Substance Advection by a Steady 2D Stream of the Viscous Fluid in a Lengthy Free-Surfaced Canal

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Abstract — Spreading of a low-concentrated admixture in the 2D (length – depth) stream of the viscous fluid in an open lengthy canal is considered; the admixture's dissipation and diffusion are taken into account. Apart from being long, the canal is assumed to be low-sloping, with a given shape of the bottom-line. A mathematical model under consideration is derived by the small parameter technique, starting from the 2D steady Navier–Stokes equations for the incompressible fluid and the unsteady diffusion equation for the moving medium.

The main feature to this model is taking account of the deep–cross structure of the stream and it lets us investigate peculiarities of the substance transfer. An interesting particular case is then a rise of the near–surface opposite flow which may be caused e.g. by the wind action.

The wide range of the main parameters to the problem does not allow to point the only one particular discretization scheme which would be superior. To our mind, in most cases some refined upwinding technique should be used to approximate the convective term. As to the time-stepping process, partially implicit (e.g., implicit with respect to the convective term) integration schemes occured to be most efficient because of an easy solvability of the corresponding equation (usually it is a tridiagonal linear system).

1. Introduction

Usually, in water ecology modelling one is restricted to considering of the passive admixture. It means that varying of the concentration is assumed to be not affecting "the hydrodynamics" of the model (i.e., the velocity and pressure fields). Such an approach is justified when the concentration is low or if physical properties of the admixture are similar to those of water (both the requirements are quite usual in water ecology modelling).

The so-called "camera" models are widely used currently [1]: the region is divided by relatively homogeneous subregions (camerae), and the characteristics are averaged and considered to be constant inside each camera. The interaction among the camerae is then described by the boundary balance relations. When the whole model is rather complicated, this simplification seems quite reasonable. However, in many cases the intracamera processes are of interest and this requires application of more detailed "distributed" models [1, 2].

In this paper, we introduce a "distributed" model for the canal (or river) ecosystem. The model under consideration is derived by the small parameter method. The starting point of the derivation is the 2D steady Navier–Stokes equations for the incompressible flow and the non-steady diffusion equation of a substance in the moving medium. As a result, for each unknown, it gives an initial boundary–value problem for the main term of the unknown's expansion with respect to a small parameter. The latter is the ratio of the stream depth to its length.

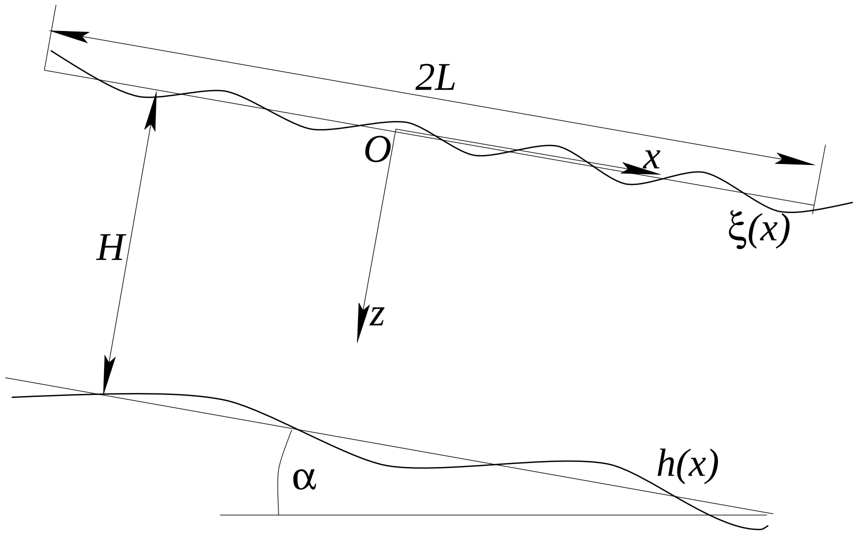

Figure 1: Disposition of the coordinate system. $\xi(x)$ is the unknown shape function of the stream free surface, $h(x)$ is the prescribed shape function of the bottom.

