $\vdash$ as follows: $\Box x_i, \Diamond y_j \vdash \Box z_k, \Diamond t_l$ iff $\wedge_i x_i \leq z_k \vee \vee_l t_l$ in D for one k or $\wedge_i x_i \wedge y_j \leq \vee_l t_l$ in D for one j.

Theorem 10. *$\vdash$ is an entailment relation on S. Furthermore, the lattice $V(D)$ generated by $S, \vdash$ is the lattice generated by abstract symbols $\Box(a), \Diamond(a),\ a \in D$ subject to the relations (see [Joh85])*

- $\Box(1) = 1,\ \Box(a_1 a_2) = \Box(a_1)\Box(a_2)$,
- $\Diamond(0) = 0,\ \Diamond(a_1 \vee a_2) = \Diamond(a_1) \vee \Diamond(a_2)$,
- $\Box(a_1)\Diamond(a_2) \leq \Diamond(a_1 a_2)$,
- $\Box(a_1 \vee a_2) \leq \Box(a_1) \vee \Diamond(a_2)$.

Proof. (R), (M) and (T) for $\vdash$ are immediate. It is also directly checked that the given relations hold in $V(D)$.

Let L be a distributive lattice with elements $t(a), m(a) \in L$ for $a \in D$ satisfying

- $m(1) = 1,\ m(a_1 a_2) = m(a_1)m(a_2)$,
- $t(0) = 0,\ t(a_1 \vee a_2) = t(a_1) \vee t(a_2)$,
- $m(a_1)t(a_2) \leq t(a_1 a_2)$,
- $m(a_1 \vee a_2) \leq m(a_1) \vee t(a_2)$.

We can define a map $f : S \to L$ by $f(\Box(a)) = t(a)$ and $f(\Diamond(a)) = m(a)$. It is direct that f preserves $\vdash$ and hence there is a unique lattice map $f' : D \to L$ such that $f'i = f$ which is the unique lattice map such that $f'\Box = t$ and $f'\Diamond = m$.

Remark. If D is generated by an entailment relation $S_0, \vdash$ another entailment relation generating $V(D)$ is given by the set of elements $\Box(X), \Diamond(X)$, where X is a finite subset of S_0, with the entailment relation: $\Box X_i, \Diamond Y_j \vdash \Box Z_k, \Diamond T_l$ iff there exists k such that $x_i, Z_k \vdash t_l$ for all choices $x_i \in X_i, t_l \in T_l$ or there exists j such that $x_i, Y_j \vdash t_l$ for all choices $x_i \in X_i, t_l \in T_l$.

4.3 Embedding of a distributive lattice in a boolean algebra

Let D be a distributive lattice. We are interested in the following problem: to find a lattice map $i : D \to B$ from D in a boolean algebra B such that, if B' is any boolean algebra and $f : D \to B'$ any lattice map there exists a unique lattice map $f' : B \to B'$ such that $f'i = f$. We say that $B, i : D \to B$ is the boolean algebra *generated by* the distributive lattice D.

Let S be the set of elements $x \in D$ or $\bar{x}$ for $x \in D$. We define the relation $\vdash$ as follows: $x_i, \bar{y_j} \vdash z_k, \bar{t_l}$ iff $\wedge_i x_i \wedge \wedge_l t_l \leq \vee_j y_j \vee \vee_k z_k$ in D.

Theorem 11. *$\vdash$ is an entailment relation on S. If $B, i : S \to B$ is the distributive lattice generated by $S, \vdash$, then B is a boolean algebra, and $x \longmapsto i(x),\ D \to B$ is the boolean algebra generated by D.*

Proof. That $\vdash$ is an entailment relation is direct.

We prove that any element of B has a complement. By construction each element $i(s)$ for $s \in S$ has a complement because $i(\bar{x})$ is the complement of $i(x)$. Furthermore the property of having a complement is closed by conjunction and disjunction, hence any element of B has a complement.

