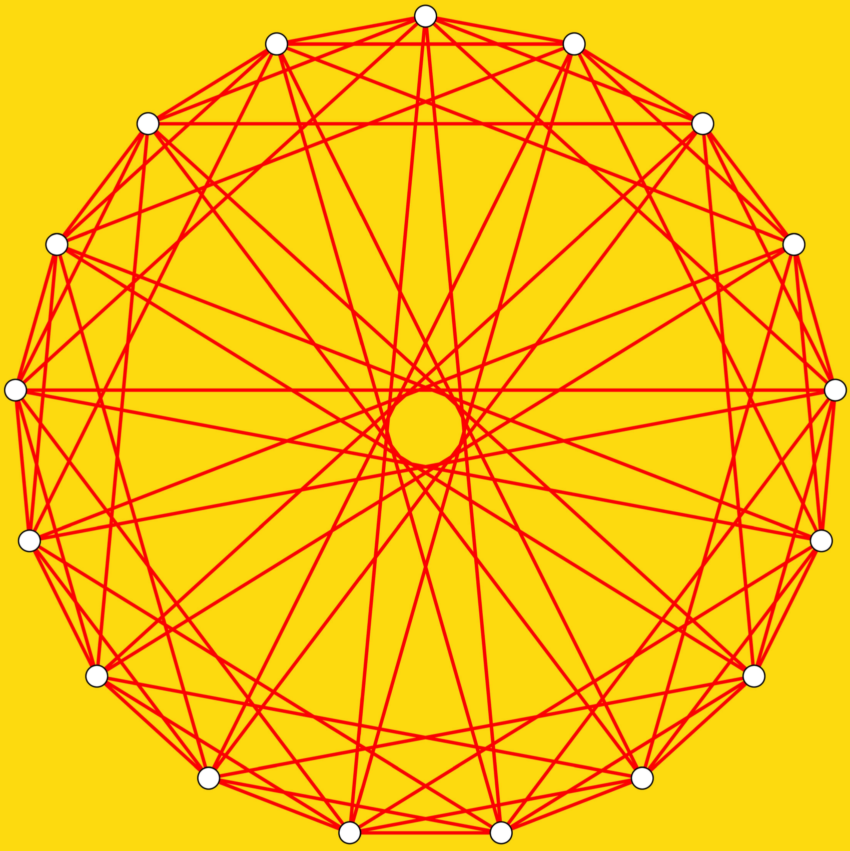
Yanbo Zhang

Generalized Ramsey Numbers for Graphs



UNIVERSITY OF TWENTE.

Generalized Ramsey Numbers for Graphs

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Preface

This thesis is the final result of several years of research that feel like a hiking tour in the mountains of Ramsey numbers. It cannot express the many long days that the author felt lost in the mountains, or the sadness he felt with each failed climbing, or the tremendous joy he felt when reaching the top, shoulder to shoulder with his advisors.

The thesis contains nine chapters with new results (Chapters 2–10), together with an introductory chapter (Chapter 1). Chapters 2, 9 and part of Chapter 6 are mainly based on research that was done while the author was working as a PhD student at Nanjing University in Nanjing, China; the other parts are mainly based on research of the author at the University of Twente. The first three of these chapters (Chapters 2, 3 and 4) deal with Ramsey numbers for cycles versus stars or wheels. The next four chapters involve Ramsey numbers for trees versus fans or wheels (Chapter 5), fans versus fans, wheels or complete graphs (Chapter 6), paths versus kipases (Chapter 7), and the union of some graphs (Chapter 8). We study planar Ramsey numbers in Chapter 9, star-critical and upper size Ramsey numbers in Chapter 10. To avoid duplication, some frequently used lemmas are stated in Chapter 1, and all references are gathered at the end of the thesis. The whole work is based on the following joint papers, which have been published or submitted to journals.

Papers underlying this research

[1] Y.B. Zhang, Y.Q. Zhang and Y.J. Chen, The Ramsey numbers of wheels versus odd cycles, Discrete Mathematics, 323 (2014) 76–80. (Chapter 2)

[2] Y.B. Zhang, S.P. Zhu and Y.Q. Zhang, Ramsey numbers for 7-cycle versus wheels with small order, Journal of Nanjing University, Mathematical Biquarterly, 30 (2013) 48–55. (Chapter 2)

[3] Y.B. Zhang and H.J. Broersma, Narrowing down the knowledge gap on exact values of cycle-star Ramsey numbers, preprint. (Chapter 3)

[4] Y.B. Zhang, H.J. Broersma and Y.J. Chen, A remark on star-C_4 and wheel-C_4 Ramsey numbers, Electronic Journal of Graph Theory and Applications, 2 (2014) 110–114. (Chapter 3)

[5] Y.B. Zhang, H.J. Broersma and Y.J. Chen, Three results on cycle-wheel Ramsey numbers, Graphs and Combinatorics, DOI 10.1007/s00373-014-1523-0. (Chapters 3 and 4)

[6] Y.B. Zhang, H.J. Broersma and Y.J. Chen, Ramsey numbers of trees versus fans, Discrete Mathematics, 338 (2015) 994–999. (Chapter 5)

[7] Y.B. Zhang, H.J. Broersma and Y.J. Chen, On fan-wheel and tree-wheel Ramsey numbers, preprint. (Chapters 5 and 6)

[8] Y.B. Zhang, H.J. Broersma and Y.J. Chen, A note on Ramsey numbers for fans, preprint. (Chapter 6)

[9] Y.B. Zhang and Y.J. Chen, The Ramsey numbers of fans versus a complete graph of order five, Electronic Journal of Graph Theory and Applications, 2 (2014) 66–69. (Chapter 6)

[10] B.L. Li, Y.B. Zhang, H.J. Broersma, H. Bielak and P. Holub, Closing the gap on path-kipas Ramsey numbers, preprint. (Chapter 7)

[11] Y.B. Zhang, H.J. Broersma and Y.J. Chen, Ramsey goodness for the union of some graphs, preprint. (Chapter 8)

[12] Y.B. Zhang, G.F. Zhou and Y.J. Chen, All complete graph-wheel planar Ramsey numbers, Graphs and Combinatorics, DOI 10.1007/s00373-014-1509-y. (Chapter 9)

[13] Y.B. Zhang, H.J. Broersma and Y.Q. Zhang, A note on some planar Ramsey numbers, preprint. (Chapter 9)

[14] Y.B. Zhang, H.J. Broersma and Y.J. Chen, On star-critical and upper size Ramsey numbers, preprint. (Chapter 10)

Notation

Let G be a graph with vertex set $V(G)$, $u,v \in V(G)$ and $U,V \subseteq V(G)$.

$\overline{G}$	the complement of G
mG	m disjoint copies of G
$E(G)$	the edge set of G
$e(G)$ (or $\lvert E(G)\rvert$)	the number of edges of G
$\lvert G\rvert$ (or $\lvert V(G)\rvert$)	the number of vertices of G
$N(v)$	the set of neighbors of v
$N[v]$	$N(v)\cup\{v\}$
$d(v)$	the number of neighbors of or the degree of v
$\delta(G)$	the minimum degree of G
$\Delta(G)$	the maximum degree of G
$\alpha(G)$	the independence number of G
$\chi(G)$	the chromatic number of G
$\kappa(G)$	the (vertex) connectivity of G
$\omega(G)$	the number of components of G
$g(G)$	the length of a shortest cycle of G
$c(G)$	the length of a longest cycle of G

$N_U(v)$	$N(v) \cap U$
$d_U(v)$	$\|N_U(v)\|$
$E(U,V)$	$\{uv \in E(G) \mid u \in U, v \in V\}$
$e(U,V)$	$\|E(U,V)\|$
$G[U]$	the subgraph induced by U in G

Contents

Chapter 1

Introduction

Graph Ramsey theory stems from a deceptively simple problem, i.e., a problem that is very easy to state and that seems easy to solve, but turns out to be very difficult. In its general form, the problem is to determine the smallest integer $r = R(m,n)$, such that at any party of r people, we can find m mutual acquaintances (each one knows all $m-1$ others) or n mutual strangers (each one does not know any of the $n-1$ others). For small values of m and n the problem is easy. It is trivial that $R(1,n) = R(m,1) = 1$, and almost trivial that $R(2,n) = n$ and $R(m,2) = m$. Even $R(3,3) = 6$ is not difficult to prove, and a nice exercise. The fact that $R(4,4) = 18$ was established by Greenwood and Gleason [67] in 1955. Perhaps surprisingly at first sight, $R(5,5)$ is still unknown. Erdős [44] explained the difficulties of this problem as follows: "It must seem incredible to the uninitiated that in the age of supercomputers $R(5,5)$ is unknown. This, of course, is caused by the so-called combinatorial explosion: there are just too many cases to be checked." Perhaps because of this, graph Ramsey theory has become a flourishing field which transcends its original motivation, just as Graham et al. [65] write (the same expression can also be found in [64]):

> A major impetus behind the early development of graph Ramsey theory was the hope that it would eventually lead to methods for determining larger values of the classical Ramsey numbers $R(m,n)$. However, as so often happens in mathematics, this hope has not been realized; rather, the field has blossomed into a discipline of its own. In fact, it is probably safe to say that the results arising from graph Ramsey theory will prove to be more valuable and interesting than knowing the exact value of $R(5,5)$ [or even $R(m,n)$].

Graph Ramsey theory can be viewed as a branch of both graph theory and

Ramsey theory. It makes extensive use of terminologies and extremal results in graph theory; and it is in a close relationship with other branches of Ramsey theory. In this chapter, we first introduce some basic concepts and notations of graph theory. Some related theorems which will be frequently used as lemmas in the following chapters are also added. Then we give a brief overview of Ramsey theory. We subsequently show some results of classical Ramsey numbers and generalized Ramsey numbers. Finally, at the end of this chapter we outline the main results of this thesis.

1.1 Graph theory

All graphs considered in this thesis are finite simple graphs. Let G be such a graph, with vertex set $V(G)$ and edge set $E(G)$. For notational simplicity, we sometimes write $|G|$ for the order $|V(G)|$ of G, i.e., the number of vertices of G. For $X \subseteq V(G)$, we let $G[X]$ and $G-X$ denote the subgraphs of G induced by X and $V(G)-X$, respectively. For a vertex $v \in V(G)$, we let $N_X(v)$ denote the set of neighbors of v that are contained in X, and we define $N_X[v] = N_X(v) \cup \{v\}$ and $d_X(v) = |N_X(v)|$. For a subgraph H of G, sometimes we write $d_H(v)$ rather than $d_{V(H)}(v)$ for simplicity. If $H = G$, we simply write $N(v)$ for $N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For two disjoint subgraphs G_1 and G_2 of G, we define $G_1 \cup G_2$ as the disjoint union of G_1 and G_2, and $G_1 + G_2$ as the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1)$ and $v \in V(G_2)\}$. We use mG to denote m vertex disjoint copies of G. Let X and Y be two subsets of $V(G)$. Then $N_X(Y) = \bigcup_{v \in Y} N_X(v)$. Moreover, we use $E(X,Y)$ to denote the set of edges between X and Y, and $e(X,Y)$ to denote the number of edges between X and Y. The minimum degree, the maximum degree, the independence number, the chromatic number, the connectivity, the number of components, the length of a shortest cycle and the length of a longest cycle in G are denoted by $\delta(G)$, $\Delta(G)$, $\alpha(G)$, $\chi(G)$, $\kappa(G)$, $\omega(G)$, $g(G)$ and $c(G)$, respectively. In the context of Ramsey theory, a k-edge coloring of G is an assignment of colors, i.e., integers from $\{1,2,\ldots,k\}$, to the edges of G, one color to each edge. This coloring is generally not proper, i.e., adjacent edges may be assigned the same color. A subgraph H of a colored G is called monochromatic if all edges of H have the same color.