The particular feature to the model is taking account of the stream cross–structure which extends the capabilities of the model, e.g., this allows consider a particular case when a near–surface opposite flow caused by the wind action arises. Note that it would be impossible in the framework of the above–mentioned "camera" modelling (where one has to deal with the averaged variables like the water consumption, velocity in the cross-section etc.). Since the admixture is supposed to be passive, we manage to find the unknown velocity and pressure fields as well as the function of the free surface shape from an independent subsystem which turns out to be solvable analytically. The remaining equation for the concentration has to be solved numerically but some analytical simplifications can be applied here.

2. Derivation of the Model

Here we derive the model for the admixture spreading in an open steady long–lengthy stream.

As a starting point, the steady Navier–Stokes equations (for the stream) and the non–steady convection–diffusion equation (for the admixture) are taken. Then the small-parameter method is applied and it leads to an initial boundary–value problem for the main terms of the asymptotic expansions with respect to the ratio of the stream depth to its length (the small parameter).

Obtained in such a way equation (which describes the admixture spreading and has to be solved numerically) has the convection term only in the length direction and the diffusion term only in the other (depth) direction.

2.1. Setting of the problem

Consider the spreading process of the nonconservative admixture in 2D steady open stream with the slope angle α (fig. 1). Let the Cartesian coordinate system Oxz be placed in such a way that the x–axis is directed downstream along the surface and the z–axis is directed to the bottom. We are interested in what is happening in the region $-L \leqslant x \leqslant L$ (fig. 1).

Initial physical model describing the transfer of the substance (admixture) in the steady stream of fluid and taking into account the diffusion and dissipation includes [3] the Navier–

Stokes equations for the incompressible flow:

$$
\begin{aligned}
&uu_x + wu_z = -\rho^{-1}p_x + \nu_1 u_{xx} + \nu_2 u_{zz} + g\sin\alpha\,,\\
&uw_x + ww_z = -\rho^{-1}p_z + \nu_1 w_{xx} + \nu_2 w_{zz} + g\cos\alpha\,,\\
&u_x + w_z = 0\,,
\end{aligned}
\qquad (1)
$$

the no-slip condition on the rigid bottom:

$$
u|_{z=h(x)} = 0\,, \qquad w|_{z=h(x)} = 0\,, \qquad (2)
$$

the kinematic and dynamic conditions on the free surface:

$$
w - \xi'(x)u|_{z=\xi(x)} = 0\,, \quad \mu\left(w_x - u_z - \frac{\xi''(x)u}{1+\xi'^2(x)}\right)\bigg|_{z=\xi(x)} = \beta\,, \quad p|_{z=\xi(x)} = 0\,, \qquad (3)
$$

the convection–diffusion equation:

$$
c_t + uc_x + wc_z = D_1 c_{xx} + D_2 c_{zz} - \gamma c\,, \qquad (4)
$$

the flux absence conditions on both lower and upper boundaries:

$$
c_z - h'(x)c_x|_{z=h(x)} = 0\,, \qquad c_z - \xi'(x)c_x|_{z=\xi(x)} = 0\,, \qquad (5)
$$

and the initial distribution of the substance:

$$
c(x,z,0) = c_0(x,z)\,. \qquad (6)
$$

Here, the unknown functions are $u = u(x,z)$, $w = w(x,z)$, $p = p(x,z)$, and $c = c(x,z,t)$, which respectively are the longitudinal and the transversal (x– and z–) components of the fluid velocity vector, the pressure and the concentration of the admixture. Furthermore, ρ is the constant density of the fluid, ν_1 and ν_2 are coefficients of the turbulent viscosity respectively for x and z directions, and α is the slope angle of the stream (fig. 1). Unknown function $\xi(x)$ describes the free surface shape by the equation $z = \xi(x)$ (fig. 1) whereas the bottom shape is defined by the function $h(x)$ in the similar way. This function $h(x)$ is assumed to be known (possibly up to an arbitrary constant — in this case the fluid consumption has to be prescribed and it will define the constant). Parameter β defines the tangential tension on the free surface which may be caused e.g. by the wind action. Next, μ is the dynamical viscosity coefficient, D_1 and D_2 are the turbulent diffusion coefficients, respectively for x and z directions, and γ is the coefficient of the substance dissipation (destruction). Finally, the known function $c_0(x,z)$ gives initial (e.g. background) distribution of the substance.

The model under consideration has the peculiarity that the system (1)–(6) can be splitted in two. First, from (1)–(3), the unknown velocity field $\vec{v} = (u,w)$ and pressure p together with the free surface shape $\xi(x)$ are defined, and then the concentration distribution c may be found from (4)–(6) (note that then the velocity and the shape function of the free surface participating respectively in (4) and (5) may be supposed to be already known). In the assumption that the fluid density and the diffusion coefficients do not depend on the concentration, the outlined splitting is valid.

2.2. Equations in dimensionless variables

There are two spatial characteristic scales in the model, they are the length L, for the x-direction, and the depth H, for the direction z. The depth may be defined e.g. by $H = h(0)$. Then the small parameter $\varepsilon = HL^{-1}$ arises in a natural way (similarly to the shallow water equations [3]); we will call it the shallowness parameter.

We take $U = \sqrt{0.5gH\sin\alpha}$ and $W = \varepsilon U$ for the scales of x- and z- components of velocity, respectively. It is natural to take the value of hydrostatic pressure on the typical depth $P = \rho g H$ as the scale for the pressure. Then, let $T = LU^{-1}$, time for a fluid particle to pass the region $-L \leqslant x \leqslant L$, be the scale to measure time. Finally, we choose $C = \max_{x,z} c_0(x,z)$ to be the scale for the concentration.

The dimensionless variables (with ˜ sign) are introduced by $x = L\tilde{x}$, $z = H\tilde{z}$, $t = T\tilde{t}$, $u = U\tilde{u}$, $w = W\tilde{w}$, $p = P\tilde{p}$, $c = C\tilde{c}$, $c_0 = C\tilde{c}_0$, $\xi = H\tilde{\xi}$, $h = H\tilde{h}$.

With the new variables substituted, the equations (1)–(6) take the form (we now omit tilde "˜" everywhere):

$$\varepsilon(uu_x + wu_z) = -\frac{2\varepsilon p_x}{\sin\alpha} + \frac{1}{Re}\left(\nu\varepsilon^2 u_{xx} + u_{zz}\right) + 2\,, \tag{7}$$

$$\varepsilon^2(uw_x + ww_z) = -\frac{2p_z}{\sin\alpha} + \frac{1}{Re}\left(\nu\varepsilon^3 w_{xx} + \varepsilon w_{zz}\right) + 2\mathrm{ctg}\alpha\,, \tag{8}$$

$$u_x + w_z = 0\,, \tag{9}$$

$$u|_{z=h(x)} = 0\,, \quad w|_{z=h(x)} = 0\,, \quad w - \xi'(x)u|_{z=\xi(x)} = 0\,, \tag{10}$$

$$\varepsilon^2 w_x - u_z - \left.\frac{\varepsilon^2\xi''(x)u}{1+\varepsilon^2\xi'^2(x)}\right|_{z=\xi(x)} = \sigma\,, \qquad p|_{z=\xi(x)} = 0\,, \tag{11}$$

$$c_t + uc_x + wc_z = \frac{1}{Pe}\left(D\varepsilon^2 c_{xx} + c_{zz}\right) - \lambda c\,, \tag{12}$$

$$c_z + \varepsilon^2 h'(x)c_x|_{z=h(x)} = 0\,, \quad c_z + \varepsilon^2\xi'(x)c_x|_{z=\xi(x)} = 0\,, \tag{13}$$