The map $i : D \to B,\ x \longmapsto i(x)$ is a lattice map. Indeed, if $x \leq y$ we have $x \vdash y$ and hence $i(x) \leq i(y)$. Since $i(x), i(y) \vdash i(xy)$ we have also $i(x)i(y) \leq i(xy)$ and hence $i(xy) = i(x)i(y)$. Similarly we prove $i(x \vee y) = i(x) \vee i(y)$ and $i(0) = 0$ and $i(1) = 1$.

Finally, if B' is a boolean algebra and $f : D \to B'$ a lattice map, then we can extend f to $g : S \to B'$ by taking $g(\bar{x}) = \overline{f(x)}$. It is direct to check that g preserves $\vdash$ and hence we have a unique lattice map $f' : B \to B'$ such that $f'i = g$. This shows that there exists a unique lattice map $f' : B \to B'$ such that $f'(i(x)) = f(x)$ for all $x \in D$ since this implies $f'(i(\bar{x})) = \overline{f(x)}$ and hence this condition is actually equivalent to $f'i = g$.

As an application of theorem 3 we get the following result.

Corollary 12. *If $B, i : D \to B$ is the boolean algebra generated by a distributive lattice D we have $a \leq b$ in D iff $i(a) \leq i(b)$ in B.*

The reader can compare this construction with the ones in [Mac37, Mac39, Per57].

Remark. If D is generated by an entailment relation $S_0, \vdash$ another entailment relation generating B is given by the set of elements $x \in S_0$ or $\bar{x}$ for $x \in S_0$ with the entailment relation: $x_i, \bar{y_j} \vdash z_k, \bar{t_l}$ iff $x_i, t_l \vdash y_j, z_k$ in S_0.

4.4 Dimension of Lattices

A. Joyal has suggested the following constructive definition of the Krull dimension of commutative rings (and distributive lattices, see [BJ81, En86]). We shall only look at the case of dimension 0. For any distributive lattice D one considers the distributive lattice D_1 solution of the following universal problem: there exist two morphisms $u_0, u_1 : D \to D_1$ such that $u_0 \leq u_1$. The dimension of D is then defined to be 0 iff $u_0 = u_1$. We can characterise the lattice D_1 as follows.

Let S be the set of formal elements $u_0(x)$ and $u_1(x)$ for $x \in D$. We consider the relation $u_0(a_i), u_1(b_j) \vdash u_0(c_k), u_1(d_l)$ defined by: there exists $x \in D$ such that $\wedge a_i \leq x \vee c_k$ *and* $\wedge a_i \wedge b_j \wedge x \leq \vee d_l$.

Theorem 13. *The relation $\vdash$ is an entailment relation on S and the distributive lattice D_1 generated by $S, \vdash$ is a solution to the universal problem: there exist two morphisms $u_0, u_1 : D \to D_1$ such that $u_0 \leq u_1$.*

Proof. That the relation $\vdash$ satisfies (R) and (M) is direct. Let us prove that it satisfies (T). If we have both $u_0(x), u_0(a), u_1(b) \vdash u_0(c), u_1(d)$ and $u_0(a), u_1(b) \vdash u_0(x), u_0(c), u_1(d)$ then there exist $z_1, z_2 \in D$ such that $xa \leq z_1 \vee c,\ xz_1ab \leq d$

and $a \leq z_2 \vee x \vee c$, $z_2ab \leq d$. Let us take $z = xz_1 \vee z_2$. We have $abz = xz_1ab \vee z_2ab \leq d$. Furthermore $xa \leq z_1x \vee c$ and $a \leq x \vee z_2 \vee c$ so that $a \leq z \vee c$. It follows that we have $u_0(a), u_1(b) \vdash u_0(c), u_1(d)$ as desired.

