Analysis of tandem fluid queues

Małgorzata M. O'Reilly*
School of Mathematics
University of Tasmania
Tas 7001, Australia
malgorzata.oreilly@utas.edu.au

Werner Scheinhardt
Department of Applied Mathematics
University of Twente
The Netherlands
w.r.w.scheinhardt@utwente.nl

ABSTRACT

We consider a model consisting of two fluid queues driven by the same background continuous-time Markov chain, such that the rates of change of the fluid in the second queue depend on whether the first queue is empty or not. We analyse this tandem model using operator-analytic methods.

Keywords: tandem, stochastic fluid model, Markov chain, Laplace-Stieltjes transform, transient analysis, limiting distribution.

1. INTRODUCTION

Stationary distributions of Markov-modulated fluid queues have been studied extensively, first using spectral methods [3], later via more efficient matrix-analytic methods [8, 9, 10, 11, 13, 17]. The analysis of networks of fluid queues is much harder, and only for a few special two-node cases the stationary joint distribution of both queue contents and the regulating Markov chain could be obtained.

However, a promising approach to find further results is the use of operator-analytic methods, studied in Bean and O'Reilly [4, 5], where a tandem model is considered, and also in Margolius and O'Reilly [16], where a time-varying queue is analysed. The operator-analytic methods generalise the matrix-analytic methods for single queues. In this work we show that this can indeed lead to good results.

The main difference with the tandem model in [4] is that here we consider fluid queues that have a lower bound, i.e., they can become empty but the content cannot become negative. The tandem model in [5] also considers queues with a lower bound, but the assumptions are slightly different and the results derived there are largely theoretical. Here, we derive numerical methods for a generalization of the tandem model in [14], for which the analytical results could be obtained by considering an embedded $M/G/1$ queue.

2. TANDEM FLUID QUEUE: MODEL AND PRELIMINARIES

In this section we first describe the model of interest and then give the stability condition. We end with some preliminary statements about the sample paths that can be taken by the model, and the implications for the shape (in particular the support) of the stationary distribution.

2.1 Model description

We consider two fluid queues, collecting fluid in buffers X and Y, with level variables recording the content at time t denoted by $X(t)$ and $Y(t)$, respectively, that are being driven by the same background continuous-time Markov chain $\{\varphi(t) : t \geq 0\}$ with some finite state space S and irreducible generator $\mathbf{T}$. The first queue behaves as a standard fluid queue $\{(\varphi(t), X(t)) : t \geq 0\}$ studied in [10], with a lower boundary at level 0, and real-valued fluid rates r_i collected in a diagonal matrix $\mathbf{R} = diag(r_i)_{i\in\mathcal{S}}$. Thus, the content $X(t)$ increases at rate r_i when $\varphi(t) = i$, unless r_i is negative and $X(t)$=0. More precisely,

$$\frac{d}{dt}X(t) = r_{\varphi(t)} \qquad \text{when } X(t) > 0,$$
$$\frac{d}{dt}X(t) = \max(0, r_{\varphi(t)}) \qquad \text{when } X(t) = 0.$$

We partition the state space $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_\bigcirc$, where $r_i > 0$ when $i \in \mathcal{S}_+$ (states in $\mathcal{S}_+$ will be called upstates), $r_i < 0$ when $i \in \mathcal{S}_-$ (states in $\mathcal{S}_-$ will be called downstates), and $r_i = 0$ when $i \in \mathcal{S}_\bigcirc$ (states in $\mathcal{S}_\bigcirc$ will be called zero-states). With the behaviour at $X(t) = 0$ in mind it will sometimes be helpful to use additional notation $\mathcal{S}_\ominus = \mathcal{S}_- \cup \mathcal{S}_\bigcirc$ for the set of "zero-states at $X(t) = 0$". After appropriately ordering the states in $\mathcal{S}$ we can write $\mathbf{T}$ as 3×3 block matrix,

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{++} & \mathbf{T}_{+-} & \mathbf{T}_{+\bigcirc} \\ \mathbf{T}_{-+} & \mathbf{T}_{--} & \mathbf{T}_{-\bigcirc} \\ \mathbf{T}_{\bigcirc+} & \mathbf{T}_{\bigcirc-} & \mathbf{T}_{\bigcirc\bigcirc} \end{bmatrix}. \qquad (1)$$

Further, we assume that the behaviour of the second fluid queue depends on both $\varphi(t)$ and $X(t)$ in the following way. Assuming fluid rates $\widehat{c}_i > 0$ and $\breve{c}_i < 0$ for all $i \in \mathcal{S}$, collected in $\widehat{\mathbf{C}} = diag(\widehat{c}_i)_{i\in\mathcal{S}}$ and $\breve{\mathbf{C}} = diag(\breve{c}_i)_{i\in\mathcal{S}}$, we have

$$\frac{d}{dt}Y(t) = \widehat{c}_{\varphi(t)} > 0 \qquad \text{when } X(t) > 0,$$
$$\frac{d}{dt}Y(t) = \breve{c}_{\varphi(t)} < 0 \qquad \text{when } X(t) = 0, Y(t) > 0,$$
$$\frac{d}{dt}Y(t) = \widehat{c}_{\varphi(t)} \cdot 1\{\varphi(t) \in \mathcal{S}_+\} \qquad \text{when } X(t) = 0, Y(t) = 0.$$

Thus, the fluid level $Y(t)$ increases when $X(t) > 0$, and decreases when $X(t) = 0$, unless both levels are at 0; in the latter case $Y(t)$ (and $X(t)$) increases as soon as $\varphi(t)$ makes a transition from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$.

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Throughout we denote by $\mathbf{1}$, $\mathbf{0}$, $\mathbf{I}$ and $\mathbf{O}$ a column vector of ones, a row vector of zeros, an identity matrix, and a zero matrix of appropriate sizes, respectively. Also, for any matrix $\mathbf{A} = [A_{ij}]$, we use notation $|\mathbf{A}|$ for a matrix collecting absolute values of the elements of $\mathbf{A}$, with $|\mathbf{A}| = [\ |A_{ij}|\ ]$.

2.2 Stability condition

The stability condition for the first queue, $\{(\varphi(t), X(t)) : t \geq 0\}$, is well-known to be

$$\sum_{i \in \mathcal{S}} r_i P(\varphi = i) < 0, \tag{2}$$

where the random variable φ is distributed according to the stationary distribution of $\varphi(t)$. Assuming this condition is satisfied, the second queue (buffer Y) will be stable when the expected increase rate of $Y(t)$ is less than the expected decrease rate, i.e.,

$$\sum_{i \in \mathcal{S}} \widehat{c}_i P(\varphi = i, X > 0) < \sum_{i \in \mathcal{S}_\ominus} |\breve{c}_i| P(\varphi = i, X = 0), \tag{3}$$

where the random vector (φ, X) is distributed according to the stationary distribution of $(\varphi(t), X(t))$.

2.3 Qualitative behaviour

In this subsection we give a short discussion of how the process $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$ behaves and what the stationary distribution looks like. Here, and in the sequel, we will sometimes write e.g. 'the process hits $x = 0$', which will be short for 'the process $(\varphi(t), X(t), Y(t))$ hits the set $\mathcal{S} \times \{0\} \times [0, \infty)$', or we will speak of 'the probability mass at $x = 0, y > 0$' meaning 'the stationary probability that the process $(\varphi(t), X(t), Y(t))$ is in the set $\mathcal{S} \times \{0\} \times (0, \infty)$'.