$$c(x,z,0) = c_0(x,z)\,. \tag{14}$$

Apart from the defined parameter $\varepsilon = HL^{-1}$, the other dimensionless parameters are: the Reynolds number $Re = HU\nu_z^{-1}$, the anisotropy parameter of the turbulent viscosity $\nu = \nu_1\nu_2^{-1}$ the wind tension parameter $\sigma = \beta H(\mu U)^{-1}$, the diffusion Peclet number $Pe = WHD_2^{-1}$, the anisotropy parameter of the turbulent diffusion $D = D_1 D_2^{-1}$, and the dissipation parameter $\lambda = \gamma T$.

2.3. Equations for the main terms and solution of the hydrodynamics subsystem

We are looking for the solutions of the system (7)–(14) in the form of the asymptotical ε–power expansion (recall that ε is the shallowness parameter):

$$\begin{aligned} u &= u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots, \\ w &= w^{(0)} + \varepsilon w^{(1)} + \varepsilon^2 w^{(2)} + \dots, \\ \xi &= \xi^{(0)} + \varepsilon \xi^{(1)} + \varepsilon^2 \xi^{(2)} + \dots, \\ p &= p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \dots, \\ c &= c^{(0)} + \varepsilon c^{(1)} + \varepsilon^2 c^{(2)} + \dots . \end{aligned} \tag{15}$$

Substitution of these series in equations (7)–(14) and collecting the similar terms lead to the following problem for the main terms of the asymptotics (15):

$$u_{zz} = -2Re\,, \qquad p_z = \cos\alpha\,, \qquad u_x + w_z = 0\,, \tag{16}$$

$$u\big|_{z=h(x)} = 0\,, \qquad w\big|_{z=h(x)} = 0\,, \tag{17}$$

$$w - \xi'(x)u\big|_{z=\xi(x)} = 0\,, \qquad u_z\big|_{z=\xi(x)} = -\sigma\,, \qquad p\big|_{z=\xi(x)} = 0\,, \tag{18}$$

$$c_t + uc_x + wc_z = \frac{1}{Pe}c_{zz} - \lambda c\,, \tag{19}$$

$$c_z\big|_{z=\xi(x)} = 0\,, \qquad c_z\big|_{z=h(x)} = 0\,, \tag{20}$$

$$c(x,z,0) = c_0(x,z)\,. \tag{21}$$

The main terms for the longitudinal and transversal velocities, as well as for the pressure and the shape function of the free surface, may be found by the straightforward integration of equations (16)–(18). Indeed, solution of the first equation in (16) accompanied by the boundary conditions (17) and (18) gives

$$u = (h(x) - z)\Big[Re\,(h(x) + z - 2\xi(x)) + \sigma\Big]\,. \tag{22}$$

As we see, the longitudinal velocity has the parabolic shape with respect to z. Depending on the direction and the strength of the wind (both values are governed by the parameter σ), the opposite flow may arise near the surface. Such a flow takes place provided that

$$\sigma < -Re\,(h(x) - \xi(x))\ .$$

Now, using (22) and boundary conditions (17), we integrate the last equation in (16) with respect to z. It results in the expression for the transversal velocity

$$w = (h(x) - z)\Big[h'(x)\Big(2Re\,(h(x) - \xi(x)) + \sigma\Big) - Re\xi'(x)\,(h(x) - z)\Big]\,. \tag{23}$$

Then, the first equation in (18) together with (22) and (23) leads to the following relation for the main term of the free surface function $\xi(x)$:

$$(h(x) - \xi(x))(h'(x) - \xi'(x))\Big[2Re\,(h(x) - \xi(x)) + \sigma\Big] = 0\,. \tag{24}$$