The second case to consider is if we have both

$$u_1(x), u_0(a), u_1(b) \vdash u_0(c), u_1(d) \quad \text{and} \quad u_0(a), u_1(b) \vdash u_1(x), u_0(c), u_1(d).$$

In this case there exist $z_1, z_2 \in D$ such that $a \leq z_1 \vee c$, $xz_1ab \leq d$ and $a \leq z_2 \vee c$, $z_2ab \leq x \vee d$. Let us take $z = z_1z_2$. We have $a \leq (z_1 \vee c)(z_2 \vee c) = z \vee c$ and $azb \leq x \vee d$, $azbx \leq d$ so that $azb \leq d$. It follows that we have $u_0(a), u_1(b) \vdash u_0(c), u_1(d)$ as desired.

The fact that we get a solution of the universal problem is then proved in the same way as in the previous examples.

Using the theorem 3 we get the following corollary.

Corollary 14. *We have $u_0(a)u_1(b) \leq u_0(c) \vee u_1(d)$ iff there exists $x \in D$ such that $a \leq c \vee x$ and $abx \leq d$.*

Another application is the following characterisation of lattices of dimension 0 [En86].

Corollary 15. *A lattice D is of dimension 0 iff it is boolean.*

Proof. In general $u_1(a) \vdash u_0(a)$ iff $u_0(1), u_1(a) \vdash u_0(a), u_1(0)$ iff a has a complement in D. Hence $u_0 = u_1$ iff D is boolean.

5 Points

Let D be a distributive lattice. As usual a *filter* of D is a subset F of D such that:

- $1 \in F$ and
- $xy \in F$ whenever $x, y \in F$ and
- $y \in F$ whenever $x \in F$ and $x \leq y$.

Each element $x \in D$ defines a filter $F_x \subseteq D$ by taking $y \in F_x$ to mean $x \vee y = 1$. Dually an *ideal* of D is a subset $I \subseteq D$ such that $0 \in D$ and $x \vee y \in D$ whenever $x, y \in D$ and $y \in D$ whenever $x \in D$ and $y \leq x$. Each element $x \in D$ defines an ideal $I_x = \{y \in D \mid y \leq x\}$.

Any distributive lattice D defines canonically a spectral space [Joh82], which, as a frame, is the frame of all ideals of D. Let $S, \vdash$ be an entailment relation and $D, i : S \to D$ the distributive lattice generated by $S, \vdash$. The following result gives a direct characterisation of points of the spectral space defined by D. We recall that these points can be defined as prime filters $\alpha \subseteq D$, that are filters α such that 0 is not in α and such that if $x \vee y \in \alpha$ then $x \in \alpha$ or $y \in \alpha$.

Proposition 16. *The points of the spectral space defined by D are completely determined by their restriction $\beta = i^{-1}(\alpha) \subseteq S$. These are exactly the subsets β of S such that if $X \vdash Y$ and $X \subseteq \beta$ then $Y \cap \beta$ is inhabited.*

Thus, if we see an entailment relation as a logical description of a spectral space, we can interpret this proposition as stating that points are *theories* compatible with the entailment relation.

6 Examples

We give three examples of entailment relations naturally associated to some mathematical structures. In each case we have a direct description of an inductively defined entailment relation.

6.1 Spectrum of a ring

Let A be a commutative ring. The relation $X \vdash Y$ is defined to mean that the product of the elements of Y belongs to the radical of the ideal generated by X.

Theorem 17. *$\vdash$ is an entailment relation on A. It is the least entailment relation on A such that:*

- $\vdash 0$,
- $1 \vdash$,
- $x \vdash xy$,
- $xy \vdash x, y$,
- $x, y \vdash x + y$.

A point for this entailment relation is exactly a prime ideal of A.