Typically the process alternates, between:

(i) periods on $x = 0$, with $Y(t)$ decreasing, possibly being halted at $x = 0, y = 0$, and $\varphi(t)$ in $\mathcal{S}_\ominus$; such a period starts at $x = 0, y > 0$, with $\varphi(t)$ in $\mathcal{S}_-$ and ends at $x = 0, y > 0$ or at $x = 0, y = 0$ as soon as $\varphi(t)$ makes a transition from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$;

(ii) periods on $x > 0$, with $Y(t)$ increasing, while $X(t)$ can either increase and decrease. Such a period starts where the previous type (i) period ended with $\varphi(t) \in \mathcal{S}_+$ and $X(t)$ increasing, and ends at $x = 0, y > 0$ with $\varphi(t)$ in $\mathcal{S}_-$ as soon as $X(t)$ decreases to 0.

Note that in stationarity, the process can not be at $y = 0$, $x > 0$, since $Y(t) = 0$ implies $X(t) = 0$ (or alternatively, $X(t) > 0$ implies $Y(t) > 0$). In fact when a type (ii) period starts from $x = 0, y = 0$, due to a transition of $\varphi(t)$ to some phase $i \in \mathcal{S}_+$, the process will move with $\frac{d}{dt}X(t) = r_i > 0$ and $\frac{d}{dt}Y(t) = \widehat{c}_i > 0$, so it will stay on the line $\{(x, y) : y = x\widehat{c}_i/r_i\}$ until some future transition of $\varphi(t)$ to some other state i'. Note that the slope of any such path leaving the origin is at least $\min_{i \in \mathcal{S}_+}\{\widehat{c}_i/r_i\}$, and also after the path has been left, the slope of the ensuing path can never be less than this value (assuming $i' \in \mathcal{S}_+$, otherwise $X(t)$ will not increase). Thus, after the process has hit the origin for the first time (which it will, due to stability), the set $\{(x, y) : y < x \cdot \min_{i \in \mathcal{S}_+}\{\widehat{c}_i/r_i\}\}$ can never be reached.

As a consequence of the above, the stationary distribution will have the following form.

- Corresponding to (i), there will be a (one-dimensional) density at $x = 0, y > 0$, denoted by $\boldsymbol{\pi}(0, y)$, and a probability point mass at $(0, 0)$, denoted by $\mathbf{p}(0, 0)$.

- Corresponding to (ii), there will be a two-dimensional density on $\{(x, y) : x > 0, y > x \cdot \min_{i \in \mathcal{S}_+}\{\widehat{c}_i/r_i\}\}$, denoted as $\boldsymbol{\pi}(x, y)$, and there will be one-dimensional densities on each of the lines $y = x\widehat{c}_i/r_i$, $i \in \mathcal{S}_+$, denoted as $\pi^i(x, x\widehat{c}_i/r_i)$. Also, define $\boldsymbol{\pi}^j(x, x\widehat{c}_j/r_j) = [\delta_{ij}\pi^j(x, x\widehat{c}_j/r_j)]_{i \in \mathcal{S}}$ for all $j \in \mathcal{S}$. There will be no other probability masses or densities, in particular there is no density at $y = 0, x > 0$.

It is important to realize that the one- and two-dimensional densities just mentioned, as well as the point mass at (0,0), are all *vectors* with $|\mathcal{S}|$ components, where the i-th component corresponds to $\varphi(t) = i$. Some of these components will be zero; in particular for $i \in \mathcal{S}_+$ we will have $[\mathbf{p}(0, 0)]_i = 0$ and $[\boldsymbol{\pi}(0, y)]_i = 0$. Also $[\boldsymbol{\pi}^j(x, x\widehat{c}_j/r_j)]_i = 0$ for all $i \neq j$.

In the next section we show how to proceed to find the stationary distribution.

3. TANDEM FLUID QUEUE: ANALYSIS

Roughly speaking, our analysis is based on the alternation between (i) stages during which $X(t) = 0$ and hence $Y(t)$ decreases, and (ii) stages during which $X(t) > 0$ and hence $Y(t)$ increases, as detailed in Section 2.3. For (parts of) both of these stages we will apply ideas from [4, 18], in order to keep track of the amount by which $Y(t)$ increases (or decreases), in much the same way as we can keep track of the amount of time that passes. We will review this in Section 3.1. In Section 3.2 we will look at the state $(\varphi(t), X(t))$ when the process hits the line $x = 0$, so that with these building blocks we can in Section 3.3 establish expressions for the stationary distribution.

3.1 Replacing time by shift

We are interested in certain behaviour of buffer X, not during some amount of *time*, but while buffer Y experiences a certain (downward/upward, virtual) *shift*. For a motivation of the expressions below we refer to [4], where the concept of shift was introduced, as well as to [18], where a generalization of this idea is discussed. We will consider two cases.

(i) The behaviour at $x = 0$, when the level in buffer Y is strictly decreasing, according to the rates in $\breve{\mathbf{C}}$;

(ii) The behaviour at $x > 0$, when the level in buffer Y is strictly increasing, according to the rates in $\widehat{\mathbf{C}}$.

First, consider the behaviour at $x = 0$, when the level in buffer Y is strictly decreasing, according to the rates in $\breve{\mathbf{C}}$. Below we define matrices $\breve{\mathbf{Q}}_{\ominus\ominus}$ and $\mathbf{Q}_{\ominus+}$ which are the key components of the analysis for this case.

Suppose $X(0) = 0$ and $\varphi(u) \in \mathcal{S}_\ominus$ for $0 \leq u \leq t$. Define the random variable $D(t)$,

$$D(t) = \int_{u=0}^{t} |\breve{c}_{\varphi(u)}| du, \tag{4}$$

interpreted as the total *downward shift* $Y(0) - Y(t)$ in buffer Y at time t when $Y(t) > 0$. Also, for any $z > 0$ define

$$t_z = \inf\{t > 0 : D(t) = z\}, \tag{5}$$

which we interpret, for any $y \geq 0$, as the first time at which the level in the buffer Y shifts from level $Y(0) = y+z$ to y.

Denote

$$\mathbf{T}_{\ominus\ominus} = \begin{bmatrix} \mathbf{T}_{--} & \mathbf{T}_{-\bigcirc} \\ \mathbf{T}_{\bigcirc-} & \mathbf{T}_{\bigcirc\bigcirc} \end{bmatrix} \quad (6)$$

and

$$\mathbf{T}_{\ominus+} = \begin{bmatrix} \mathbf{T}_{-+} \\ \mathbf{T}_{\bigcirc+} \end{bmatrix}, \quad \mathbf{T}_{\pm\bigcirc} = \begin{bmatrix} \mathbf{T}_{+\bigcirc} \\ \mathbf{T}_{-\bigcirc} \end{bmatrix}, \quad (7)$$

and let $\breve{\mathbf{C}}_{\ominus} = diag(\breve{c}_i)_{i\in\mathcal{S}_{\ominus}}$ be a diagonal matrix partitioned according to $\mathcal{S}_{\ominus} = \mathcal{S}_{-} \cup \mathcal{S}_{\bigcirc}$.

We define the generator matrix

$$\breve{\mathbf{Q}}_{\ominus\ominus} = (|\breve{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus\ominus}, \quad (8)$$

which has the following physical interpretation. By the analysis in [10, Lemmas 1-2], for $i,j \in \mathcal{S}_{\ominus}$, and $z>0$, we have

$$\begin{aligned}[e^{\breve{\mathbf{Q}}_{\ominus\ominus}z}]_{ij} &= P(\varphi(t_z) = j, \varphi(u) \in \mathcal{S}_{\ominus}, 0 \leq u \leq t_z \\ &\quad \mid \varphi(0) = i, X(0) = 0), \end{aligned} \quad (9)$$

which, for any $y>0$, we interpret as the probability that the process is in phase j at time t_z and the phase remains in the set $\mathcal{S}_{\ominus}$ at least until time t_z, given the process starts from phase i with empty buffer X and level $y+z$ in buffer Y.