Evidently, due to the setting of the problem, $h > \xi$. Hence, at least for positive σ, the expression in square brackets does not vanish, therefore

$$h'(x) - \xi'(x) \equiv 0\,.$$

The last means that in the 2D setting the main term $\xi(x)$ for the shape function of the free surface coincides with the shape function of the bottom $h(x)$ up to an additive constant. Due to the choice of the transversal scale, this constant can be taken equal to 1:

$$\xi(x) = h(x) - 1\,. \tag{25}$$

Next, substitution of the last relation in (22) and (23) gives

$$u = (h(x) - z)\left[\mathit{Re}\,(2 - h(x) + z) + \sigma\right], \tag{26}$$

$$w = (h(x) - z)\,h'(x)\left[\mathit{Re}\,(2 - h(x) + z) + \sigma\right]. \tag{27}$$

As it can be seen from (27), if the bottom is flat (i.e. $h(x) \equiv \mathrm{const}$) then the transversal component of the velocity w is zero. Thus, for this approximation (when we are interested only in the main terms of the asymptotics), the transversal motion of the fluid are caused only by irregularities of the bottom.

Finally, with (25), the second equation in (16) accompanied by the last relation in (18) has a solution

$$p = (z + 1 - h(x))\cos\alpha\,. \tag{28}$$

By (25)–(28), the main asymptotical terms of the hydrodynamics subsystem (7)–(11) are defined. Now, the diffusion subsystem (19)–(21) remains to be treated.

2.4. The diffusion subsystem

Making the change in (19)–(21):

$$z = y + h(x) - 1\,,$$

with (25)–(27), we arrive at

$$c_t + u(y)c_x = \frac{1}{\mathit{Pe}}c_{yy} - \lambda c\,, \tag{29}$$

$$u(y) = (1 - y)\left[\mathit{Re}(1 + y) + \sigma\right], \tag{30}$$

$$c_y\big|_{y=0} = 0\,, \qquad c_y\big|_{y=1} = 0\,, \tag{31}$$

$$c(x, y, 0) = c_0(x, y)\,, \tag{32}$$

where $-1 < x < 1$ and $0 < y < 1$. The initial boundary–value problem above is not solvable analytically. Numerical technique for solution of (29) is briefly discussed in the next section.

3. Numerical Solution of the Diffusion Subsystem

3.1. Spatial discretization

Applying the finite difference method to solve (29)–(32), we choose the grid

$$\begin{aligned}\omega_h = \{(x_i, y_k) \mid x_i = ih_1,\ i = 1, \ldots, n_1 - 1;\\ y_k = (k - 1/2)h_2,\ k = 1, \ldots, n_2 - 1\}\,.\end{aligned} \tag{33}$$

The diffusion term $Pe^{-1}c_{yy}$ is approximated with the second order accuracy by the central differences:

$$\left[Pe^{-1}c_{yy}\right]_{i,k} \cong \begin{cases} Pe^{-1}\left[-c_{i,k} + c_{i,k+1}\right]h_2^{-2}, & k = 1\,,\\ Pe^{-1}\left[c_{i,k-1} - 2c_{i,k} + c_{i,k+1}\right]h_2^{-2}, & 2 \leqslant k \leqslant n_2 - 1\,,\\ Pe^{-1}\left[c_{i,k-1} - c_{i,k}\right]h_2^{-2}, & k = n_2 - 1\,,\end{cases} \tag{34}$$

where (second order) approximations of boundary conditions (31) are incorporated.

We avoid the use of the central differences to approximate the convective term (for more discussion see e.g. [4]). Instead, the upwinding of the first or the second order is applied.

The regular upwind differences give the first order approximation

$$\begin{aligned}&[u(y)c_x]_{i,k} \cong 0.5\{(u_k + |u_k|)\left[c_{i,k} - c_{i-1,k}\right] + (u_k - |u_k|)\left[c_{i+1,k} - c_{i,k}\right]\}h_1^{-1}\,,\\ &\quad i = 1, \ldots, n_1 - 1\,,\end{aligned} \tag{35}$$

which severes from the massive artificial diffusion destroying the accuracy [5]. However, for this particular problem, the artificial diffusion is added only streamline.