6.2 Real Spectrum of a ring

Let A be a commutative ring. A *cone* of A is a subset $C \subseteq A$ closed by addition, mutiplication and which contains all square elements $x^2,\ x \in A$. The following claims are directly checked: the smallest cone S of A is the set of sum of squares; if C is a cone and $a \in A$ the cone generated by C and a, that is the least cone containing C and a is the set $C + aC$ of elements $u + va,\ u, v \in C$. The relation $X \vdash Y$ is defined to mean that there exists a relation of the form $m + p = 0$ where m is in the monoid generated by X and p is in the positive cone generated by X and $\{-y \mid y \in Y\}$.

Theorem 18. *$\vdash$ is an entailment relation on A. It is the least entailment relation on A such that:*

- $\vdash 1$,
- $x, -x \vdash$,

- $x + y \vdash x, y$,
- $x, y \vdash xy$,
- $xy \vdash x, -x$,
- $xy \vdash x, -y$.

A point for this entailment relation defines a total ordering over A.

6.3 Space of Valuations

Let K be a field, that is a ring in which any element is 0 or is invertible, and let S be the set of its invertible elements. If x_i, y_j are in S, we define $x_i \vdash y_j$ to mean that there exist q_j polynomials in y_j^{-1} and x_i with integer coefficients such that $\Sigma y_j^{-1} q_j = 1$.

Theorem 19. *$\vdash$ is an entailment relation on S. It is the least entailment relation on S such that:*

- $\vdash x, x^{-1}$,
- $x \vdash -x$,
- $x, y \vdash xy$,
- $\vdash x, y$ *if* $xy = x + y$.

For a proof, see [CP98]. In [CP98] this description of the space of valuation is used to give a constructive version of a proof of a theorem of Kronecker which uses valuation rings. A point for this entailment relation defines a valuation ring of K. Finally, notice that $X \vdash y$ means that y is *integral* over the set X.

6.4 Total Ordering of a Vector Space

Let E be a vector space over the field Q of rational numbers. We define $x_i \vdash y_j$ as meaning that there exists $r_i \geq 0, s_j \geq 0$ such that $\Sigma r_i x_i = \Sigma s_j y_j$ and $\Sigma r_i = 1$. Another equivalent formulation is that $X \vdash Y$ mean that the convex hull of X meets the positive cone generated by Y.

Theorem 20. *$\vdash$ is an entailment relation on E. It is the least entailment relation on E such that:*

- $x, -x \vdash$,
- $x + y \vdash x, y$.

Notice that a consequence of these two entailment is $x + y, -y \vdash x$ and so $x, y \vdash x + y$. It follows that $x \vdash px$ and $px \vdash x$ for any natural number $p > 0$ and so $tx \vdash x$ if t is a rational > 0.

A point for this entailment relation defines a strict ordering $<$ on E such that $tx < ty$ implies $x < y$ for $t > 0$ and $x + z < y + t$ implies $x < y$ or $z < t$.

7 Normal Lattices and Compact Regular Spaces

7.1 Normal Lattices

For $x, y \in D$, we let $x \ll y$ mean that there exists m such that $xm = 0$ and $y \vee m = 1$. We say that a filter $F \subseteq D$ is *regular* iff whenever $x \in F$ there exists $x' \in F$ such that $x' \ll x$ [Mul90]. Dually we say that an ideal $I \subseteq D$ is *regular* iff $x \in I$ whenever $x' \in I$ for all $x' \ll x$. The following results are proved directly.

Lemma 21. *$F_0 = \{1\}$ and $F_1 = D$ are regular filters. Furthermore if F_x, F_y are regular then so are F_{xy} and $F_{x \vee y}$. Finally $x' \ll x$ and $y' \ll y$ imply both $x'y' \ll xy$ and $x' \vee y' \ll x \vee y$.*

We say that a distributive lattice D is *normal* iff all filters F_x are regular. An equivalent definition is given by the following result.

Lemma 22. *D is normal iff whenever $x \vee y = 1$ there exist $a, b \in D$ such that $a \vee x = 1$, $b \vee y = 1$ and $ab = 0$.*

We say that an entailment relation $\vdash$ on a set S is *normal* iff whenever $\vdash b, X$ there exist $b', m \in S$ such that $\vdash b', X$ and $\vdash b, m$ and $b', m \vdash$.