Now we introduce the most commonly studied graphs in graph Ramsey theory, and we use the most commonly used names for them. A complete graph of order n is denoted by K_n, and a complete graph of order n with one arbitrary edge deleted is denoted by $K_n - e$. $K_{m,n}$ stands for a complete bipartite graph with bipartition classes of cardinality m and n. A cycle, a path, a star and a tree of order n are denoted by C_n, P_n, S_n and T_n, respectively. Here, $S_n = K_{1,n-1}$, and T_n can denote any of the possible trees on n vertices. A wheel $W_n = K_1 + C_n$ is a graph of order $n+1$, where the K_1 is called the hub of the wheel (Note that in the literature, sometimes W_n is used to denote a wheel of order n). A kipas $\widehat{K}_n = K_1 + P_n$ is a graph of order $n+1$. A fan $F_n = K_1 + nK_2$ is a graph of order $2n+1$. Here we note that obviously S_{2n+1} is a subgraph of F_n, F_n is a subgraph of $\widehat{K}_{2n}$, and all of them are subgraphs of W_{2n}. A book $B_n = K_2 + nK_1$ is a graph of order $n+2$. Let $C_{p,t}$ be a graph on $p+t$ vertices obtained from C_p by joining exactly one vertex of C_p to all vertices of tK_1. A broom $B_{p,t}$ is a tree on $p+t$ vertices obtained from P_p by joining exactly one end of P_p to all vertices of tK_1. See Figure 1.1 for examples of the above graphs on five vertices.

Let G be a graph on n vertices. A path in G of order n is called a Hamilton path (of G) and a cycle of order n in G is called a Hamilton cycle (of G). We say that G is hamiltonian if G contains a Hamilton cycle, and that G is Hamilton-connected if any two distinct vertices of G are connected by a Hamilton path. A graph G is pancyclic if it contains cycles of every length between 3 and n. A graph G is weakly pancyclic if it contains cycles of every length between $g(G)$ and $c(G)$. For a cycle C with a given orientation $\overrightarrow{C}$, we denote by $\overleftarrow{C}$ the cycle C with the reverse orientation. We use z_i^+ to denote the immediate successor of z_i, and z_i^- to denote its immediate predecessor on $\overrightarrow{C}$. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$; if $u = v$, then $u\overrightarrow{C}v = \{u\}$. By xP_my we mean a path from x to y on m vertices. An (X,Y) path is a path that starts at a vertex of X and terminates at a vertex of Y. A subdivision of an edge e is obtained by deleting e, adding a new vertex, and joining it to the former end vertices of e. Any graph derived from a graph G by a sequence of (edge) subdivisions is called a subdivision of G.

As we will see later in this thesis, graph Ramsey theory is closely associated with extremal graph results. For instance, to prove the existence of a cycle with a given length in a graph G (or its complement), we will often try to verify that

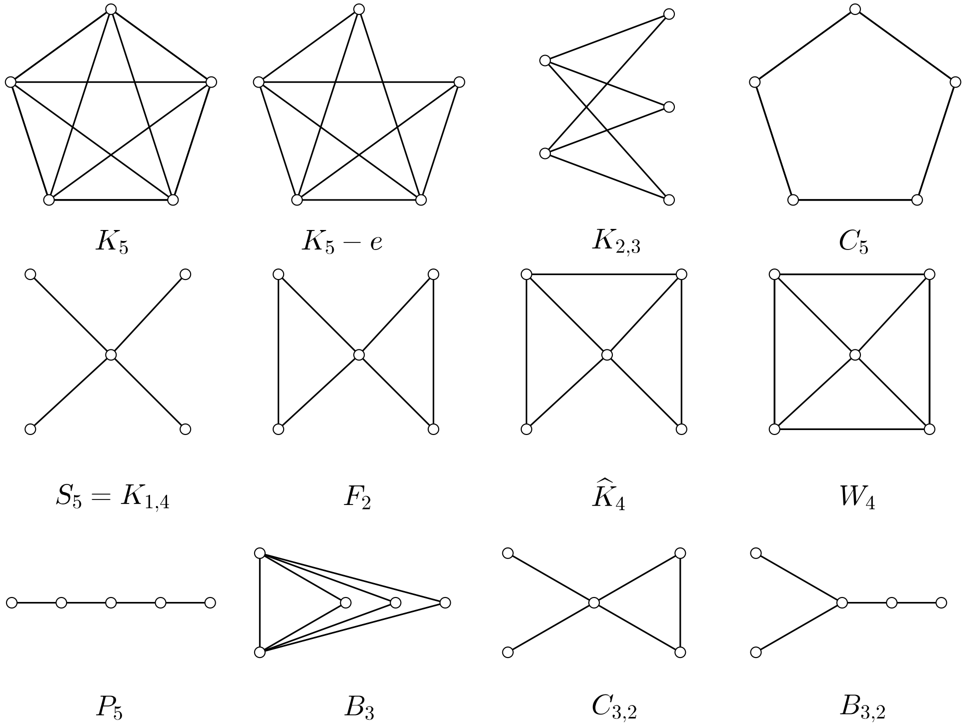

Figure 1.1: Some of the most commonly studied graphs in graph Ramsey theory

G is weakly pancyclic, and that $c(G)$ is not smaller than the length of the cycle we are looking for. At the end of this part, we therefore give a brief exposition of some conditions guaranteeing that a graph contains cycles of (at least) a given length. All the listed results in this section will be used as lemmas in some of the following chapters more than once. The following fundamental result was obtained by Dirac in 1952.

Theorem 1.1 (Dirac [38]). *Let G be a graph with $\delta(G) \geq 2$. Then $c(G) \geq \delta(G)+1$. If G is a 2-connected graph, then $c(G) \geq \min\{2\delta(G), |V(G)|\}$.*

In 1959, Erdős and Gallai established an edge condition providing a lower bound on $c(G)$. That is, $c(G) \geq 2e(G)/(|V(G)|-1)$.

Theorem 1.2 (Erdős and Gallai [50]). *Let G be a graph of order n and $3 \leq c \leq n$. If $e(G) \geq ((c-1)(n-1)+1)/2$, then $c(G) \geq c$.*

The investigation of pancyclic graphs was initiated by Bondy [12], who established several sufficient conditions for a graph to be pancyclic. A typical degree condition is given below.

Theorem 1.3 (Bondy [12]). *Let G be a graph with $\delta(G) \geq |V(G)|/2$. Then G is pancyclic, or $G = K_{r,r}$ (hence $|V(G)| = 2r$).*

The closure of a graph G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$ until no such pair remains. This closure operation was introduced by Bondy and Chvátal [13]. They showed that the closure is unique and that it preserves the existence of Hamilton cycles, in the following sense.

Theorem 1.4 (Bondy and Chvátal [13]). *Let G be a graph on at least three vertices. Then G is hamiltonian if and only if its closure is hamiltonian. Thus, if the closure of G is a complete graph on at least three vertices, then G is hamiltonian.*

Just as there are both degree conditions and edge conditions that determine lower bound on $c(G)$, Brandt established several sufficient degree and edge conditions for a nonbipartite graph to be weakly pancyclic. The most classical two are as follows.

Theorem 1.5 (Brandt et al. [16]). *Every nonbipartite graph G of order n with $\delta(G) \geq (n+2)/3$ is weakly pancyclic with $g(G) = 3$ or 4.*

Theorem 1.6 (Brandt [15]). *Every nonbipartite graph G of order n with $e(G) > (n-1)^2/4 + 1$ is weakly pancyclic with $g(G) = 3$.*

1.2 Ramsey theory

Ramsey theory, named after F.P. Ramsey and his eponymous theorem, is basically the study of the existence of some given substructure in a large structure. The theme of Ramsey theory is well described by Motzkin [101] by the following short statement: complete disorder is impossible. Ramsey theory intersects with a wide range of mathematical areas, including set theory, graph theory, combinatorial number theory, probability theory, analysis and also with theoretical computer science. Rosta's survey [115] reveals the Ramsey-type principles in many fields.

Three famous Ramsey-type theorems that are sometimes called the milestones of Ramsey theory [102], are Schur's theorem, Van der Waerden's theorem and Ramsey's theorem, in accordance with chronology. In an attempt to solve Fermat's last theorem over finite fields, Schur obtained Schur's theorem in 1916.

Theorem 1.7 (Schur [118]). ***Schur's theorem***
If the positive integers are partitioned into a finite number of classes, then one of the classes contains x, y, z with $x + y = z$.

Schur's theorem is in fact a lemma in Schur's original paper. With this theorem, Schur deduced that, for every $n \in \mathbb{N}$, the equation $x^n + y^n \equiv z^n \pmod p$ has a non-trivial (positive integers) solution for every sufficiently large prime p. The theorem is so concise and beautiful that Schur theory was one of the two deserving candidates for its name when Ramsey theory was created [121]. The finite version of Schur's theorem is: given $r \geq 1$, there exists a smallest positive integer $s = s(r)$ such that, for any r-coloring of $[1, s]$, there exists a monochromatic solution to $x + y = z$. Like with edge-coloring, in this r-coloring each integer is assigned one color from a set of r colors, without any restrictions. A monochromatic solution to $x + y = z$ is one in which x, y and z are assigned the same color. The numbers $s(r)$ are called Schur numbers. As it turns out, the only Schur numbers that are currently known are $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, and $s(4) = 45$.

Van der Waerden's theorem was proved by Van der Waerden in 1927.

Theorem 1.8 (Van der Waerden [138]). ***Van der Waerden's theorem***
If the positive integers are partitioned into a finite number of classes, then one of the classes contains arithmetic progressions of arbitrary length.

The finite version of Van der Waerden's theorem is: given $k, r \geq 1$, there exists a smallest positive integer $w(k;r)$ such that, for any r-coloring of $[1, w(k;r)]$, there exists a monochromatic arithmetic progression of length k. The numbers $w(k;r)$ are called Van der Waerden numbers. The only known nontrivial Van der Waerden numbers are $w(3;2) = 9$, $w(3;3) = 27$, $w(3;4) = 76$, $w(4;2) = 35$, $w(5;2) = 178$ and $w(6;2) = 1132$.