The second order approximation of the convective term may be obtained with [6]

$$[u(y)c_x]_{i,k} \cong \begin{cases} 0.5(u_{i+1} - u_{i-1})h_1^{-1} - C_F(u_{i+1} - 3u_i + 3u_{i-1} - u_{i-2})h_1^{-1}, & u_k \geqslant 0\,,\\ 0.5(u_{i+1} - u_{i-1})h_1^{-1} + C_F(u_{i-1} - 3u_i + 3u_{i+1} - u_{i+2})h_1^{-1}, & u_k < 0\,.\end{cases} \tag{35$'$}$$

The choice $C_F = 0.5$ leads to the second order upwinding, whereas for $C_F = 1/8$ one gets the so-called QUICK-scheme [6].

The problem (29)–(32) does not have any boundary conditions for x (at the ends of the canal region). The use of the upwind differences allows to get around this formal difficulty. Indeed, if we assume that there are no sources of the admixture outside the region $0 < x < 1$ then, on the in-flow ends of the region, the concentration can be set to zero:

$$c_{0,k} = c_{-1,k} = 0 \quad \text{for } v_k \geqslant 0\,, \qquad c_{n_1,k} = c_{n_1+1,k} = 0 \quad \text{for } v_k < 0\,. \tag{36}$$

Now it is easy to see that the upwinding schemes (35) and (35′) (with $C_F = 0.5$) do not contain any other values $c_{i,k}$ outside the grid (33).

3.2. The time–stepping process

Equation (29) may be rewritten as

$$c_t = -\mathcal{L}[c]\,, \tag{29$'$}$$

$$\text{and} \qquad \mathcal{L}[c] \equiv \left(-Pe^{-1}c_{yy} + \lambda c\right) + u(y)c_x \cong A_{\text{diff}}\boldsymbol{c} + A_{\text{conv}}\boldsymbol{c}\,.$$

where $\mathcal{L}$ is the spatial convection–diffusion operator, $\boldsymbol{c} = \{c_{i,k}(t)\}$, A_{diff} and A_{conv} are the discrete diffusion and convection operators respectively defined by (34) and (35) (or (35′)).

If the first order time accuracy is sufficient for the time–stepping then, to reduce the computational work, it is better to use partly implicit schemes. For example, for the case of dominating convection the time–stepping process

$$\boldsymbol{c}^{m+1} = [I + \tau A_{\text{conv}}]^{-1} [I - \tau A_{\text{diff}}] \boldsymbol{c}^m, \quad m = 0, 1, \dots, \tag{37}$$

can be appropriate (here $\boldsymbol{c}^m = \{c_{i,k}(t_m)\}$). With (34), (35), the scheme (37) gives the restriction for the step size [7]

$$\tau \leqslant \tau'' = h_2^2 Pe \big/ \big(2 + \lambda h_2^2 Pe\big) \ .$$

The linear equation with matrix $I + \tau A_{\text{conv}}$ can be solved very efficiently: for the natural "i-first" ordering of the grid nodes A_{conv} has its (at most four) non-main diagonals next to the main diagonal. Similarly, one can use the time–stepping process with implicitly taken diffusion term.

If the second order accurate time–stepping is required, we suggest use e.g. the BDF2 scheme [8] (in this case the iterative solution of the arising linear system allows many possibilities to decrease the computational expences, see e.g. [9] and citation therein). If the large step size is not an issue then one can use explicit stabilized RK methods [10].

3.3. Analytical integrability

There are two specific cases where the problem (29)–(32) can be simplified and easily solved analytically. The analytical solution then may be used for testing the discretization schemes. These cases are (i) absence of the diffusion term and (ii) the constant velocity $u(y) \equiv \text{const}$.