Also, define

$$\breve{\mathbf{Q}}_{\ominus+} = (|\breve{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus+}, \quad (10)$$

which by [10, Lemma 2], is a matrix of transition rates, w.r.t. level, to phases in $\mathcal{S}_{+}$, corresponding to the moments at which the level in buffer Y begins to increase.

Second, consider the behaviour at $x>0$, when the level in buffer Y is strictly increasing according to the rates in $\widehat{\mathbf{C}}$. The key components of the analysis are matrices $\widehat{\mathbf{Q}}(s)$ and $\widehat{\mathbf{\Psi}}(s)$ to be defined below and interpreted afterwards.

Let

$$\theta = \inf\{t>0 : X(t) = 0\}, \quad (11)$$

be the first time at which the level in buffer X reaches 0.

Suppose $X(0)>0$, or $X(0)=0$ and $\varphi(0) \in \mathcal{S}_{+}$; and $t \leq \theta$. Define the random variable $U(t)$,

$$U(t) = \int_{u=0}^{t} \widehat{c}_{\varphi(u)}du, \quad (12)$$

interpreted as the total *upward shift* $Y(t) - Y(0)$ in buffer Y at time t.

We define the key generator matrix $\widehat{\mathbf{Q}}(s)$,

$$\widehat{\mathbf{Q}}(s) = \begin{bmatrix} \widehat{\mathbf{Q}}(s)_{++} & \widehat{\mathbf{Q}}(s)_{+-} \\ \widehat{\mathbf{Q}}(s)_{-+} & \widehat{\mathbf{Q}}(s)_{--} \end{bmatrix}, \quad (13)$$

with

$$\widehat{\mathbf{Q}}(s)_{++} = (\mathbf{R}_{+})^{-1}\left(\mathbf{T}_{++} - s\widehat{\mathbf{C}}_{+} - \mathbf{T}_{+\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc+}\right),$$
$$\widehat{\mathbf{Q}}(s)_{+-} = (\mathbf{R}_{+})^{-1}\left(\mathbf{T}_{+-} - \mathbf{T}_{+\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc-}\right),$$
$$\widehat{\mathbf{Q}}(s)_{-+} = (|\mathbf{R}_{-}|)^{-1}\left(\mathbf{T}_{-+} - \mathbf{T}_{-\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc+}\right),$$
$$\widehat{\mathbf{Q}}(s)_{--} = (|\mathbf{R}_{-}|)^{-1}\left(\mathbf{T}_{--} - s\widehat{\mathbf{C}}_{-} - \mathbf{T}_{-\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_{\bigcirc})^{-1}\mathbf{T}_{\bigcirc-}\right),$$
$$\widehat{\mathbf{C}}_{+} = diag(\widehat{c}_i)_{i\in\mathcal{S}_{+}}, \widehat{\mathbf{C}}_{-} = diag(\widehat{c}_i)_{i\in\mathcal{S}_{-}}, \widehat{\mathbf{C}}_{\bigcirc} = diag(\widehat{c}_i)_{i\in\mathcal{S}_{\bigcirc}}.$$

The physical interpretation of $\widehat{\mathbf{Q}}(s)$ was established in [4, Theorem 2]. For completeness, we state this result in Theorem 1 below. Now, for any $s>0$, we can find the minimum nonnegative solution $\widehat{\mathbf{\Psi}}(s)$ of the Riccati equation

$$\widehat{\mathbf{Q}}(s)_{+-} + \widehat{\mathbf{Q}}(s)_{++}\widehat{\mathbf{\Psi}}(s) + \widehat{\mathbf{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{--} + \widehat{\mathbf{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{-+}\widehat{\mathbf{\Psi}}(s) = \mathbf{O}, \quad (14)$$

which has the following interpretation, by the analysis in [4, Theorem 3]. For all $i \in \mathcal{S}_{+}$ and $j \in \mathcal{S}_{-}$,

$$[\widehat{\mathbf{\Psi}}(s)]_{ij} = E(e^{-sU(\theta)}1\{\varphi(\theta) = j\} \mid \varphi(0) = i, X(0) = 0), \quad (15)$$

is the Laplace-Stieltjes transform of the distribution of the upward shift in buffer Y at the moment the level in buffer X first returns to 0 and does so in phase j, given start from phase i and empty buffer X. We can write

$$\widehat{\mathbf{\Psi}}(s) = \int_{z=0}^{\infty} e^{-sz}\widehat{\boldsymbol{\psi}}(z)dz, \quad (16)$$

where the entry $[\widehat{\boldsymbol{\psi}}(z)]_{ij}$, for $i \in \mathcal{S}_{+}$ and $j \in \mathcal{S}_{-}$, is the corresponding probability density, which can be derived by numerically inverting $[\widehat{\mathbf{\Psi}}(s)]_{ij}$ using the algorithm by Abate and Whitt [1], for any $z>0$. That is, the matrix $\widehat{\boldsymbol{\psi}}(z)$ is an $|\mathcal{S}_{+}| \times |\mathcal{S}_{-}|$ matrix of densities, the (i,j)-th component of which records the density of an upward shift of z in the buffer Y, from some y to $y+z$, during a busy period of the buffer X, ending in phase $j \in \mathcal{S}_{-}$, starting at phase $i \in \mathcal{S}_{+}$.

In the remainder of this section we will give a slightly enhanced proof of Theorem 2 in [4]. This theorem gives the matrix recording the Laplace-Stieltjes transforms of the distribution of the shift in buffer Y, during the time that an amount x has flown into or out of the buffer X, ending up in phase j given that it starts in i. In [4] this matrix was called[1] $\tilde{\Delta}^{y}(s)$, while in the current paper we will write it as $\mathbf{U}^{(x)}(s)$. But more importantly, we will modify its definition somewhat, to reflect the fact that the value of the shift in buffer Y does not only depend on the initial phase i, the ending phase j, and the time duration, but on the whole sample path of $\varphi(t)$ in between. For the moment we will assume that, in our current context, $Y(t)$ can only increase, so that the shift in buffer Y, expressed as $Y(t) - Y(0)$, is always nonnegative[2].

Let, as in [4], $f(t) = \int_0^t |r_{\varphi(u)}|du$ be the total amount of fluid that flowed into or out of buffer X during $(0,t)$, referred to as the *in-out fluid* of X, and let $\omega(x) = \inf\{t > 0 : f(t) = x\}$ be the first time this in-out fluid reaches level x. Moreover, let now $V^x = \{\varphi(u), 0 \leq u \leq \omega(x)\}$ denote the whole path of $\varphi(t)$ during this interval, and let V_i^x be the set of all such paths that can be taken, starting from $\varphi(0) = i$, such that the total in-out fluid in buffer X is precisely x.

Denoting the duration of any path v by $|v|$, let $U(|v|)$ be the total shift in the second buffer during $(0, |v|)$; note that this random variable is completely determined by the path v. Then we formally define the matrix $\mathbf{U}^{(x)}(s)$ via its (i,j)-th

[1] with superscript y rather than x, since unfortunately the monotonously increasing (or decreasing) buffer, in which the shift is measured, was there called X, so the notations for X and Y are interchanged.

[2] i.e. we only consider the case $X(t) > 0$; the case $X(t) = 0$ is similar, except that we should replace the word 'shift' by 'virtual shift', as if the buffer Y had no lower boundary at 0.

entry as follows,

$$[\mathbf{U}^{(x)}(s)]_{ij} = \int_{v\in V_i^x} e^{-sU(|v|)} 1\{\varphi(|v|)=j\}dP(V=v), \tag{17}$$

where the integral incorporates the (countable) number of all possible successive states that $\varphi(t)$ visits, as well as all the corresponding sojourn times during all of these visits (adding up to $\omega(x)$). Using this definition we can prove the following result.