Being interested in decision procedures for logical systems, Ramsey [112] obtained what is now commonly known as Ramsey's theorem in 1930. In the next part, we shortly describe the infinite and finite versions of Ramsey's theorem.

Theorem 1.9 (Ramsey [112]). ***Ramsey's theorem, infinite version***
For any positive integers k and r, if the collection of all r-element subsets of an infinite set S is colored using k colors, then S contains an infinite subset S_1 such that all r-element subsets of S_1 are assigned the same color.

Theorem 1.10 (Ramsey [112]). ***Ramsey's theorem, finite version***
For any positive integers r, n and k, there is a smallest integer $m_0 = R(r,n,k)$ such that if $m \geq m_0$ and the collection of all r-element subsets of an m-element set S_m is colored using k colors, then S_m contains an n-element subset S_n such that all r-element subsets of S_n are assigned the same color.

Note that the notion color can be replaced by partition or class, which is the language of Ramsey's original paper. If $r = 1$, then Ramsey's theorem is in fact the Pigeonhole Principle. Thus, Ramsey theory is often considered as a generalization of it. If $r = k = 2$, then it is the graph case of Ramsey's theorem, which models the problem as it was stated at the beginning of this chapter. That is, if a graph contains sufficiently many vertices (depending on r,n,k), then it must contain either a complete set or an independent set of vertices of size n. Or, alternatively, any 2-coloring of its edges contains a monochromatic K_n. Also, $R(2,3,k)-1$ is the maximum number of vertices in a complete graph the edges of which can be decomposed into k triangle-free graphs.

Thinking of Schur's theorem as a theorem about the equation $x + y - z = 0$, can we obtain an analogous result by changing $x + y - z = 0$ into another homogeneous linear equation? Rado, a PhD student of Schur, generalized Schur's theorem by adopting the above idea. To describe Rado's theorem, we need the following definition.

Definition 1. Let S be a (system of) linear homogeneous equation(s). We say that S is regular if, for any finite partition of the positive integers, there is always a solution to S in one of the classes.

Using the above definition, Schur's theorem can be restated as: the equation $x+y-z=0$ is regular. We now state Rado's theorem for a single equation, which is an obvious generalization of Schur's theorem.

Theorem 1.11 (Rado [109]). ***Rado's single equation theorem***
Let E be a linear equation $\Sigma_{i=1}^{n} a_i x_i = 0$, where all a_i are nonzero integers and $n \geq 2$. Then E is regular if and only if some nonempty subset of the coefficients a_i sums up to zero.

Thinking more deeply, we may find that Van der Waerden's theorem can be restated as: the system $Ax = 0$ is regular, where A is the following matrix:

$$A = \begin{pmatrix} 1 & -1 & & & & 1 \\ & 1 & -1 & & & 1 \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & -1 & 1 \end{pmatrix}$$

The full version of Rado's theorem can also be viewed as a generalization of Van der Waerden's theorem. To state the theorem, we need one more definition.

Definition 2. Let $\overrightarrow{c_i}$ denote the i-th column of A. The matrix A satisfies the columns condition provided that there exists a partition $\mathrm{C_1, C_2, \ldots, C_n}$ of the column indices such that if $\overrightarrow{s_i} = \Sigma_{j \in \mathrm{C}_i} \overrightarrow{c_j}$, then

(1) $\overrightarrow{s_1} = \overrightarrow{0}$; and
(2) for all $i \geq 2$, $\overrightarrow{s_i}$ can be written as a rational linear combination of the $\overrightarrow{c_j}$'s in the C_k with $k < i$.

Theorem 1.12 (Rado [109]). ***Rado's theorem***
A system $Ax = 0$ is regular if and only if the matrix A satisfies the columns condition.

Ramsey expressed his theorem in a completely mathematical language, making it one of the first combinatorial results that attracted attention of mathematicians in general. Perhaps it is for this reason that Ramsey's theorem was never regarded as a puzzle or a combinatorial curiosity. However, it was largely through the efforts of Erdős that the subject enjoys the current high level of popularity and research activity. Erdős together with Szekeres initiated the applications of Ramsey's theorem in geometry by proving the following in 1935.

Theorem 1.13 (Erdős and Szekeres [51]). ***Erdős-Szekeres' theorem***
For any positive integer $n \geq 3$ there is an integer m_0 such that any set of at least m_0 points in the plane in general position (no three points lie on a line) contains n points that form a convex polygon.

Erdős-Szekeres' theorem is also called the Happy Ending theorem because it led to the marriage of George Szekeres and Esther Klein (who proposed the theorem for $n = 4$). For many other Ramsey-type theorems like Hales-Jewett's theorem and Graham-Leeb-Rothschild's theorem, and an introduction to nearly all areas related to Ramsey theory, see the monograph [65]. An exciting history of the discovery of Ramsey theory can be found in [121]. See [86, 102, 103, 122] for more information.

1.3 Graph Ramsey theory

1.3.1 Classical Ramsey numbers

The problem that was introduced at the beginning of this chapter is a special case of the problem of determining exact values of (classical) Ramsey numbers. Now we state the following unambiguous definition.

Definition 3. (Classical Ramsey number) For positive integers $m_1, m_2, \ldots, m_k$, the (classical) Ramsey number $R(m_1, m_2, \ldots, m_k)$ is the smallest positive integer N such that for any k-edge coloring of the complete graph K_N, there is a monochromatic subgraph K_{m_i} with color i.

For $k \geq 3$, the only known nontrivial classical Ramsey number is $R(3,3,3)$, which is 17, as shown by Greenwood and Gleason [67]. For $k = 2$, they also established the initial values $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$ and $R(4,4) = 18$ in 1955. In fact, proving $R(3,3) \leq 6$ was a problem in Putnam Mathematical Competition in March 1953. Kéry [83] proved that $R(3,6) = 18$ in 1964, and Graver and Yackel [66] found that $R(3,7) = 23$ in 1968. The determination of other classical Ramsey numbers required the use of computers. Grinstead and Roberts [68] obtained $R(3,9) = 36$ in 1982; Mckay and Zhang [98] determined $R(3,8) = 28$ in 1992; Mckay and Radziszowski [96] computed $R(4,5) = 25$ in 1995. It has been proved by Exoo [52] that $R(5,5) \geq 43$ and by Mckay and Radziszowski [97] that $R(5,5) \leq 49$. To explain the difficulty of determining

$R(m,n)$, Erdős [44] came up with the following famous joke, which had a few different variants during his talks.

> Suppose an evil spirit would tell us: unless you give me the value of $R(5,5)$ in a year, I will exterminate the human race. Our best strategy probably would be to have our computers working on $R(5,5)$ and we could have the value of $R(5,5)$ in time. If he would ask for $R(6,6)$ our best strategy would be to try to destroy him/her/it before he destroys us.

All known nontrivial Ramsey numbers $R(m,n)$ are given below.

m	3	3	3	3	3	3	3	4	4
n	3	4	5	6	7	8	9	4	5
$R(m,n)$	6	9	14	18	23	28	36	18	25

The following three inequalities are the classical ones on Ramsey numbers, which can be found in many combinatorics textbooks. The two upper bounds are due to Erdős and Szekeres [51], the lower bound on diagonal Ramsey numbers is due to Erdős [43].

$$R(m,n) \le R(m-1,n) + R(m,n-1) \text{ for } m,n \ge 2$$

$$R(m,n) \le \binom{m+n-2}{m-1}$$

$$R(n,n) \ge 2^{n/2}$$

With regard to asymptotic bounds for Ramsey numbers, the behavior of $R(3,k)$ for large k is one of the most important results. We see that $R(3,k) \le n$ if for every triangle-free graph G of order n, there exists an independent set I with $|I| \ge k$; and $R(3,k) > n$ if there exists a triangle-free graph G of order n which does not have an independent set I with $|I| \ge k$. Generally, there is a big difference between proofs of lower and upper bounds for Ramsey numbers. Both the lower and upper bounds of $R(3,k)$ have been improved several times, and the order of $R(3,k)$ as a function of k was finally obtained by Kim [84] in 1995, when he showed:

$$R(3,k) = \Theta(\frac{k^2}{\ln^2 k}).$$

Kim resolved this 64-year-old problem by using many advanced and sophisticated probabilistic methods, and was awarded the Fulkerson Prize for this achievement.

1.3.2 Generalized Ramsey numbers

The study of generalized Ramsey numbers dates from 1967. In the 1970s, Chvátal and Harary introduced the term *Generalized Ramsey Theory for Graphs* and started an impressive series of papers under this title. They generalized the notion of the Ramsey number by including in the study the existence of subgraphs other than complete graphs. Since then the subfield has grown in a vigorous way and has attracted remarkable attention. The generalized Ramsey numbers are defined as follows.

Definition 4. (Generalized Ramsey number) Given two graphs G_1 and G_2, the (generalized) Ramsey number $R(G_1, G_2)$ is the smallest positive integer N such that, for any graph G of order N, either G_1 is a subgraph of G, or G_2 is a subgraph of the complement of G.

We can also define generalized Ramsey number in coloring language. That is, $R(G_1, G_2)$ is the smallest integer N such that, for any 2-edge coloring of K_N with red and blue, there exists either a red subgraph G_1 or a blue subgraph G_2. It is easy to check that $R(G_1, G_2) = R(G_2, G_1)$ and $R(K_m, K_n) = R(m, n)$. The definition of multicolor Ramsey numbers is an obvious generalization. Given graphs $G_1, G_2, \ldots, G_k$, the (multicolor) Ramsey number $R(G_1, G_2, \ldots, G_k)$ is the smallest positive integer N such that, for any k-edge coloring of K_N, there exists a monochromatic subgraph G_i with color i.

In the classical case, only ten nontrivial Ramsey numbers are known, including $R(3, k)$ for $3 \le k \le 9$, $R(4, 4)$, $R(4, 5)$ and $R(3, 3, 3)$. But in the generalized case, many more exact values are known, in which the most studied (sub)graphs include cycles, wheels, paths, stars, trees, books and fans. We will encounter many examples of results involving these graphs throughout this thesis.

Now we introduce a general lower bound which often yields the exact values. Let F be a connected graph of order p, and let $\chi(H)$ be the chromatic number of a graph H. Chvátal and Harary [36] proved that $R(F, H) \ge (p-1)(\chi(H)-1)+1$. This result is based on a canonical coloring of K_{N-1}, where $N = (p-1)(\chi(H)-1)+1$: we color the edges of a $(\chi(H)-1)K_{p-1}$ red, and all other edges blue. Clearly there is no red F and no blue H, and hence the inequality holds. Burr [17] generalized this lower bound by adding another parameter $s(H)$, which is the chromatic surplus of H. In other words, it is the minimum number of vertices in some color class under all proper vertex colorings of H with $\chi(H)$ colors. The following theorem was established.