Proposition 23. *If $\vdash$ is normal then so is the distributive lattice $D, i : S \to D$ generated by $\vdash$.*

Proof. Using lemma 21 and the fact that any element of D is a disjunction of conjunctions of elements in $i(S)$, it is enough to show that each filter $F_{i(x)}$ is normal for $x \in S$. Let $a \in F_{i(x)}$. We can write $a = \wedge_j a_j$ with $a_j = \vee_k i(y_{jk})$ for some family (y_{jk}) in S. We have then $a_j \in F_{i(x)}$ for each j and hence, by theorem 3 $\vdash x, y_{jk}$ for each j. Using the normality of $\vdash$ we find then $a'_{jk} \ll i(y_{jk})$ such that $a'_j = \vee_k a'_{jk} \in F_{i(x)}$. Using the lemma 21 again, we have $a'_j \ll a_j$ for all j and hence $a' = \wedge a'_j \ll \wedge a_j = a$. Since $a' \in F_{i(x)}$, this shows that $F_{i(x)}$ is regular.

Corollary 24. *Let D_j be a family of lattices and $D, \sigma_j : D_j \to D$ its coproduct. If each D_j is normal then so is D.*

Corollary 25. *If D is a normal lattice then so is $V(D)$.*

Proof. Suppose $\vdash \Box a, X$ with $X = \Box a_1, \ldots, \Box a_m, \Diamond b_1, \ldots, \Diamond b_n$ then, by definition, $a_i \vee b_1 \vee \ldots \vee b_n = 1$, for some i or $a \vee b_1 \ldots \vee b_n = 1$. If we have $a_i \vee b_1 \vee \ldots \vee b_n = 1$, then we have $\vdash \Diamond 0, X$ and $\vdash \Box 1, \Box a$ and $\Box 1, \Diamond 0 \vdash$. If we have $a \vee b_1 \ldots \vee b_n = 1$ then since D is normal there exists $a' \ll a$ such that $a' \vee b_1 \vee \ldots \vee b_n = 1$, and hence $\vdash \Box a', X$. We then have $m \in D$ such that $a'm = 0$ and $a \vee m = 1$ and this implies $\vdash \Box a, \Diamond m$ and $\Box a', \Diamond m \vdash$.

We do a similar reasoning if $\vdash \Diamond a, X$.

7.2 Compact Hausdorff Spaces

Any distributive lattice defines a formal spectral space [Joh82] which can be defined as the frame of all ideals of this lattice.

Lemma 26. *For any distributive lattice D if $x \leq y$ and $y \ll y'$ then $x \ll y'$ and if $x \ll x'$ and $x' \leq y$ then $x \ll y$. If furthermore D is normal then the relation $\ll$ is* dense*: if $x \ll y$ then there exists z such that $x \ll z \ll y$. If D is normal and $x \ll y_1 \vee y_2$ then there exists $y_1' \ll y_1$ and $y_2' \ll y_2$ such that $x \leq y_1' \vee y_2'$.*

The importance of the notion of normal lattices comes from the following result.

Theorem 27. *If D is a normal lattice, the* regular *ideals of D defines a frame which is compact regular [Joh82]. Furthermore, any compact regular space can be presented in this way.*

Proof. If U is a subset of D define

$$j(U) = \{x \in D \mid \forall x' \ll x \; x' \in U\}.$$

From the lemma it follows that $j(U)$ is a regular ideal of D whenever U is an ideal of D. It follows that j defines a nucleus [Joh82] on the frame of ideals of D whose fixed points are exactly the regular ideals. Hence [Joh82] regular ideals form a frame. As a space, it is compact because $1 \ll 1$, and hence $1 \in \bigvee U_i$ iff 1 is in the ideal generated by $\cup_i U_i$.