THEOREM 1. *(Theorem 2 in Bean and O'Reilly [4])*

$$\mathbf{U}^{(x+h)}(s) = \mathbf{U}^{(x)}(s)\mathbf{U}^{(h)}(s),$$

from which it follows that

$$\mathbf{U}^{(x)}(s) = e^{\widehat{\mathbf{Q}}(s)x}.$$

Proof. First note that any path $v \in V_i^{x+h}$ can be seen as a concatenation of two paths, $v_1 \in V_i^x$, ending in some phase k, and $v_2 \in V_k^h$ representing the in/outflow increase in buffer X from x to $x+h$. Due to the Markov property these paths are independent, conditional on v_2 starting in the same phase k as where v_1 finished. Since in that case clearly we also have $U(|v|) = U(|v_1|) + U(|v_2|)$, we arrive at

$$\begin{aligned} & e^{-sU(|v|)}1\{\varphi(|v|)=j\}dP(V=v) \\ = & \sum_k e^{-sU(|v_1|)}1\{\varphi(|v_1|)=k\}dP(V=v_1) \\ & \times e^{-sU(|v_2|)}1\{\varphi(|v_2|)=j\}dP(V=v_2), \end{aligned}$$

from which we find

$$\begin{aligned} & \int_{v\in V_i^{x+h}} e^{-sU(|v|)}1\{\varphi(|v|)=j\}dP(V=v) \\ = & \sum_k \int_{v\in V_i^x} e^{-sU(|v|)}1\{\varphi(|v|)=k\}dP(V=v) \\ & \times \int_{v\in V_k^h} e^{-sU(|v|)}1\{\varphi(|v|)=j\}dP(V=v), \end{aligned}$$

and hence the first statement follows. For the proof of the second statement we can simply refer to [4]. □

3.2 Embedded discrete-time Markov chain

Let θ_k be the k-th time that $(\varphi(t), X(t), Y(t))$ hits the line $x=0$, and let the discrete-time Markov chain $J_k = (\varphi(\theta_k), Y(\theta_k))$ with discrete/continuous state space $\mathcal{S}_- \times (0,\infty)$, record the position of $(\varphi(t), Y(t))$ at time θ_k. Also, let $\tau_k > \theta_k$ be the k-th time the process *leaves* the boundary $x=0$.

LEMMA 1. *The transition kernel of J_k is given by*

$$\begin{aligned} \mathbf{P}_{z,y} \;=\; & \int_{u=[z-y]^+}^{z} \left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right] e^{\breve{\mathbf{Q}}_{\ominus\ominus}u}\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\psi}}(y-z+u)du \\ & + \left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right] e^{\breve{\mathbf{Q}}_{\ominus\ominus}z}(-\breve{\mathbf{Q}}_{\ominus\ominus})^{-1}\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\psi}}(y). \end{aligned} \tag{18}$$

where $[x]^+$ denotes $\max(0,x)$, and $\left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right]$ is a $|\mathcal{S}_-|\times|\mathcal{S}_\ominus|$ matrix.

Proof. We apply the physical interpretations of the quantities analysed in Section 3.1. Essentially, the process J_k satisfies a Lindley-type recursion, since for its second component $Y(\theta_k)$ we can write

$$Y(\theta_{k+1}) = [Y(\theta_k) - D_k]^+ + U_k, \tag{19}$$

where

$$D_k = \int_{u=\theta_k}^{\tau_k} |\breve{c}_{\varphi(u)}|du, \quad U_k = \int_{u=\tau_k}^{\theta_{k+1}} \widehat{c}_{\varphi(u)}du \tag{20}$$

are appropriately chosen random variables. More precisely, starting from time θ_k, with $X(\theta_k)=0$ and $\varphi(\theta_k) = i \in \mathcal{S}_-$, we recall the two consecutive stages described in Section 2.3.

First, (i) the process $Y(t)$ will make a negative shift of size $-D$, say, as long as $\varphi(t) \in \mathcal{S}_\ominus$ (while $X(t)$ remains at zero during this stage). Then, after a transition of $\varphi(t)$ from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$, the second stage (ii) commences, during which the process $Y(t)$ will make a positive shift of size U, say, during a busy period of the first queue (i.e., during a first return time of $X(t)$ back to level zero, starting at level zero).

There are two alternatives. The first alternative is that the chain J_k transitions from (i,z) to (j,y) without the level in the buffer Y returning to 0 during time interval (θ_k, θ_{k+1}). Assume $y \geq z$. In this case,

- first the phase remains in the set $\mathcal{S}_\ominus$ at least until the level in buffer Y shifts down by u units (from z to $z-u$), for some u with $0 \leq u \leq z$; this occurs according to the probability matrix $e^{\breve{\mathbf{Q}}_{\ominus\ominus}u}$;
- then the process makes a transition to some phase in $\mathcal{S}_+$, which starts the busy period in buffer X; this occurs according to the rate matrix $\breve{\mathbf{Q}}_{\ominus+}$;
- finally, the busy process in buffer X ends and the level y is observed in buffer Y; this occurs according to the density matrix $\widehat{\psi}(y-z+u)$ since the shift in buffer Y during the busy period in X must be exactly $y-(z-u) = y-z+u$.

The transition kernel of the first alternative, when $y \geq z$, is therefore

$$I(y\geq z)\left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right]\int_{u=0}^{z} e^{\breve{\mathbf{Q}}_{\ominus\ominus}u}\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\psi}}(y-z+u)du, \tag{21}$$

and by analogous argument, when $y < z$,

$$I(y<z)\left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right]\int_{u=z-y}^{z} e^{\breve{\mathbf{Q}}_{\ominus\ominus}u}\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\psi}}(y-z+u)du. \tag{22}$$

The second alternative is that the chain J_k transitions from (i,z) to (j,y) with the level in the buffer Y returning to 0 some time during time interval (θ_k, θ_{k+1}). In this case,

- first the phase remains in the set $\mathcal{S}_\ominus$ at least until the level in buffer Y shifts down by z units (from z to 0); this occurs according to the probability matrix $\int_{u=z}^{\infty} e^{\breve{\mathbf{Q}}_{\ominus\ominus}u}du = e^{\breve{\mathbf{Q}}_{\ominus\ominus}z}(-\breve{\mathbf{Q}}_{\ominus\ominus})^{-1}$;
- then the process makes a transition to some phase in $\mathcal{S}_+$, which starts the busy period in buffer X; this occurs according to the rate matrix $\breve{\mathbf{Q}}_{\ominus+}$;
- finally, the busy process of buffer X ends at level y; this occurs according to the density matrix $\widehat{\psi}(y)$.

The transition kernel of the second alternative is

$$\left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right] e^{\breve{\mathbf{Q}}_{\ominus\ominus}z}(-\breve{\mathbf{Q}}_{\ominus\ominus})^{-1}\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\psi}}(y), \tag{23}$$

and so the result follows by summing (21)–(23). □

We could work with the above directly, but in Section 4 we prefer to determine the following Laplace-Stieltjes transforms, which can then be inverted using the algorithm in Abate and Whitt [1]. We note that $\mathbf{P}_{z,y}$ is continuous w.r.t. $y > 0$, and it is easy to check that $\int_{y=0}^{\infty} \mathbf{P}_{z,y} dy \mathbf{1} = \mathbf{1}$.