Theorem 1.14 (Burr [17]). *$R(F,H) \geq (p-1)(\chi(H)-1)+s(H)$ for $p \geq s(H)$.*

Moreover, Burr defined F to be H-good if the equality holds in Theorem 1.14.

The rest of this section is devoted to a list of known results that give general exact values of $R(F,H)$ for whole classes of graphs. We present them in chronological order. In most cases, we see that F is H-good for two given graphs F and H.

Theorem 1.15 (Gerencsér and Gyárfás [62]). *For $m \geq n \geq 2$, $R(P_m,P_n) = m + \lfloor n/2 \rfloor - 1$.*

Theorem 1.16 (Harary [73]). *$R(K_{1,n},K_{1,m}) = n+m-\varepsilon$, where $\varepsilon = 1$ for n,m even, and $\varepsilon = 0$ otherwise.*

The Ramsey number problem for cycles versus cycles was solved by Rosta, and also by Faudree and Schelp independently, as shown by the following result. A simpler proof for this result was later provided by Károlyi and Rosta [82].

Theorem 1.17 (Rosta [114], Faudree and Schelp [58] independently).

$$R(C_m,C_n) = \begin{cases} 6 & \text{for } (m,n)=(3,3) \text{ or } (4,4), \\ 2n-1 & \text{for } 3 \leq m \leq n,\ m \text{ odd},\ (m,n) \neq (3,3), \\ n-1+m/2 & \text{for } 4 \leq m \leq n,\ m,n \text{ even},\ (m,n) \neq (4,4), \\ \max\{n-1+m/2, 2m-1\} & \text{for } 4 \leq m < n,\ m \text{ even},\ n \text{ odd}. \end{cases}$$

We see that either C_m is C_n-good, or C_n is C_m-good, except for $(m,n)=(3,3)$ and $(4,4)$. The following theorem states that, for any $m \geq 3$ and $n \geq 2$, either C_m is P_n-good, or P_n is C_m-good.

Theorem 1.18 (Faudree et al. [54]).

$$R(C_m,P_n)=\begin{cases} 2n-1 & \textit{for } 3\le m\le n,\ m\ \textit{odd},\\ n-1+m/2 & \textit{for } 4\le m\le n,\ m\ \textit{even},\\ \max\{m-1+\lfloor n/2\rfloor,2n-1\} & \textit{for } 2\le n\le m,\ m\ \textit{odd},\\ m-1+\lfloor n/2\rfloor & \textit{for } 2\le n\le m,\ m\ \textit{even}.\end{cases}$$

In 1977, Chvátal gave an ingenious proof of the result that T_n is K_m-good for any positive integers m,n.

Theorem 1.19 (Chvátal [32]). $R(T_n,K_m)=(n-1)(m-1)+1$ *for all positive integers m and n.*

For more detailed information on graph Ramsey numbers, and open problems on this topic, we refer the reader to the dynamic survey of Radziszowski [110].

1.4 Outline of results

The thesis contains nine chapters with many new results on generalized Ramsey numbers for graphs. We mainly concentrate on two-color Ramsey numbers, and always refine or generalize known results. Planar Ramsey numbers, star-critical Ramsey numbers and upper size Ramsey numbers are also included.

There are three chapters involving Ramsey numbers for cycles versus wheels. For small odd cycles versus large wheels, Zhou [145] showed that $R(W_n,C_m)=2n+1$ for odd m and $n\ge 5m-7$. Even though the correctness of the proof is questionable, there is no doubt on the conclusion. We improve the result by reducing the lower bound on n from $n\ge 5m-7$ to $n\ge 3(m-1)/2$. That is, $R(W_n,C_m)=2n+1$ for m odd, $n\ge 3(m-1)/2$ and $(m,n)\ne(3,3),(3,4)$. If m,n are odd and $m<n\le 3(m-1)/2$, then W_n is not C_m-good any more. Instead, C_m is W_n-good, which means $R(W_n,C_m)=3m-2$ for m,n odd and $m<n\le 3(m-1)/2$. Moreover, we show that $R(C_7,W_n)=2n+1$ for $8\le n\le 10$, where the Ramsey number $R(C_7,W_8)$ has not been included in the former expressions. Then we give an alternative proof of $R(C_7,W_n)=2n+1$ for $n\ge 11$ by an inductive argument.

Chapter 3 concerns the Ramsey numbers for small even cycles versus large stars or wheels. The well-known theorem on cycle-star Ramsey numbers is due to Lawrence [87], who proved that $R(C_m,K_{1,n}) = 2n+1$ for odd $m \le 2n-1$, and $R(C_m,K_{1,n}) = m$ for $m \ge 2n$. For even $m < 2n$, not many results on exact values of these Ramsey numbers are known. In fact, all generic results we know of deal with the case that $m = 4$. We prove that $R(C_m,K_{1,n}) = 2n$ for even m with $n < m \le 2n$; and $R(C_m,K_{1,n}) = 2m-1$ for even m with $3n/4+1 \le m \le n$. Moreover, we calculate $R(C_6,K_{1,n})$ for $7 \le n \le 11$ and $R(C_6,W_9)$. When comparing the two Ramsey numbers for $R(W_n,C_4)$ and $R(S_{n+1},C_4)$, we find that there is an infinite number of values of n for which they are equal. In this way, we demonstrate that $R(W_n,C_4) = R(S_{n+1},C_4)$ for $n \ge 6$. In addition, we have another result, which is $R(W_n,C_m) = 3m-2$ for n odd, m even and $m < n < 3m/2$.

Chapter 4 deals with the Ramsey numbers for small wheels versus large cycles. There are three main results in this chapter. The first one is, $R(C_m,W_n) = 2m-1$ for even n and $m \ge n+502$. This outcome refines a theorem by Chen et al. [26], which is $R(C_m,W_n) = 2m-1$ for even n and $m \ge 3n/2+1$. The second result considers generalized odd wheels and confirms that $R(C_m,W_{2,n}) = 4m-3$ for n odd, $m \ge 9n/8+1$. The last result is based on the definition of Ramsey unsaturated graph, which improves the results of Ali and Surahmat [2]. We prove that C_4 is Ramsey saturated with respect to W_n; C_5 is Ramsey saturated with respect to W_4; W_4 is Ramsey saturated with respect to C_4. The unsaturated cases are, C_m with respect to W_n for $m \ge \max\{n+1,6\}$; W_n with respect to C_m for $n \ge \max\{m,5\}$; C_5 with respect to W_3; and W_4 with respect to C_3.

We study the Ramsey numbers for trees versus fans or wheels in Chapter 5. It is first showed that $R(S_n,F_m) = 2n-1$ for $n \ge m(m-1)+1$ and $m \ne 3,4,5$, and the lower bound $n \ge m(m-1)+1$ is best possible. $R(S_n,F_m) = 2n-1$ for $n \ge 6(m-1)$ and $m = 3,4,5$. Then for an arbitrary tree of order n, we prove that $R(T_n,F_m) = 2n-1$ for all integers $n \ge 3m^2-2m-1$. We may obtain by induction that $R(T_n,K_{\ell-1}+mK_2) = \ell(n-1)+1$ for $\ell \ge 2$ and $n \ge 3m^2-2m-1$. The rest of this chapter deals with ES-trees. A tree T of order n is called an ES-tree if every graph $G = (V,E)$ with $|E(G)| > |V(G)|(n-2)/2$ contains T as a subgraph. We show that $R(T_n,C_m) = 2n-1$ for $T_n \in \mathscr{T}$, odd $m \ge 3$ and $n \ge m-1$; $R(T_n,W_m) = 3n-2$ for $T_n \in \mathscr{T}$, odd $m \ge 3$ and $n \ge m-2$. Here, $\mathscr{T}$ denotes the set of all ES-trees.

In Chapter 6, we take up the Ramsey numbers for fans versus fans, wheels and complete graphs. Lin and Li [91] calculated that $R(F_n,F_2)=4n+1$ for $n\geq 2$; and $R(F_n,F_m)\leq 4n+2m$ for $n\geq m\geq 2$. We have a general result: $R(F_n,F_m)=4n+1$ for $n\geq \max\{m^2-m/2, 11m/2-4\}$. Moreover, F_n is not F_m-good for $m\leq n< m(m-1)/2$. For Ramsey numbers of fans versus wheels of even order, Surahmat et al. [129] proved that $R(F_n,W_3)=6n+1$ for $n\geq 3$. We generalize this result by showing that $R(F_n,W_m)=6n+1$ for odd $m\geq 3$ and $n\geq (5m+3)/4$. On the other hand, taking W_3 as K_4, we give an analogous conclusion for K_5 with a different method: $R(F_n,K_5)=8n+1$ for $n\geq 5$.

In Chapter 7, we give a short proof of Ramsey numbers for paths versus kipases. Very recently, all path-wheel Ramsey numbers were determined by Li and Ning in [88]. Since $R(P_n,\widehat{K}_m)$ can be easily determined for $m\geq 2n$, we here give a short proof for $m\leq 2n-1$, by discussing the length of a longest cycle in the neighborhood of a vertex with maximum degree. The result is $R(P_n,\widehat{K}_m)=\max\{2n-1, \lceil 3m/2\rceil-1, 2\lfloor m/2\rfloor+n-2\}$ for $m\leq 2n-1$ and $m,n\geq 2$.

In Chapter 8, we discuss Ramsey goodness for the union of some graphs. We establish the following result. Let $c(F)$ denote the order of a largest component of a graph F, and $k_i(F)$ the number of components of order i in F. Let G be a graph with $\chi(G)=m\geq 2$, $s(G)$ the chromatic surplus of G, F a graph with G-good components, $f(j)=(j-1)(m-2)+\sum_{i=j}^{c(F)} ik_i(F)$, and $f(j_0)=\max_{1\leq j\leq c(F)} f(j)$. Let H be a graph with k components $H_1,\ldots,H_k$ such that for $1\leq t\leq k$, $|H_t|\geq j_0$ and $R(H_t,G)\leq f(j_0)+s(G)-1+\sum_{i=1}^{t}|H_i|$. Then $R(F\cup H,G)=f(j_0)+s(G)-1+|H|$. Since the components of H are not necessarily G-good components, we obtain an extension to the results of Bielak [10].

In the same paper, Bielak proved that $R(C_{5,t},C_5)=2t+9$ for $t\geq 0$; we prove the generalization that $R(C_{p,t},C_q)=2(p+t)-1$ for odd q, $p+t\geq q$ and $(p,q,t)\neq(3,3,0)$. Bielak also showed that $R(C_{5,t},2C_5)=2t+10$ for $t\geq 2$; we prove the generalization that $R(C_{p,t},C_q\cup C_r)=2(p+t)$ for q,r odd, $p\geq q\geq r\geq 5$ and $t\geq (r-1)/2$. Two other results are about $R(B_{p,t},C_q)$ for odd q; and $R(B_{p,t},W_n)$ for odd n and $p+t\geq (n+1)/2$.