4. Numerical Experiments

Assume that the accident ejection of a pollution occured at the specific point (x_*, y_*) of the canal. Then we may model the spreading of the pollution by solving the problem (29)–(32) with the initial distribution of the form

$$c_0(x, y) = C_0 e^{-a(x-x_*)^2 - b(y-y_*)^2} , \tag{38}$$

where a,b and C_0 are parameters depending e.g. on the mass of the ejected pollution.

We present results of the numerical experiments for such a model situation for the following choice of parameters in (29)–(32):

$$Re = 100\,, \quad \sigma = -130\,, \quad \lambda = 0\,, \quad Pe = 10^2\,, \quad \text{and} \tag{39}$$

$$Re = 100\,, \quad \sigma = -130\,, \quad \lambda = 0\,, \quad Pe = 10^4\,. \tag{40}$$

Both sets of parameter values here correspond to the dominating convection, besides, for (40) there exists an intensive transverse intermixing in the stream. For both cases there is the near–surface opposite flow ($u(y)$ is negative near the surface). In (38), we set $a = b = 64$, $C_0 = 1$.

We used the spatial grid (33 with $n_1 = 300$ and $n_2 = 51$ and the second order scheme (34),(35′) with $C_F = 1/8$ in all grid nodes except in the nodes with indices $i = 1$ and $i = n_1 - 1$ where C_F was set to 0.5. For the time–stepping process, the convectionally implicit scheme (37) was employed, with $\tau = 0.02$.

The pollution distribution for several moments of time is shown on fig. 2.

5. Conclusions

In this paper, we introduce a distributed model for the advection process of a substance in the lengthy canal (or river) 2D stream. The pleasant feature to the model seems to be its simplicity; the hydrodynamical part of the model turns out to be integrable analytically, so that the problem is reduced to the non-steady convection–diffusion problem with the only longitudinal convection and the only transversal diffusion presented. The model can be used as a part of the more comprehensive ecosystem model.

References

1. *Estuarine Water Quality Management: Monitoring, Modelling and Research.* W. Michaelis Ed. Springer Series in Coastal and Estuarine Studies, 36. Springer–Verlag, Berlin, 1990.
2. *Terrain Analysis and Distributed Modelling in Hydrology.* Part 3. K.J. Beven, I.D. Moore Eds. John Wiley, Chichester, 1993.
3. J.J. Stoker. *Water Waves: The Mathematical Theory with Applications*. Interscience, NY, 1966.
4. B.P. Leonard. A Stable and Accurate Convective Modelling Procedure Based on Quadratic Upstream Interpolation. *Computer Methods in Applied Mechanics and Engineering*, 19:59–98, 1979.
5. C.A.J. Fletcher. *Computational Techniques for Fluid Dynamics.* Springer, Berlin, 1988.
6. B.P. Leonard and J.E. Drummond. Why You Should Not Use ‘Hybrid’, ‘Power Law’, and Related Exponential Schemes — There Are Much Better Alternatives. *International Journal for Numerical Methods in Fluids,* 20:421–442, 1995.
7. M.A. Botchev. On Ad-hoc Upwind Implicit Difference Schemes for Parabolic P.D.E. of the Convection–Diffusion Type. *Z. Angew. Math. Mech.,* 76 (Suppl. 1):361–362, 1996.
8. E. Hairer, S.P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer Series in Computational Mathematics 8, Springer–Verlag, Berlin, 1987.
9. T.F. Chan and K.R. Jackson. The Use of Iterative Linear–Equation Solvers in Codes for Large Systems of Stiff IVPs for ODEs. *SIAM J. Sci. Stat. Comput.*, 7(2):378–417, 1986.
10. J.G. Verwer. Explicit Runge–Kutta Methods Parabolic Partial Differential Equations. *Applied Numerical Mathematics,* 22:359–379, 1996.