COROLLARY 1. *The Laplace-Stieltjes transform of $\mathbf{P}_{z,y}$ w.r.t. y is given by the matrix*

$$\begin{aligned}\mathbf{P}_{z,\cdot}(s) &= \left[\begin{array}{cc}\mathbf{I} & \mathbf{O}\end{array}\right] e^{-sz}\left(\breve{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I}\right)^{-1}\left(e^{(\breve{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I})z}-\mathbf{I}\right)\\ &\quad\times\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\Psi}}(s)\\ &\quad+\left[\begin{array}{cc}\mathbf{I} & \mathbf{O}\end{array}\right] e^{\breve{\mathbf{Q}}_{\ominus\ominus}z}(-\breve{\mathbf{Q}}_{\ominus\ominus})^{-1}\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\Psi}}(s). \qquad (24)\end{aligned}$$

Proof. By straightforward computation of $\int_{y=0}^{\infty} e^{-sy}\mathbf{P}_{z,y}dy$, or by using (19) directly as follows. Letting $Y_k = Y(\theta_k)$ and $\varphi_k = \varphi(\theta_k)$ for notational convenience, we have

$$\begin{aligned}&E[e^{-sY_{k+1}}1\{\varphi_{k+1}=j\} \mid Y_k = z, \varphi_k = i]\\ &\quad= E[e^{-s(z-D_k+U_k)}1\{\varphi_{k+1}=j\}1\{D_k\le z\} \mid Y_k=z,\varphi_k=i]\\ &\qquad+E[e^{-sU_k}1\{\varphi_{k+1}=j\}1\{D_k> z\} \mid Y_k=z,\varphi_k=i].\end{aligned}$$

By conditioning on the phases m and ℓ just before and after the time when the process leaves $x = 0$, we rewrite the first term as

$$\begin{aligned}&E[e^{-s(z-D_k+U_k)}1\{\varphi_{k+1}=j\}1\{D_k\le z\} \mid Y_k=z,\varphi_k=i]\\ &= \sum_{m\in\mathcal{S}_\ominus}\sum_{\ell\in\mathcal{S}_+} e^{-sz}\cdot E[e^{sD_k}1\{\varphi(\tau_k-)=m\}\\ &\quad\times 1\{D_k\le z\} \mid Y_k=z,\varphi_k=i]\\ &\quad\times E[1\{\varphi(\tau_k)=\ell\} \mid \varphi(\tau_k-)=m]\\ &\quad\times E[e^{-sU_k}1\{\varphi_{k+1}=j\} \mid \varphi(\tau_k)=\ell]\\ &= \sum_{m\in\mathcal{S}_\ominus}\sum_{\ell\in\mathcal{S}_+} e^{-sz}\int_{u=0}^{z}\left[\left[\begin{array}{cc}\mathbf{I} & \mathbf{O}\end{array}\right] e^{\breve{\mathbf{Q}}_{\ominus\ominus}u}\right]_{im}\\ &\quad\times e^{su}du\,[\breve{\mathbf{Q}}_{\ominus+}]_{m\ell}\,[\widehat{\boldsymbol{\Psi}}(s)]_{\ell j}\\ &= \Big[\left[\begin{array}{cc}\mathbf{I} & \mathbf{O}\end{array}\right] e^{-sz}\left(\breve{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I}\right)^{-1}\\ &\quad\times\left(e^{(\breve{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I})z}-\mathbf{I}\right)\breve{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\Psi}}(s)\Big]_{ij}.\end{aligned}$$

A similar expression can be given for the second term, by which the statement follows. $\square$

We denote the stationary distribution of J_k by a row vector $\boldsymbol{\xi}_z = [\xi_{i,z}]_{i\in\mathcal{S}_-}$ of densities, satisfying

$$\left\{\begin{array}{lcl}\int_{z=0}^{\infty}\boldsymbol{\xi}_z\mathbf{P}_{z,y}dz &=& \boldsymbol{\xi}_y\\ \int_{y=0}^{\infty}\boldsymbol{\xi}_y dy\mathbf{1} &=& 1,\end{array}\right. \qquad (25)$$

and proceed in the next section to express the stationary distribution of the process $(\varphi(t), X(t), Y(t))$ at level $x = 0$ in terms of $\boldsymbol{\xi}_z$.

REMARK 1. Instead of (19) we could also have worked with the true Lindley recursion

$$Y(\tau_{k+1}) = [Y(\tau_k) + U_k - D_{k+1}]^+. \qquad (26)$$

This is the approach that was followed in [14]. There, the stationary distribution of the chain, embedded at these times, in fact gave immediately also the stationary distribution of the whole process at $x = 0$, due to a PASTA-like argument related to the workload in an $M/G/1$ queue. However, in the more general model at hand, with possibly multiple phases being visited while $X(t) = 0$, this need not be true; e.g. there may be phases in $\mathcal{S}_\ominus$ from which it is impossible to jump to a state in $\mathcal{S}_+$. Moreover, one disadvantage would be that the stationary distribution of the embedded Markov chain besides having a density for $y > 0$, also has a mass at $y = 0$. Hence we decided to embed at *hitting times* of $x = 0$, in a manner similar to the analysis in [5].

3.3 Stationary distribution

In the following subsections we show how to find the various densities and probability masses that define the joint stationary distribution of the process.

3.3.1 Density at $\mathbf{x = 0, y > 0}$ and mass at $\mathbf{x = 0, y = 0}$

Recall from Section 2.3 that we need expressions for the vectors $\boldsymbol{\pi}(0, y)$ and $\mathbf{p}(0, 0)$, which we give in the following.

LEMMA 2. *We have $\boldsymbol{\pi}(0,y) = \left[\begin{array}{cc}\mathbf{0} & \boldsymbol{\pi}(0,y)_\ominus\end{array}\right]$, where*

$$\boldsymbol{\pi}(0,y)_\ominus = \alpha\int_{z=y}^{\infty}\left[\begin{array}{cc}\boldsymbol{\xi}_z & \mathbf{0}\end{array}\right] e^{\breve{\mathbf{Q}}_{\ominus\ominus}(z-y)}(|\breve{\mathbf{C}}_\ominus|)^{-1}dz, \qquad (27)$$

and $\mathbf{p}(0,0) = \left[\begin{array}{cc}\mathbf{0} & \mathbf{p}(0,0)_\ominus\end{array}\right]$, where

$$\mathbf{p}(0,0)_\ominus = \alpha\int_{z=0}^{\infty}\left[\begin{array}{cc}\boldsymbol{\xi}_z & \mathbf{0}\end{array}\right] e^{\breve{\mathbf{Q}}_{\ominus\ominus}z}dz(-\mathbf{T}_{\ominus\ominus})^{-1}. \qquad (28)$$

Here, α is a normalization constant that satisfies

$$\begin{aligned}1 &= \mathbf{p}(0,0)\mathbf{1} + \int_{y=0}^{\infty}\boldsymbol{\pi}(0,y)dy\mathbf{1} + \sum_{j\in\mathcal{S}_+}\int_{x=0}^{\infty}\pi^j(x, x\widehat{c}_j/r_j)dx\\ &\quad+\int_{x=0}^{\infty}\int_{y=0}^{\infty}\boldsymbol{\pi}(x,y)dydx\mathbf{1}, \qquad (29)\end{aligned}$$

given by

$$\begin{aligned}\alpha &= \Big\{\left[\begin{array}{cc}\boldsymbol{\xi} & \mathbf{0}\end{array}\right](-\mathbf{T}_{\ominus\ominus})^{-1}\Big(\mathbf{1}\\ &\quad+\mathbf{T}_{\ominus+}\mathbf{K}^{-1}\left[\begin{array}{cc}(\mathbf{R}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{R}_-|)^{-1}\end{array}\right]\\ &\quad\times\left(\mathbf{1}+\mathbf{T}_{\pm\bigcirc}(-\mathbf{T}_{\bigcirc\bigcirc})^{-1}\mathbf{1}\right)\Big)\Big\}^{-1}, \qquad (30)\end{aligned}$$

where, $\boldsymbol{\xi} = \int_{z=0}^{\infty}\boldsymbol{\xi}_z dz$, $\boldsymbol{\Psi} = \widehat{\boldsymbol{\Psi}}(s)|_{s=0}$ and $\mathbf{K} = \widehat{\mathbf{K}}(s)|_{s=0}$ with

$$\widehat{\mathbf{K}}(s) = \widehat{\mathbf{Q}}(s)_{++} + \widehat{\boldsymbol{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{-+}. \qquad (31)$$

Proof. In *(i)–(iii)* we prove (27)–(30) respectively.