In Chapter 9, we consider planar Ramsey numbers. For two given graphs G_1 and G_2, the planar Ramsey number $R(G_1,G_2)$ is the smallest integer N such that for any planar graph G of order N, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. We confirm all planar Ramsey numbers for complete graphs versus wheels, which generalize the result of Zhou et al. [144]

who determined the triangle-wheel planar Ramsey numbers. Our first result on planar Ramsey numbers is:

$$PR(K_m, W_n) = \begin{cases} 13 & \text{for } m \geq 4 \text{ and } 3 \leq n \leq 6, \\ 14 & \text{for } m \geq 4 \text{ and } n = 7, \\ n+6 & \text{for } m \geq 4 \text{ and } n \geq 8. \end{cases}$$

Furthermore, we determine all planar Ramsey numbers $PR(K_m^-, W_n)$, and give a result on 2-connected graphs versus wheels: Let F be a 2-connected graph with $|V(F)| \geq 13$. Then $PR(F, W_n) = n+6$ for $n \geq 45$ or $n = 18$, 19, 29, 30, 31, 40, 41, 42, 43. The Ramsey numbers $PR(K_m^-, W_n)$ are as below.

<table>
<tr><th>n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>$n \geq 11$</th></tr>
<tr><td>$PR(K_3^-, W_n)$</td><td>7</td><td colspan="8">$2\lceil n/2 \rceil + 1$</td></tr>
<tr><td>$PR(K_4^-, W_n)$</td><td colspan="3">10</td><td colspan="4">$2\lfloor n/2 \rfloor + 5$</td><td>$n+4$</td><td>$n+5$</td></tr>
<tr><td>$PR(K_m^-, W_n)$ for $m \geq 5$</td><td colspan="4">13</td><td>14</td><td colspan="4">$n+6$</td></tr>
</table>

In Chapter 10, we study the upper size Ramsey number $u(G_1, G_2)$, defined by Erdős and Faudree [46]; and the star-critical Ramsey number $r_*(G_1, G_2)$, defined by Hook and Issak [77]. We define Ramsey-full graphs and size Ramsey good graphs, and perform a detailed study on the two definitions. For example, we obtain $u(nK_k, mK_l)$ and $r_*(nK_k, mK_l)$ for $k, l \geq 3$ and large m, n; $u(C_n, C_m)$ for m odd, $n > m \geq 3$; and $r_*(C_n, C_m)$ for m odd, $n \geq m \geq 3$ and $(m, n) \neq (3, 3)$.

Chapter 2

Small odd cycles versus wheels

2.1 Introduction

Because of the Hamilton-connected property of wheels and existing conclusions, the research of Ramsey numbers for cycles versus wheels is meaningful and has a prospective future to be fully solved. In this chapter we investigate the Ramsey numbers of small odd cycles versus large wheels, which is $R(W_n, C_m)$ with m odd and $n > m$.

Recall the definition of Ramsey goodness which was created by Burr [17]: a connected graph F is H-good, if

$$R(F,H) = (|F|-1)(\chi(H)-1) + s(H) \text{ for } |F| \geq s(H),$$

where $\chi(H)$ denotes the chromatic number of H and $s(H)$ the chromatic surplus of H.

The smallest case considered in this chapter is $R(W_4, C_3)$. For $|V(G_1)| \leq 4$ and $|V(G_2)| = 5$, Clancy [37] calculated almost all of the pairs $R(G_1, G_2)$, including $R(W_4, C_3) = 11$. Radziszowski and Jin [111] proved that the graph in Figure 2.1 is the only graph of order 10 which contains no W_4 and its complement contains no C_3. Obviously, W_4 is not C_3-good.

For $n = 5$, Faudree et al. [56] calculated that $R(W_5, C_3) = 11$. For $n \geq 6$, Burr and Erdős proved that all W_n is C_3-good by induction on n, which was the first

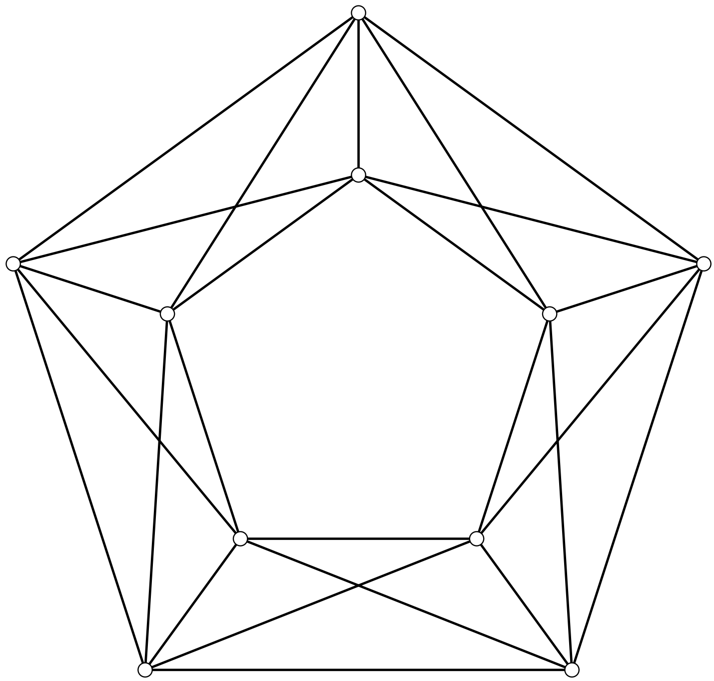

Figure 2.1: G contains no W_4 and $\overline{G}$ contains no C_3

paper involving a general case for cycle-wheel Ramsey numbers.

Theorem 2.1 (Burr and Erdős [20]). $R(W_n, C_3) = 2n + 1$ *for* $n \geq 5$.

For odd $m \geq 3$, Zhou [145] showed that W_n is C_m-good if $n \geq 5m - 7$. Unfortunately, the correctness of the proof is questionable since the author didn't give the proofs for the two key claims in the paper, and so whether W_n is C_m-good for odd $m \geq 5$ is still open. Sun and Chen [126] considered the wheels which are C_5-good by discussing the independent number of a graph G, and obtained the following.

Theorem 2.2 (Sun and Chen [126]). $R(W_n, C_5) = 2n + 1$ *for* $n \geq 6$.

In this chapter, we determine the values of $R(W_n, C_m)$ for odd m in a more general situation. The main results of this chapter are as follows.

Theorem 2.3. *$R(W_n, C_m) = 2n + 1$ for m odd, $n \geq 3(m-1)/2$ and $(m,n) \neq (3,3),(3,4)$.*

Theorem 2.4. *$R(W_n, C_m) = 3m - 2$ for m, n odd and $m < n \leq 3(m-1)/2$.*

Instead of proving the two theorems separately, we derive them from one single proof. The relations between the size and the weakly pancyclic property in a graph and its complement are of great importance in the proof.

Clearly, Theorem 2.3 says that W_n is C_m-good for odd $m \geq 3$, $n \geq 3(m-1)/2$ and $(m,n) \neq (3,3),(3,4)$, and Theorem 2.4 shows that C_m is W_n-good for odd m, n and $m < n \leq 3(m-1)/2$.

By now, on Ramsey numbers for small odd cycles versus wheels, all of them have been determined except the case when n is even and $m < n < 3(m-1)/2$. For this case, we have the following conjecture.

Conjecture 1. $R(W_n, C_m) = 2n + 1$ for m odd, n even and $m < n < 3(m-1)/2$.

Clearly, if there exists an even n such that $m < n < 3(m-1)/2$, then $m \geq 7$. If $m = 7$, then $3(m-1)/2 = 9$ and so $n = 8$. We may verify that Conjecture 1 is true for $m = 7$. A somewhat stronger theorem is as below.

Theorem 2.5. *$R(W_n, C_7) = 2n + 1$ for $8 \leq n \leq 10$.*

With the above initial step, we can prove that W_n is C_7-good for $n \geq 11$ by induction. Even though the conclusion is already contained in Theorem 2.3, the alternative proof itself is of interest.

Theorem 2.6. *$R(W_n, C_7) = 2n + 1$ for $n \geq 8$.*

Now the smallest case needing to be settled in Conjecture 1 is $R(C_9, W_{10})$.

2.2 Proof of Theorems 2.3 and 2.4

In order to prove Theorems 2.3 and 2.4, we need some preliminaries including Theorems 1.1, 1.2, 1.3, 1.5, 1.6 and the following lemmas. The first is due to Faudree et al., who showed a lower bound for the circumference of a graph and its complement.

Lemma 2.1 (Faudree et al. [55]). *Let G be a graph of order $n \geq 6$. Then* $\max\{c(G), c(\overline{G})\} \geq \lceil 2n/3 \rceil$.

The second lemma is almost a trivial corollary of Theorem 1.3.

Lemma 2.2 (Lawrence [87]). $R(C_m, K_{1,n}) = m$ *for* $m \geq 2n$.

Lemma 2.3. *Let C be a longest cycle of a graph G and $v_1, v_2 \in V(G) - V(C)$. Then* $|N_C(v_1) \cup N_C(v_2)| \leq \lfloor |C|/2 \rfloor + 1$.

Proof. Let $C = u_1 u_2 \cdots u_l u_1$. If there exist $u_i u_{i+1}, u_j u_{j+1}, u_k u_{k+1} \in E(C)$ with $i < j < k$ such that $u_i, u_{i+1}, u_j, u_{j+1}, u_k, u_{k+1} \in N_C(v_1) \cup N_C(v_2)$, where the subscripts are taken modulo l, then v_1 or v_2 has at least two neighbours in $\{u_i, u_j, u_k\}$. By symmetry, we may assume that $u_i, u_j \in N_C(v_1)$. By the maximality of C, $u_{i+1}, u_{j+1} \notin N_C(v_1)$ which implies that $u_{i+1}, u_{j+1} \in N_C(v_2)$. Thus, there is a cycle $v_1 u_j \overleftarrow{C} u_{i+1} v_2 u_{j+1} \overrightarrow{C} u_i v_1$, the length of which is longer than C, a contradiction. Therefore, C has at most two edges whose ends are contained in $N_C(v_1) \cup N_C(v_2)$, and hence $|N_C(v_1) \cup N_C(v_2)| \leq \lfloor |C|/2 \rfloor + 1$. □

Lemma 2.4. *Let $m \geq 5$ be an odd integer and (X, Y) a partition of $V(G)$ of a graph G such that $|Y| \geq (m+1)/2$ and $|X - (N(y_i) \cup N(y_j))| \geq (m-1)/2$ for any $y_i, y_j \in Y$. If $\overline{G}$ contains no C_m, then $G[Y]$ is a complete graph.*

Proof. Set $Y = \{y_1, y_2, \ldots, y_k\}$ and $l = (m+1)/2$, then $k \geq l$. If $G[Y]$ is not a complete graph, say, $y_1 y_2 \notin E(G)$, then since $|X - (N(y_i) \cup N(y_j))| \geq (m-1)/2$ for any $y_i, y_j \in Y$, we can choose $x_1, x_2, \ldots, x_{l-1} \in X$ such that $y_i, y_{i+1} \notin N(x_{i-1})$ for $i = 2, \ldots, l-1$ and $y_1, y_l \notin N(x_{l-1})$, which implies that $y_1 y_2 x_1 y_3 x_2 y_4 \cdots x_{l-2} y_l x_{l-1} y_1$ is a C_m in $\overline{G}$, a contradiction. □

We first prove the following upper bound, then Theorems 2.3 and 2.4 can easily be shown by providing the lower bounds.