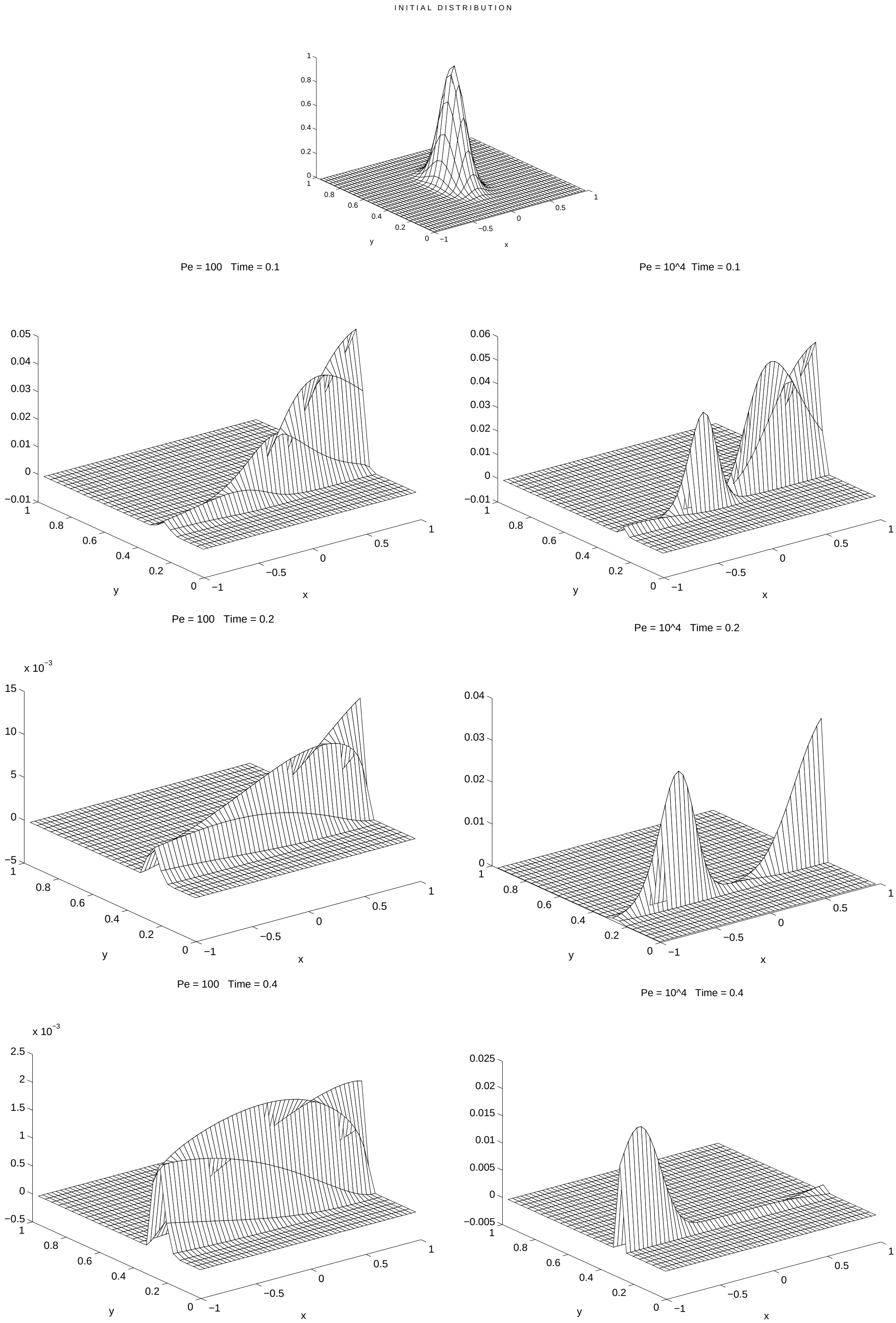

Figure 2: Pollution at the moments $t = 0$ (top), and $t = 0.1$, 0.2, and 0.4 for parameter values (39) (left) and (40) (right)