(i) Observe the process whenever the level in buffer X hits 0. Denote by α the corresponding rate such that $E^* = \alpha^{-1}$ is the average time between two hits.

Let $E^*_{z,i}(j, 0, u)$ be the derivative w.r.t. y of the expected time in phase j, $x = 0$ and $y \le u$ until the next hit given start from state $(i, 0, z)$.

Consider the process $\{(\varphi(t), X(t), Y(t)) : t \ge 0\}$ in stationarity. By the argument analogous to [15, Theorem 4.1],

$$P(\phi = j, X = 0, Y \in dy) = \alpha\sum_{i\in\mathcal{S}_-}\int_{z=y}^{\infty}\xi_{z,i}E^*_{z,i}(j,0,y)dz\cdot dy, \qquad (32)$$

where the integral starts at y since for $z < y$ it is not possible to reach $(j, 0, y)$ from $(i, 0, z)$ without leaving $x = 0$ in between. Since, by adapting the argument in [2, Theorem 3.2.1] to the analysis here,

$$E^*_{z,i}(j,0,y) = 1 \cdot [e^{\breve{\mathbf{Q}}_{\ominus\ominus}(z-y)}]_{ij}/|\breve{c}_j|, \tag{33}$$

equation (27) for $\boldsymbol{\pi}(0,y)$ follows.

(ii) Similar arguments show the expression for (28); for ending up in $(j,0,0)$ from $(i,0,z)$ with $i \in \mathcal{S}_-$ and $z \geq 0$, the process $\varphi(t)$ now needs to stay in $\mathcal{S}_\ominus$ for an amount of 'shift' (rather than time) of $z+w$ for some $w \geq 0$, and end up in phase $j \in \mathcal{S}_\ominus$. We have

$$\begin{aligned}[\mathbf{p}(0,0)]_j &= \alpha \sum_{i\in\mathcal{S}_-} \int_{z=y}^{\infty} \xi_{z,i} \int_{w=0}^{\infty} [e^{\breve{\mathbf{Q}}_{\ominus\ominus}(z+w)}]_{ij}/\breve{c}_j dwdz \\ &= \alpha \sum_{i\in\mathcal{S}_-} \int_{z=y}^{\infty} \xi_{z,i} [e^{\breve{\mathbf{Q}}_{\ominus\ominus}z}(-\mathbf{T}_{\ominus\ominus})^{-1}]_{ij} dz.\end{aligned} \tag{34}$$

(iii) To find α, since this is a constant that does not depend on buffer Y, we only need to consider the process $(\varphi(t), X(t))$, together with the distribution of $\{\varphi(t)\}$ upon hitting $x=0$, which is $\boldsymbol{\xi} = \int_{z=0}^{\infty} \boldsymbol{\xi}_z dz$. The vector $\boldsymbol{\xi}$ is the stationary distribution of the corresponding discrete-time Markov chain with state space $\mathcal{S}_-$ which records the position of $\varphi(t)$ at time θ_k. The vector $\boldsymbol{\xi}$ is is the unique solution of the set of equations

$$\begin{aligned}\begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1} \mathbf{T}_{\ominus+} \boldsymbol{\Psi} &= \boldsymbol{\xi}, \\ \boldsymbol{\xi}\mathbf{1} &= 1.\end{aligned} \tag{35}$$

The stationary distribution for the SFM has been derived in the literature in [5, 6, 11, 13, 15, 17] in slightly different contexts. For completeness, we summarize here the results required for the derivation of the stationary distribution of $(\varphi(t), X(t))$, including the probability mass vector at level zero, $\mathbf{p} = \begin{bmatrix} \mathbf{0} & \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix}$, and the probability density vector, $\boldsymbol{\pi}(x) = \begin{bmatrix} \boldsymbol{\pi}(x)_+ & \boldsymbol{\pi}(x)_- & \boldsymbol{\pi}(x)_\bigcirc \end{bmatrix}$, for all $x > 0$. By conditioning on the last time the SFM $(\varphi(t), X(t))$ hits level zero from above, in a manner similar to [5, Theorem 2],

$$\begin{bmatrix} \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1}, \tag{36}$$

and

$$\begin{aligned}\begin{bmatrix} \boldsymbol{\pi}(x)_+ & \boldsymbol{\pi}(x)_- \end{bmatrix} &= \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix} \mathbf{T}_{\ominus+} e^{\mathbf{K}x} \\ &\quad \times \begin{bmatrix} (\mathbf{R}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{R}_-|)^{-1} \end{bmatrix}, \\ \boldsymbol{\pi}(x)_\bigcirc &= \begin{bmatrix} \boldsymbol{\pi}(x)_+ & \boldsymbol{\pi}(x)_- \end{bmatrix} \mathbf{T}_{\pm\bigcirc} \\ &\quad \times (-\mathbf{T}_{\bigcirc\bigcirc})^{-1}.\end{aligned} \tag{37}$$

Alternatively, (36) can be found by integrating (27) w.r.t. y and adding to (28). Similarly, (37) can be found by integrating $\boldsymbol{\pi}(x,y)$ w.r.t. y and adding $\sum_{j\in\mathcal{S}_+} \boldsymbol{\pi}^j(x, x\widehat{c}_j/r_j)$; the expressions for these quantities will be derived in sections that follow.

Since α is a normalizing constant that solves

$$\mathbf{p}\mathbf{1} + \int_{x=0}^{\infty} \boldsymbol{\pi}(x) dx \mathbf{1} = 1, \tag{38}$$

we have

$$\begin{aligned}\alpha^{-1} &= \begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1} \Big(\mathbf{1} \\ &\quad + \mathbf{T}_{\ominus+} \mathbf{K}^{-1} \begin{bmatrix} (\mathbf{R}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{R}_-|)^{-1} \end{bmatrix} \\ &\quad \times \left(\mathbf{1} + \mathbf{T}_{\pm\bigcirc}(-\mathbf{T}_{\bigcirc\bigcirc})^{-1}\mathbf{1} \right) \Big),\end{aligned} \tag{39}$$

and so the expression (30) for α follows. □

Note that α can also be interpreted as the total (stationary) rate of leaving $x=0$, since by (36),

$$\begin{aligned}\begin{bmatrix} \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix} \mathbf{T}_{\ominus+}\mathbf{1} &= -\begin{bmatrix} \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix} \mathbf{T}_{\ominus\ominus}\mathbf{1} \\ &= \alpha \begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} \mathbf{1} \\ &= \alpha,\end{aligned} \tag{40}$$

and also as the total (stationary) rate of hitting $x=0$, since by (37) and $\boldsymbol{\Psi}\mathbf{1} = \mathbf{1}$,

$$\begin{aligned}\lim_{x\to 0^+} \boldsymbol{\pi}(x)_- |\mathbf{R}_-| \mathbf{1} &= \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix} \mathbf{T}_{\ominus+}\mathbf{1} \\ &= \alpha,\end{aligned} \tag{41}$$

with the two forms equivalent, as expected in stationarity.

For the Laplace-Stieltjes transform vector of the density part, denoted as $\boldsymbol{\pi}(0,\cdot)(s) = \int_{z=0}^{\infty} e^{-sy} \boldsymbol{\pi}(0,y) dy$, we have the following.