Theorem 2.7. $R(W_n, C_m) \leq \max\{2n+1, 3m-2\}$ *for odd* $m \geq 5$ *and* $n > m$.

Proof. Let G be a graph of order $N = \max\{2n+1, 3m-2\}$ and $v_0 \in V(G)$ with $d(v_0) = \Delta(G) = d$. Set $H = G[N(v_0)]$ and $Z = V(G) - N[v_0]$. Suppose to the contrary that neither G contains a W_n nor $\overline{G}$ has a C_m.

If $\overline{G}$ is bipartite, then $\alpha(\overline{G}) \geq \lceil N/2 \rceil \geq n+1$, which implies that G has a K_{n+1} and thus G has a W_n, a contradiction. Hence $\overline{G}$ is nonbipartite. If $\delta(\overline{G}) \geq \lceil (N+2)/3 \rceil$, then since $\lceil (N+2)/3 \rceil \geq m$, $\overline{G}$ contains a C_m by Theorems 1.5 and 1.1, a contradiction. Therefore, we have

$$d = \Delta(G) \geq \lfloor (2N-2)/3 \rfloor = \max\{\lfloor 4n/3 \rfloor, 2m-2\}. \tag{2.1}$$

Noting that $N = 2n+1$ implies that $n \geq 3(m-1)/2$ and $N = 3m-2$ implies that $n \leq 3(m-1)/2$, after an easy calculation, we can deduce that

$$N - m \geq 4n/3 \text{ and } d - n \geq (m-1)/2. \tag{2.2}$$

If H is bipartite, let $H = (X,Y)$. Noting that $\overline{G}$ has no C_m and $|H| = d$, we have $|X| = |Y| = m-1$ by (2.1). For the same reason, we have $e(X,Y) \geq |X| \cdot |Y| - 1$, $d_X(z) \geq |X| - 1$ and $d_Y(z) \geq |Y| - 1$ for any $z \in Z$. If $E(G[Z]) = \emptyset$, then since $|G - N(v_0)| \geq m$, we see that $\overline{G}[Z \cup \{v_0\}]$ contains a C_m, and so we may assume that $z_1 z_2 \in E(G[Z])$. Let $X_1 \cup Y_1 \subseteq N(z_1)$ with $X_1 \subseteq X$, $Y_1 \subseteq Y$ and $|X_1| = |Y_1| = m-2$. Since $G[X_1 \cup Y_1] = K_{m-2,m-2}$ or $K_{m-2,m-2} - e$, $d_{X_1}(z_2) \geq m-3 \geq 2$ and $d_{Y_1}(z_2) \geq m-3 \geq 2$, it is not difficult to see that $G[X_1 \cup Y_1 \cup \{z_2\}]$ is pancyclic. By (2.1), $2(m-2)+1 = d-2+1 \geq \lfloor 4n/3 \rfloor - 2 + 1 \geq n$. Thus, $G[X_1 \cup Y_1 \cup \{z_2\}]$ contains a C_n, which together with z_1 forms a W_n with the hub z_1, a contradiction. Hence,

$$H \text{ is nonbipartite.} \tag{2.3}$$

Suppose that $\overline{H} = (X,Y)$ is bipartite. We first show the following three claims.

Claim 1. $(m+1)/2 < |X| \leq n-2$ and $(m+1)/2 < |Y| \leq n-2$.

Proof. If G contains a $K = K_n$, then $d_K(v) \leq 2$ for any $v \in V(G) - V(K)$ since otherwise G has a W_n. Thus, $n - d_K(v') - d_K(v'') \geq n-4$ for any $v', v'' \in V(K) - (N(v') \cup N(v''))$. Since $n > m$ and $m \geq 5$, $n - 4 \geq (m-1)/2$. By Lemma 2.4, $G - V(K)$ is a complete graph of order $N - n \geq n+1$ and so G contains a W_n, a contradiction. Thus, G has no K_n, which implies that $|X| \leq n-2$ and $|Y| \leq n-2$.

Noting that $|X| + |Y| = d$, we have $|X| \geq d - (n-2) > (m+1)/2$ by (2.2). By the symmetry of X and Y, $|Y| > (m+1)/2$. □

By (2.1), $|H| = d = |X| + |Y| \geq \lfloor 4n/3 \rfloor$. Since $\overline{H}$ is bipartite, $G[X]$ and $G[Y]$ are complete graphs in G, which implies that $E(X,Y)$ contains no two independent edges for otherwise H has a C_n and hence G has a W_n with the hub v_0.

Consequently, there is some vertex $v \in V(H)$, such that $E(X-\{v\}, Y-\{v\}) = \emptyset$, that is, $\overline{H}-v$ is complete bipartite graph. Let $v_1 \in V(H)$ be given and $\overline{H}-v_1$ a complete bipartite graph. Set $X' = X - \{v_1\}$ and $Y' = Y - \{v_1\}$.

Claim 2. There exists some $v \in V(G)-\{v_0\}$ such that $E(X_1, Z_1) = |X_1|\cdot|Z_1|$ and $E(Y_1, Z_1) = |Y_1|\cdot|Z_1|$, where $X_1 = X' - \{v\}$, $Y_1 = Y' - \{v\}$ and $Z_1 = Z - \{v\}$.

Proof. By Claim 1, $|X'| > (m-1)/2$ and $|Y'| > (m-1)/2$. Since $\overline{H}-v_1 = (X', Y')$ is a complete bipartite graph, $\overline{H}-v_1$ has an (x,y)-path of order l with $l = m-1, m-3$ for any $x \in X'$ and $y \in Y'$, an (x_1,x_2)-path of order $m-4$ for any $x_1, x_2 \in X'$ and a (y_1,y_2)-path of order $m-4$ for any $y_1, y_2 \in Y'$.

Let $x \in X'$ and $y \in Y'$. If $z_1 \in Z$ and $x, y \notin N(z_1)$, then xz_1y is a P_3 in $\overline{G}$ and if $z_1, z_2 \in Z$ with $z_1x, z_2y \notin E(G)$, then $xz_1v_0z_2y$ is a P_5 in $\overline{G}$. Thus, the P_3 and P_5 together with an (x,y)-path of order $m-1$ and $m-3$ in $\overline{H}-v_1$, respectively, form a C_m in $\overline{G}$, a contradiction. Hence we have $X' \subseteq N(z)$ for any $z \in Z$ or $Y' \subseteq N(z)$ for any $z \in Z$.

Assume without loss of generality that $X' \subseteq N(z)$ for any $z \in Z$. Suppose that $y_1, y_2 \in Y'$, $z_1, z_2 \in Z$ and $y_1z_1, y_2z_2 \notin E(G)$. If $z \in Z - \{z_1, z_2\}$ such that $\{z_1, z_2\} \not\subseteq N(z)$, say $z_1z \notin E(G)$, then $y_1z_1zv_0z_2y_2$ is a P_6 in $\overline{G}$, which together with a (y_1,y_2)-path of order $m-4$ in $\overline{H}-v_1$ gives a C_m in $\overline{G}$, a contradiction. Hence, $z_1, z_2 \in N(z)$ for any $z \in Z - \{z_1, z_2\}$. Noting that $d(z) \le d = |H|$, we have $Y' \not\subseteq N(z)$ for any $z \in Z - \{z_1, z_2\}$. By Claim 1, $|Z| \ge 4$. Let $z_3, z_4 \in Z - \{z_1, z_2\}$ and $z_3y_3 \notin E(G)$. Noting that $y_1 \neq y_2$, we assume that $y_3 \neq y_1$. If $z_1z_2 \notin E(G)$, then $y_1z_1z_2v_0z_3y_3$ is a P_6 in $\overline{G}$ and if $z_3z_4 \notin E(G)$, then $y_1z_1v_0z_4z_3y_3$ is a P_6 in $\overline{G}$, which together with a (y_1,y_3)-path of order $m-4$ in $\overline{H}-v_1$ produce a C_m in $\overline{G}$, a contradiction. Thus, $G[Z]$ is a complete graph. By Claim 1, $G[X' \cup Z]$ is a complete graph of order at least $n+1$ and so G has a W_n, again a contradiction. Therefore, $\overline{G}$ has no two independent edges between Y' and Z. This is to say that there is some vertex $v \in Z \cup Y'$ such that $E(Y'-\{v\}, Z-\{v\}) = |Y_1|\cdot|Z_1|$. Clearly, v is the vertex as required. □

Assume that v_2 is a given vertex as required in Claim 2. Set $X_1 = X' - \{v_2\}$, $Y_1 = Y' - \{v_2\}$ and $Z_1 = Z - \{v_2\}$.

Claim 3. $|Z| \le m-2$.

Proof. If $|Z| \ge m-1$, then $|Z \cup \{v_0\}| \ge m$. Let $z_0 \in Z$ with $d_Z(z_0) = \Delta(G[Z])$. By Lemma 2.2, $d_Z(z_0) \ge \lfloor m/2 \rfloor \ge 2$. Let $z_1, z_2 \in N(z_0)$. If $m \ge 7$, then $d_Z(z_0) \ge 3$. By Claim 2, we have $z_0 = v_2$ and for any $z \in Z-\{z_0\}$, $d_Z(z) \le 1$ for otherwise we have $d = \Delta(G) \ge d+1$. Let $Z' = \{z_1, z_2, \cdots, z_{m-2}\} \subseteq Z - \{z_0\}$. Since $\delta(\overline{G}[Z']) \ge m-4 \ge$

$(m-2)/2$, $\overline{G}[Z']$ has a C_{m-2} by Theorem 1.3, which implies that $\overline{G}[Z' \cup \{v_0\}]$ contains a W_{m-2}. Noting that $z_1, z_2 \notin N(v_1)$, we see that $\overline{G}[Z' \cup \{v_0, v_1\}]$ contains a C_m, a contradiction. Hence, $m = 5$. By Claim 1, $|Z| \geq 4$. Let $z_3 \in Z - \{z_0, z_1, z_2\}$. If $v_2 \notin Z$, then $d_Z(z_0) = 2$ for otherwise $d(z_0) \geq d+1$ by Claim 2. If $d_Z(z_i) = 2$ for some $i \in \{1,2\}$, then since $z_0, z_i \notin N(v_1)$, $v_0 z_i v_1 z_0 z_3 v_0$ is a C_5 in $\overline{G}$, and if $d_Z(z_i) = 1$ for $i = 1,2$, then $v_0 z_1 z_2 z_3 z_0 v_0$ is a C_5 in $\overline{G}$, a contradiction. If $v_2 \in Z$, then by Claim 2, we have $v_2 = z_0$, $d_Z(z_i) = 1$ for $i = 1,2$ and $z_1, z_2 \notin N(v_1)$ since otherwise $\Delta(G) \geq d+1$. Thus, $v_0 z_1 v_1 z_2 z_3 v_0$ is a C_5 in $\overline{G}$, again a contradiction. Therefore, $|Z| \leq m-2$. □