Let U be a regular ideal. We have $U = \bigvee_{u \in U} j(I_u)$ and we show that $j(I_u) = \bigvee_{v \ll u} j(I_v)$ and that $j(I_v) \ll j(I_u)$ if $v \ll u$. This will show that regular ideals define a regular space. The first assertion follows directly from the definition of j. If $v \ll u$ there exists m such that $m \vee u = 1$ and $vm = 0$. We then have $j(I_m) \vee j(I_u) = 1$ and $j(I_v)j(I_m) = 0$ and hence $j(I_v) \ll j(I_u)$.

The corollaries 24 and 25 give an alternative way of defining the product of a family of compact regular spaces and the Vietoris space associated to a compact regular space respectively and of showing that in both cases we get a compact regular space [Joh82, Joh85]. For the product, this is a special case of Tychonoff's theorem [Joh82].

Theorem 28. *The compact regular space associated to the coproduct of a family of normal lattices is the product of the family of associated spaces.*

Theorem 29. *If D is a normal lattice, the space associated to the normal lattice $V(D)$ is the Vietoris powerlocale of the compact regular space associated to D.*

The proofs are omitted here.

8 Example: Linear functionals of norm ≤ 1

Let E is a seminormed space [MP91] and S be the vector space $Q \times E$. Let us write $p < x$ an element $(p, x) \in S$. Using lemma 2 and theorem 20 we consider the entailment relation over S generated by the axioms $\vdash -1 < x$ for $x \in N(1)$. A direct definition is that $p_i < x_i \vdash q_j < y_j$ holds iff there exists $r_i \geq 0, r \geq 0, s_j \geq 0$ and $z \in N(1)$ such that $r + \Sigma r_i = 1$ and $r(z, -1) + \Sigma r_i(x_i, p_i) = \Sigma s_j(y_j, q_j)$. Notice that we can suppose z to be in the vector space generated by the elements x_i and y_j.

Theorem 30. *$\vdash$ is an entailment relation on S. It is the least entailment relation such that, writing $x < r$ for $-r < -x$:*

- $x < r, r < x \vdash$,
- $r + s < x + y \vdash r < x, s < y$,
- $\vdash -1 < x$ *if* $x \in N(1)$.

Notice that $\vdash r < x, x < s$ is a consequence of these axioms for $r < s$.

Theorem 31. *The entailment relation $\vdash$ is* normal.

Proof. If $\vdash r < x, X$ it can be checked directly that there exists $r' > r$ such that $\vdash r' < x, X$. Furthermore we have then $x < r', r' < x \vdash$ and $\vdash x < r', r < x$.

The *points* of the associated compact Hausdorff space are exactly the linear functionals over E of norm ≤ 1.

Let $E_1 \subseteq E_2$ be two spaces. We have now two entailment relations $\vdash_1, \vdash_2$ on E_1, E_2 respectively. The following result, which is a direct consequence of the direct description of $\vdash_1$ and $\vdash_2$, can be seen as the localic version of the theorem of Hahn-Banach [MP91].

Theorem 32. *The entailment relation $\vdash_2$ is a conservative extension of $\vdash_1$: if $x_i, y_j \in E_1$ then $r_i < x_i \vdash_2 s_j < y_j$ iff $r_i < x_i \vdash_1 s_j < y_j$.*

Related work and Acknowledgement

A Gentzen style sequent calculus is studied in [JKM97]. There a category of coherent sequent calculi with *compatible consequence relations* as arrows are defined, this category is equivalent to the category of *strong proximity lattices* and *weak approximable relations.* The sequents do not necessarily satisfy reflexivity and this makes it possible to have different logical systems on the left and right of the turnstile.

Direct descriptions of powerlocales in terms of presentational schemes have been carried through in a number of contexts: for algebraic dcpos in [Plo83], continuous dcpos in [Vic93], completions of quasimetric spaces in [Vic97], and for strongly algebraic (SFP) domains in [Abr91].

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