Corollary 2. *We have* $\boldsymbol{\pi}(0,\cdot)(s) = \begin{bmatrix} \mathbf{0} & \boldsymbol{\pi}(0,\cdot)(s)_\ominus \end{bmatrix}$, *where*

$$\begin{aligned}\boldsymbol{\pi}(0,\cdot)(s)_\ominus &= \alpha \int_{z=0}^{\infty} \begin{bmatrix} \boldsymbol{\xi}_z & \mathbf{0} \end{bmatrix} e^{\breve{\mathbf{Q}}_{\ominus\ominus}z} (\breve{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \\ &\quad \times \left(\mathbf{I} - e^{-(\breve{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I})z} \right) (|\breve{\mathbf{C}}_\ominus|)^{-1} dz.\end{aligned} \tag{42}$$

Proof. Since

$$\begin{aligned}\boldsymbol{\pi}(0,\cdot)(s)_\ominus &= \int_{y=0}^{\infty} e^{-sy} \alpha \int_{z=y}^{\infty} \begin{bmatrix} \boldsymbol{\xi}_z & \mathbf{0} \end{bmatrix} \\ &\quad \times e^{\breve{\mathbf{Q}}_{\ominus\ominus}(z-y)} (|\breve{\mathbf{C}}_\ominus|)^{-1} dzdy \\ &= \alpha \int_{z=0}^{\infty} \begin{bmatrix} \boldsymbol{\xi}_z & \mathbf{0} \end{bmatrix} e^{\breve{\mathbf{Q}}_{\ominus\ominus}z} \\ &\quad \times \int_{y=0}^{z} e^{-(\breve{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I})y} (|\breve{\mathbf{C}}_\ominus|)^{-1} dydz \\ &= \alpha \int_{z=0}^{\infty} \begin{bmatrix} \boldsymbol{\xi}_z & \mathbf{0} \end{bmatrix} e^{\breve{\mathbf{Q}}_{\ominus\ominus}z} \\ &\quad \times \left(-e^{-(\breve{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I})y} (\breve{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \Big|_{y=0}^{z} \right) (|\breve{\mathbf{C}}_\ominus|)^{-1} dz,\end{aligned}$$

the result follows. □

3.3.2 *Density at* $\mathbf{x} > \mathbf{0}, \mathbf{y} > \mathbf{0}$

We now proceed to the density vector $\boldsymbol{\pi}(x,y)$ as a function of y for fixed value of x.

Define the Laplace-Stieltjes transform $\boldsymbol{\pi}(x,\cdot)(s)$ such that, $[\boldsymbol{\pi}(x,\cdot)(s)]_i = \int_{y=0}^{\infty} e^{-sy} [\boldsymbol{\pi}(x,y)]_i dy$ for $i \in \mathcal{S}_\ominus$, and $[\boldsymbol{\pi}(x,\cdot)(s)]_i = \int_{y=0}^{\infty} e^{-sy} [\boldsymbol{\pi}(x,y)]_i dy + e^{-sx\widehat{c}_i/r_i} \pi^i(x, x\widehat{c}_i/r_i)$ for $i \in \mathcal{S}_+$.

Lemma 3. *We have*

$$\boldsymbol{\pi}(x,\cdot)(s) = \begin{bmatrix} \boldsymbol{\pi}(x,\cdot)(s)_+ & \boldsymbol{\pi}(x,\cdot)(s)_- & \boldsymbol{\pi}(x,\cdot)(s)_\bigcirc \end{bmatrix}$$

with

$$\left[\begin{array}{cc} \boldsymbol{\pi}(x,\cdot)(s)_+ & \boldsymbol{\pi}(x,\cdot)(s)_- \end{array}\right] = (\boldsymbol{\pi}(0,\cdot)(s)_\ominus + \mathbf{p}(0,0)_\ominus) \times \mathbf{T}_{\pm\bigcirc} e^{\widehat{\mathbf{K}}(s)x} \times \left[\begin{array}{cc} (\mathbf{R}_+)^{-1} & \widehat{\boldsymbol{\Psi}}(s)(|\mathbf{R}_-|)^{-1} \end{array}\right], \quad (43)$$

and

$$\boldsymbol{\pi}(x,\cdot)(s)_\bigcirc = \left[\begin{array}{cc} \boldsymbol{\pi}(x,\cdot)(s)_+ & \boldsymbol{\pi}(x,\cdot)(s)_- \end{array}\right] \times \mathbf{T}_{\pm\bigcirc}(s\widehat{\mathbf{C}}_\bigcirc - \mathbf{T}_{\bigcirc\bigcirc})^{-1}. \quad (44)$$

Proof. The result follows immediately by a partitioning of the sample paths argument, analogous to the one used in the derivation of (37). □

COROLLARY 3. *Letting* $\boldsymbol{\pi}(\cdot,\cdot)(v,s) = \int_{x=0}^{\infty} e^{-vx}\boldsymbol{\pi}(x,\cdot)(s)dx$, *we have*

$$\boldsymbol{\pi}(\cdot,\cdot)(v,s) = \left[\begin{array}{ccc} \boldsymbol{\pi}(\cdot,\cdot)(v,s)_+ & \boldsymbol{\pi}(\cdot,\cdot)(v,s)_- & \boldsymbol{\pi}(\cdot,\cdot)(s)_\bigcirc \end{array}\right]$$

with

$$\left[\begin{array}{cc} \boldsymbol{\pi}(\cdot,\cdot)(v,s)_+ & \boldsymbol{\pi}(\cdot,\cdot)(v,s)_- \end{array}\right] = (\boldsymbol{\pi}(0,\cdot)(s)_\ominus + \mathbf{p}(0,0)_\ominus) \times \left[\begin{array}{c} \mathbf{T}_{-+} \\ \mathbf{T}_{\bigcirc+} \end{array}\right] (-\widehat{\mathbf{K}}(s) + v\mathbf{I})^{-1} \left[\begin{array}{cc} (\mathbf{R}_+)^{-1} & \widehat{\boldsymbol{\Psi}}(s)(|\mathbf{R}_-|)^{-1} \end{array}\right], \quad (45)$$

and

$$\boldsymbol{\pi}(\cdot,\cdot)(s)_\bigcirc = \left[\begin{array}{cc} \boldsymbol{\pi}(\cdot,\cdot)(s)_+ & \boldsymbol{\pi}(\cdot,\cdot)(s)_- \end{array}\right] \mathbf{T}_{\pm\bigcirc} \times (s\widehat{\mathbf{C}}_\bigcirc - \mathbf{T}_{\bigcirc\bigcirc})^{-1}. \quad (46)$$

3.3.3 Density at $y = x\widehat{c}_i/r_i$

Finally, we state the result for the one-dimensional densities on each of the lines $y = x\widehat{c}_i/r_i$, $i \in \mathcal{S}_+$.

LEMMA 4. *For all* $i \in \mathcal{S}_+$,

$$\pi^i(x, x\widehat{c}_i/r_i) = \sum_{j\in\mathcal{S}_\ominus} \mathbf{p}_j(0,0) T_{ji} \exp(-(T_{ii}/r_i)x)/r_i. \quad (47)$$

Proof. This result essentially follows by arguments analogous to the proof of the first equation in (37), in a slightly different environment.

By conditioning on the most recent time the process leaves the point $(0,0)$, in order to observe the process in stationarity at the point $(x, x\widehat{c}_i/r_i)$, the following must occur.

- First, the process starts from state $(j,0,0)$ for some $j \in \mathcal{S}_\ominus$, with probability $\mathbf{p}_j(0,0)$, and instantaneously transitions to phase i at a rate T_{ji}.

- Next, the process remains in phase i at least for the duration of time x/r_i, with probability $\exp(-(T_{ii}/r_i)x)$.

Denote by $E(i,x,x\widehat{c}_i/r_i)$ the expected number of visits to state $(i,x,x\widehat{c}_i/r_i)$ given the process starts in state $(i,0,0)$ and avoids returning to level 0 in both buffer X and Y. Clearly, $E(i,x,x\widehat{c}_i/r_i) = 1 \cdot \exp(-(T_{ii}/r_i)x)$.