Assume without loss of generality that $|X| \geq |Y|$. By Claim 3 and (2.2), $|X| \geq (N - |Z| - 1)/2 \geq (N - m + 1)/2 \geq 2n/3 + 1/2$. By Claim 1, $|X \cup Z \cup \{v_0\}| \geq N - |Y| \geq N - n + 2 \geq n + 3$. Thus we can choose $Z_0 \subseteq Z - \{v_2\}$ such that $|Z_0| = n - |X|$. Let $x_0 \in X - \{v_1, v_2\}$ and $X_0 = X - \{x_0\}$. We now show that $G[X_0 \cup Z_0 \cup \{v_0\}]$ contains a C_n. Clearly, $|X_0| \geq \lceil 2n/3 - 1/2 \rceil$ and $|Z_0| \leq \lfloor n/3 - 1/2 \rfloor$. By Claim 2, $X_0 - \{v_1, v_2\} \subseteq N(z)$ for any $z \in Z_0$. Noting that $n \geq 6$, we have $|X_0 - \{v_1, v_2\}| \geq \lceil 2n/3 - 1/2 \rceil - 2 \geq \lfloor n/3 - 1/2 \rfloor + 1 \geq |Z_0| + 1$. Since $G[X_0 \cup \{v_0\}]$ is a complete graph and some $|Z_0| + 1$ vertices of X_0 are adjacent to all vertices of Z_0, we see that $G[X_0 \cup Z_0 \cup \{v_0\}]$ is hamiltonian, that is, $G[X_0 \cup Z_0 \cup \{v_0\}]$ contains a C_n. By Claim 2, $Z_0 \subseteq N(x_0)$. Thus, $G[X \cup Z_0 \cup \{v_0\}]$ has a W_n with the hub x_0, a contradiction. Therefore,

$$\overline{H} \text{ is nonbipartite.} \tag{2.4}$$

Claim 4. If $(m,n) \neq (5,6)$, then H is weakly pancyclic with $g(H) = 3$ and $c(H) \geq m$.

Proof. By (2.1), $\lfloor d/2 \rfloor + 1 \geq m$. If $e(H) \geq d(d-1)/4 + 1$, then the result holds by Theorems 1.6 and 1.2. Thus we may assume that $e(\overline{H}) > d(d-1)/4 - 1$. Since $\overline{H}$ has no C_m, by Theorems 1.6 and 1.2, $e(\overline{H}) < d(d-1)/4 + 1$ which implies that $e(H) > d(d-1)/4 - 1$. Because $(m,n) \neq (5,6)$, we have $d \geq 9$ by (2.1) and hence $d(d-1)/4 - 1 \geq (d-1)^2/4 + 1$. By Theorem 1.6, (2.3) and (2.4), H and $\overline{H}$ are weakly pancyclic with girth 3. Noting that $\overline{H}$ has no C_m, we have $c(\overline{H}) < m$. Thus we have $c(H) > m$ by Lemma 2.1, and so the result follows. □

If $(m,n) = (5,6)$, then Theorem 2.7 holds by Theorem 2.2.

If $(m,n) \neq (5,6)$, then let $C = x_1 x_2 \ldots x_s x_1$ be a longest cycle in H and $U = V(H) - V(C) = \{u_1, u_2, \ldots, u_t\}$, where $t = d - s$. By Claim 4, $m \leq s \leq n-1$. By (2.2), $t = d - s \geq d - n + 1 \geq (m+1)/2$. By Lemma 2.3, $|N_C(u_i) \cup N_C(u_j)| \leq \lfloor |C|/2 \rfloor + 1$

for any $u_i, u_j \in U$, which implies that $|V(C) - (N_C(u_i) \cup N_C(u_j))| \geq \lceil |C|/2 \rceil - 1 \geq \lceil m/2 \rceil - 1 \geq (m-1)/2$. Thus, by Lemma 2.4, $G[U] = K_t$.

If $E(V(C), U) = \emptyset$, then $G[V(C)] = K_s$ by Lemma 2.4, and so $\overline{H}$ is bipartite, which contradicts (2.4). Thus we may assume that $x_1 u_1 \in E(H)$. Let $U_1 = U - \{u_1\}$ and $X_1 = \{x_2, x_3, \ldots, x_{t+1}, x_{s-t+1}, \ldots, x_{s-1}, x_s\}$. Since C is a longest cycle of H and $G[U] = K_t$, we have $E(X_1, U_1) = \emptyset$. Because $|X_1| > t \geq (m+1)/2$ and $|U_1| = t - 1 \geq (m-1)/2$, $G[X_1]$ is a complete graph by Lemma 2.4. If $G[V(C) - \{x_1\}] \neq K_{s-1}$, then there exists some i with $t+2 \leq i \leq s-t$ such that $G[X_1 \cup \{x_{t+1}, \ldots, x_{i-1}\}]$ is a complete graph and $G[X_1 \cup \{x_{t+1}, \ldots, x_{i-1}, x_i\}]$ is not a complete graph. By the maximality of C, $N_{U_1}(x_i) = \emptyset$. Since $\overline{H}$ has no C_m, by Lemma 2.4, $G[X_1 \cup \{x_{t+1}, \ldots, x_{i-1}, x_i\}]$ is a complete graph, a contradiction. Hence, $G[V(C) - \{x_1\}] = K_{s-1}$. By the maximality of C, $E(V(C) - \{x_1\}, U_1) = \emptyset$. Noting that $\overline{H}$ has no C_m, we have $U \subseteq N(x_1)$ or $V(C) - \{x_1\} \subseteq N(x_1)$. This is to say that $\overline{H}$ is bipartite, which contradicts (2.4). Therefore, $R(W_n, C_m) \leq \max\{2n+1, 3m-2\}$ for odd $m \geq 5$ and $n > m$.

The proof of Theorem 2.7 is completed. □

Proof of Theorem 2.3.

Proof. If $m = 3$, then Theorem 2.3 holds by Theorem 2.1. Since $n \geq 3(m-1)/2$, we have $\max\{2n+1, 3m-2\} = 2n+1$. If $m \geq 5$, then $R(W_n, C_m) \leq 2n+1$ by Theorem 2.7. Because $2K_n$ has no W_n and its complement has no C_m for odd m, $R(W_n, C_m) \geq 2n+1$. Thus we have $R(W_n, C_m) = 2n+1$ for $n \geq 3(m-1)/2$. □

Proof of Theorem 2.4.

Proof. Since n is odd, it is easy to see that $K_{m-1,m-1,m-1}$ has no W_n and its complement contains no C_m. Thus we have $R(W_n, C_m) \geq 3m-2$. If m, n are odd and $m < n \leq 3(m-1)/2$, then $m \geq 7$. By Theorem 2.7, $R(W_n, C_m) \leq 3m-2$ since $n \leq 3(m-1)/2$ implies that $\max\{2n+1, 3m-2\} = 3m-2$. Therefore, $R(W_n, C_m) = 3m-2$ for m, n odd and $m < n \leq 3(m-1)/2$. □

2.3 Proof of Theorem 2.5

In order to prove Theorem 2.5, we need Theorem 1.18 and the following two lemmas.

Lemma 2.5. *Let $C = x_1x_2\ldots x_sx_1$ be a cycle in a graph G of order n, $6 \le 2l \le s \le 2l+1$ and $Y = V(G)-V(C) = \{y_1, y_2, \ldots, y_{n-s}\}$ with $n-s \ge l+1$. Suppose that $\overline{G}$ contains no C_7 and $d_C(y) \le l-1$ for any $y \in Y$. Then, $G[Y] = K_{n-s}$. Moreover, if $d_C(y_1) \ge 2$, then G has a C_{s+1}, and if $d_C(y_i) \ge 2$ for $i = 1,2$, then G contains a C_{s+2}.*

Proof. Firstly, we prove that $G[Y]$ is a complete graph. Suppose to the contrary that $y_1y_2 \notin E(G)$. Since $d_C(y) \le l-1$ for any $y \in Y$ and $s \ge 2l$, any two vertices of Y have at least two common nonadjacent vertices in C. Assume that $x_i \notin N(y_1)\cup N(y_3)$ and $x_j \notin N(y_2)\cup N(y_4)$ with $x_i \ne x_j$. If there exists some $x_k \in V(C)-\{x_i, x_j\}$ such that $x_k \notin N(y_3)\cup N(y_4)$, then $y_1y_2x_jy_4x_ky_3x_iy_1$ is a C_7 in $\overline{G}$, and so we have $V(C)-N(y_3)\cup N(y_4) = \{x_i, x_j\}$. In this case, $s = 2l$ and $d_C(y_3) = d_C(y_4) = l-1$, which implies that either $x_k \notin N(y_3)$ or $x_k \notin N(y_4)$ for any $x_k \in V(C)-\{x_i, x_j\}$. Since y_1 has at least $l+1 \ge 4$ nonadjacent vertices in C, there is some $x_k \in V(C)-\{x_i, x_j\}$ such that $x_k \notin N(y_1)\cup N(y_3)$ or $x_k \notin N(y_1)\cup N(y_4)$. Thus, $x_ky_1y_2x_jy_4x_iy_3x_k$ or $x_ky_1y_2x_jy_3x_iy_4x_k$ is a C_7 in $\overline{G}$, a contradiction.

If $d_C(y_1) \ge 2$, we let $\{x_1, x_i\} \subseteq N(y_1)$. Suppose that G contains no C_{s+1}. Obviously, $i \ne 2, s$. Assume that $y_kx_p, y_mx_q \in E(Y, V(C))$ are two independent edges. If $x_px_q \notin E(C)$, then assume that $|x_p\overrightarrow{C}x_q| \le |x_q\overrightarrow{C}x_p|$ and P is a (y_k, y_m)-path of order $|x_p\overrightarrow{C}x_q|-1$ in $G[Y]$, we see that $x_py_k\overrightarrow{P}y_mx_q\overrightarrow{C}x_p$ is a C_{s+1}, and hence $x_px_q \in E(C)$. If $yx_j \in E(Y-\{y_1\}, V(C))$, then we must have $x_j \in \{x_2, x_s\}\cap\{x_{i-1}, x_{i+1}\}$ by the argument above. This means that $i = 3$ and $j = 2$, or $i = s-1$ and $j = s$. By symmetry, we assume that $i = 3$ and $j = 2$. If $x_3x_5 \notin E(G)$, then $\overline{G}[\{y_2, y_3, y_4, x_3, x_4, x_5, x_6\}]$ contains a C_7 and if $x_3x_5 \in E(G)$, then $x_1y_1yx_2x_3x_5\overrightarrow{C}x_1$ is a C_{s+1}, a contradiction. If $E(Y-\{y_1\}, V(C)) = \emptyset$, then G contains a C_{s+1} or $\overline{G}$ has a C_7 according to $G[V(C)] = K_s$ or not, again a contradiction.