Further, we note that, by [2, Theorem 3.2.1],

$$\pi^i(x, x\widehat{c}_i/r_i) = \sum_{j\in\mathcal{S}_\ominus} \mathbf{p}_j(0,0) T_{ji} E(i,x,x\widehat{c}_i/r_i)/r_i, \quad (48)$$

and the result (47) follows. □

3.4 Main Result

We now summarize the results for the stationary distribution of the process $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$.

THEOREM 2. *The probability mass components of the stationary distribution, corresponding to* $x = 0$, *are*

$$\boldsymbol{\pi}(0,y) \quad \text{and} \quad \mathbf{p}(0,0),$$

given in Lemma 2. The Laplace-Stieltjes transforms of $\boldsymbol{\pi}(0,y)$ *w.r.t.* y *are given in Corollary 2.*

The one-dimensional density components of the stationary distribution, corresponding to $y = x\widehat{c}_j/r_j$, *are*

$$\boldsymbol{\pi}^j(x, x\widehat{c}_j/r_j) = [\delta_{ij}\pi^j(x, x\widehat{c}_j/r_j)]_{i\in\mathcal{S}}, \quad j \in \mathcal{S}_+,$$

given in Lemma 4.

The Laplace-Stieltjes transforms of the two-dimensional density components of the stationary distribution, $\boldsymbol{\pi}(x,y)$, *corresponding to* $x > 0$, *w.r.t.* y, *are*

$$[\boldsymbol{\pi}(x,\cdot)(s)]_i, \quad i \in \mathcal{S}_\ominus$$

and

$$[\boldsymbol{\pi}(x,\cdot)(s)]_i - e^{-sx\widehat{c}_i/r_i}\pi^i(x, x\widehat{c}_i/r_i), \quad i \in \mathcal{S}_+,$$

given in Lemma 3. The corresponding Laplace-Stieltjes transforms w.r.t. x *and* y *are given in Corollary 3.*

4. TANDEM FLUID QUEUE: NUMERICAL TREATMENT

In order to evaluate the stationary distribution of the model using the theoretical results of Section 3, we apply discretization and truncation with appropriate parameters Δu, and L, $\ell = 0,1,2,\ldots L$. The key points of the methodology are summarized below.

Step 1. Construct discretized version of the process J_k discussed in Section 3.2, with a truncated level variable as follows.

Fix some small $\Delta u > 0$ and some large integer $L > 0$, and consider a discrete-time Markov chain $\{\bar{J}_k : k = 0,1,2,\ldots\}$ with state space $\{(i,\ell) : i \in \mathcal{S}_-, \ell = 0,1,2,\ldots L\}$, with the interpretation that when $J_k = (j,z)$ for some z with $\ell\Delta u \leq z < (\ell+1)\Delta u$, $\ell = 0,1,2,\ldots(L-1)$, then we have $\bar{J}_k = (j,\ell)$, and when $J_k = (j,z)$ with $z \geq L\Delta u$, we let $\bar{J}_k = (j,L)$.

(i). Approximate the corresponding one-step transition probabilities $P_{i,\ell;j,m} = P(\bar{J}_{k+1} = (j,m)|\bar{J}_k = (i,\ell))$, which are collected in matrix $\mathbf{P} = [\mathbf{P}_{\ell m}]_{\ell,m=0,1,2,\ldots,L}$ made of block matrices $\mathbf{P}_{\ell m} = [P_{i,\ell;j,m}]_{i,j\in\mathcal{S}_-}$ as follows.

First, for $\ell, m = 0,1,2,\ldots L$, evaluate

$$\tilde{\mathbf{P}}_{\ell m} = \int_{y=m\Delta u}^{(m+1)\Delta u} \mathbf{P}_{\ell\Delta u, y}dy, \quad (49)$$

and then normalize $\tilde{\mathbf{P}}_{\ell m}$ to obtain $\mathbf{P}_{\ell m}$ so that

$$\sum_{m=0}^{L} \mathbf{P}_{\ell m}\mathbf{1} = \mathbf{1}. \quad (50)$$

(ii). Next, with the notation $\lim_{k\to\infty} P(\bar{J}_k = (j,\ell)) = \bar{\xi}_{j;\ell}$ whenever the limits exist, denote by $\bar{\boldsymbol{\xi}} = [\bar{\boldsymbol{\xi}}_\ell]_{\ell=0,1,2,\ldots L}$, $\bar{\boldsymbol{\xi}}_\ell = [\bar{\xi}_{j;\ell}]_{j\in\mathcal{S}_-}$, the stationary distribution vector of the process $\{\bar{J}_k : k = 0, 1, 2, \ldots\}$. Derive $\bar{\boldsymbol{\xi}}$ by solving the set of equations, using standard methods,

$$\bar{\boldsymbol{\xi}}\mathbf{P} = \bar{\boldsymbol{\xi}}, \quad \bar{\boldsymbol{\xi}}\mathbf{1} = \mathbf{1}. \tag{51}$$

Step 2. Approximate the values of stationary distribution of the process $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$ as follows.

(i). For any z with $\ell\Delta u \leq z < (\ell+1)\Delta u$, $\ell = 0, 1, 2, \ldots L$, approximate

$$\boldsymbol{\xi}_z \approx \frac{\bar{\boldsymbol{\xi}}_\ell}{\Delta u}. \tag{52}$$

(ii). Using (28), apply

$$\begin{aligned}
\mathbf{p}(0,0)_\ominus &= \alpha \int_{z=0}^{\infty} \boldsymbol{\xi}_z e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1} \\
&= \alpha \sum_{\ell=0}^{\infty} \int_{z=\ell\Delta u}^{(\ell+1)\Delta u} \boldsymbol{\xi}_z e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1} \\
&\approx \alpha \sum_{\ell=0}^{L} \int_{z=\ell\Delta u}^{(\ell+1)\Delta u} \frac{\bar{\boldsymbol{\xi}}_\ell}{\Delta u} e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1} \\
&\approx \alpha \sum_{\ell=0}^{L} \bar{\boldsymbol{\xi}}_\ell e^{\check{\mathbf{Q}}_{\ominus\ominus} \ell\Delta u} (-\mathbf{T}_{\ominus\ominus})^{-1}.
\end{aligned} \tag{53}$$

Apply analogous approximation idea to calculating $\boldsymbol{\pi}(0, y)$, $y > 0$, and $\boldsymbol{\pi}(x, y)$, $x > 0$, $y > 0$, using (42), (45)-(47) and the inversion method of Abate and Whitt in [1].

Work on the numerical application of the above methodology is in progress.

5. CONCLUSION

We considered a tandem fluid queue model consisting of two queues, in which the first queue, $\{(\varphi(t), X(t)) : t \geq 0\}$, is a standard stochastic fluid model with a finite buffer and real rates r_i, and the second queue, $\{(\varphi(t), Y(t)) : t \geq 0\}$, is a stochastic fluid model with a finite buffer and rates $\widehat{c}_i > 0$ and $\check{c}_i < 0$, such that the rates of change of level depend on whether the first queue is empty or not. Specifically, we assumed that the rates of change of level in the second queue are negative ($dY(t)/dt = \check{c}_i$) when the first queue is empty, and positive ($dY(t)/dt = \widehat{c}_i$) otherwise.

We derived theoretical results for the stationary analysis of such tandem fluid queue, and summarized the key points of the methodology for the numerical evaluation of the stationary distribution of the process based on these results.

As future work we are also interested in the analysis of a dual tandem fluid queue model, with the difference that the rates of change of level in the second queue are positive ($dY(t)/dt = \widehat{c}_i$) when the first queue is empty, and negative ($dY(t)/dt = \check{c}_i$) otherwise. Work on the theoretical analysis of the dual model is in progress.

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