If $d_C(y_i) \ge 2$ for $i = 1,2$, we may assume that $y_1x_1, y_2x_i \in E(Y, V(C))$ with $i \ne 1$. Suppose that $|x_1\overrightarrow{C}x_i| \le |x_i\overrightarrow{C}x_1|$ and Q is a (y_1, y_2)-path of order $|x_1\overrightarrow{C}x_i|$ in $G[Y]$. Obviously, $x_1y_1\overrightarrow{Q}y_2x_i\overrightarrow{C}x_1$ is a C_{s+2}. □

Let G be a graph and C a longest cycle in G. Suppose G is not hamiltonian and H any component of $G-C$. Set $N_C(H) = \{z_1, z_2, \ldots, z_k\}$, where indices following the orientation of C, $A = \{a_1, a_2, \ldots, a_k\}$, where $a_i = z_i^+$, and $B = \{b_1, b_2, \ldots, b_k\}$, where $b_i = z_i^-$. We have the following lemma on Hamilton theory.

Lemma 2.6. *Both $A \cup \{h\}$ and $B \cup \{h\}$ are independent sets for any $h \in V(H)$.*

Now we prove Theorem 2.5.

Since $2K_n$ contains no W_n and its complement contains no C_7, $R(W_n, C_7) \geq 2n+1$. In the following, we need only to show that $R(W_n, C_7) \leq 2n+1$ for $8 \leq n \leq 10$.

Let G be a graph of order $2n+1$ with $8 \leq n \leq 10$. Suppose to the contrary that neither G contains a W_n nor $\overline{G}$ contains a C_7. We distinguish the following two cases.

Case 1. $\delta(\overline{G}) \leq 5$.

Let v_0 be a vertex with $d(v_0) = \Delta(G)$, then $d(v_0) = \Delta(G) \geq 2n-5$. By Theorem 1.18, $G[N(v_0)]$ contains a C_{n-2}. Set $X = V(C_{n-2})$, $Y = N(v_0) - X = \{y_1, y_2, \ldots, y_l\}$ and $Z = V(G) - N[v_0]$. Clearly, $l \geq n-3$. Assume without loss of generality that $d_X(y_1) \geq d_X(y_2) \geq \cdots \geq d_X(y_l)$.

Subcase 1.1. $d_X(y_2) \geq 2$.

If $d_X(y_2) \geq 2$, then $G[X \cup \{y_1, y_2\}]$ is a 2-connected subgraph in $G[N(v_0)]$. Let H be a 2-connected subgraph of order n in $G[N(v_0)]$ such that $c(H)$ is as large as possible. Then $n-2 \leq c(H) \leq n-1$.

If $c(H) = n-2$, we let $C = x_1x_2 \ldots x_{n-2}x_1$ be a C_{n-2} in $G[X]$ and $H = G[X \cup \{y_1, y_2\}]$. By Lemma 2.6, $d_X(y_1) \leq \lfloor (n-2)/2 \rfloor$. If $d_X(y_1) \leq \lfloor (n-2)/2 \rfloor - 1$, then since $d_X(y_2) \geq 2$, by Lemma 2.5, $G[N(v_0)]$ contains a C_n, which implies G contains a W_n, a contradiction. Hence, $d_X(y_1) = \lfloor (n-2)/2 \rfloor$. By symmetry, we may assume $N_C(y_1) = \{x_1, x_3, \ldots, x_k\}$, where $k = n-3$ if n is even and $k = n-4$ if n is odd.

If $n = 8$, then set $X_1 = \{x_2, x_4, x_6, y_1\}$, $X_2 = \{x_1, x_3, x_5\}$, $Y_1 = \{y_i \mid d_{X_1}(y_i) = 0\}$ and $Y_2 = \{y_i \mid d_{X_1}(y_i) = 1 \text{ and } d_{X_2}(y_i) = 0\}$. By Lemma 2.6, X_1 is an independent set. Since $c(H) = 6$, we see that either $y_i \in Y_1$ or $y_i \in Y_2$ for any $2 \leq i \leq l$. Because $d_X(y_2) \geq 2$, we have $y_2 \in Y_1$, which implies that $\overline{G}[X_1 \cup \{y_2\}] = K_5$ by Lemma 2.6. Noting that $d_{X_1}(y_i) \leq 1$ for any y_i with $i > 2$, we see that $\overline{G}[X_1 \cup \{y_2, y_3, y_4\}]$ contains a C_7, a contradiction.

If $n = 9$, we set $X_1 = \{x_2, x_4, x_6, x_7, y_1\}$. By Lemma 2.6, $\overline{G}[X_1] = K_5 - e$. Since $c(H) = 7$, we have $d_{X_1}(y_i) \leq 1$ for $2 \leq i \leq l$. Thus, $\overline{G}[X_1 \cup \{y_2, y_3\}]$ contains a C_7, a contradiction.

If $n=10$, let $X_1=\{x_2,x_4,x_6,x_8,y_1\}$. By Lemma 2.6, $\overline{G}[X_1]=K_5$. Since $c(H)=8$, we have $d_{X_1}(y_i)\le 1$ for $2\le i\le l$, which implies that $\overline{G}[X_1\cup\{y_2,y_3\}]$ has a C_7, also a contradiction.

If $c(H)=n-1$, let $C=v_1v_2\ldots v_{n-1}v_1$ be a longest cycle in H and $V(H)-V(C)=\{v_n\}$. Choose H such that $d_C(v_n)$ is as large as possible. Noting that $d(v_0)\ge 2n-5$, we let $s_1,s_2,\ldots,s_{n-5}\in N(v_0)-V(H)$. If $d_C(v_n)\le\lfloor(n-1)/2\rfloor-1$, then by the choice of H, we have $d_C(s)\le\lfloor(n-1)/2\rfloor-1$ for any $s\in N(v_0)-V(H)$. By Lemma 2.5, $G[N(v_0)]$ contains a C_n, a contradiction. Thus we have $d_C(v_n)\ge\lfloor(n-1)/2\rfloor$. By Lemma 2.6, $d_C(v_n)\le\lfloor(n-1)/2\rfloor$, and hence $d_C(v_n)=\lfloor(n-1)/2\rfloor$. By symmetry, we may assume $N_C(v_n)=\{v_1,v_3,\ldots,v_k\}$, where $k=n-2$ if n is odd and $k=n-3$ if n is even.

If $n=8$, then by Lemma 2.6, both $\{v_2,v_4,v_6,v_8\}$ and $\{v_2,v_4,v_7,v_8\}$ are independent sets. Set $U=\{v_2,v_4,v_6,v_7\}$. Since $G[N(v_0)]$ contains no C_8, $d_U(s_i)\le 2$ for $i=1,2,3$, and if the equality holds, then $N_C(s_i)=\{v_2,v_4\}$. If $d_U(s_i)\le 1$ for some $i\in\{1,2\}$, then $\overline{G}[U\cup\{s_1,s_2,v_8\}]$ contains a C_7, and so we have $d_U(s_i)=2$ for any $i\in\{1,2\}$. In this case, $v_2s_1s_2v_4v_5v_6v_7v_1v_2$ is a C_8 in $G[N(v_0)]$ if $s_1s_2\in E(G)$ and $v_6s_1s_2v_7v_2v_8v_4v_6$ is a C_7 in $\overline{G}$ if $s_1s_2\notin E(G)$, a contradiction.

If $n=9$, we set $X_1=\{v_1,v_3,v_5,v_7\}$, $X_2=\{v_2,v_4,v_6,v_8,v_9\}$, $S_1=\{s_i\mid d_{X_1}(s_i)=0\}$ and $S_2=\{s_i\mid d_{X_2}(s_i)=0\}$. By Lemma 2.6, X_2 is an independent set. Since $G[N(v_0)]$ has no C_9, we see that either $s\in S_1$ or $s\in S_2$ for any $s\in N(v_0)-V(H)$. If $|S_2|\ge 2$, then $\overline{G}[X_2\cup S_2]$ contains a C_7, a contradiction. Noting that $|N(v_0)-V(H)|\ge n-5\ge 4$, we have $|S_1|\ge 3$, which implies $G[X_1]$ is a complete graph for $\overline{G}$ contains no C_7. Thus, for any $s\in N(v_0)-V(H)$, $d_{X_2}(s)\le 1$ since $G[N(v_0)]$ contains no C_9, and hence $\overline{G}[X_2\cup\{s_1,s_2\}]$ contains a C_7, a contradiction.

If $n=10$, we let $X_1=\{v_2,v_4,v_6,v_8,v_9,v_{10}\}$. By Lemma 2.6, $\overline{G}[X_1]=K_6-e$. If $d_{X_1}(s_1)\le 4$, then $\overline{G}[X_2\cup\{s_1\}]$ contains a C_7, and if $d_{X_1}(s_1)\ge 5$, then $G[N(v_0)]$ has a C_{10}, a contradiction.

Subcase 1.2. $d_X(y_2)\le 1$.

In this case, $d_X(y_i)\le 1$ for $2\le i\le l$. By Lemma 2.5, $G[Y-\{y_1\}]=K_{l-1}$. Let $C=x_1x_2\ldots x_{n-2}x_1$ be a C_{n-2} in $G[X]$. Because $G[N(v_0)]$ has no C_n, $E(Y-\{y_1\},X)$ contains no two independent edges. Thus, there exists some $x\in X$, say $x=x_1$, such that $E(Y-\{y_1\},X-\{x_1\})=\emptyset$. Moreover, $G[X-\{x_1\}]=K_{n-3}$ for otherwise $\overline{G}$ contains a C_7. For the same reason, we have $X-\{x_1\}\subseteq N(u)$ or $Y-\{y_1\}\subseteq N(u)$ for any $u\in Z\cup\{x_1,y_1\}$. If $v_1,v_2\in Z\cup\{y_1\}$ such that $X-\{x_1\}\subseteq N(v_i)$ for $i=1,2$, then $C' = x_1v_0x_3v_1x_4v_2x_5\overrightarrow{C}x_{n-2}x_1$ is a C_n and $\{x_2\}+C'$ is a W_n, a contradiction. Thus, there exist $n-l+2$ vertices $u_1,\dots,u_{n-l+2} \in Z\cup\{y_1\}$ such that $Y-\{y_1\} \subseteq N(u_i)$ and $X-\{x_1\} \not\subseteq N(u_i)$ for $1 \le i \le n-l+2$. Now, let $G_0 = G[(Y-\{y_1\})\cup\{u_1,\dots,u_{n-l+2}\}]$. Obviously, $|G_0| = n+1$. If $G_0 = K_{n+1}$, then G_0 has a W_n and so we may assume that $u_1u_2 \notin E(G)$. If there exists $x_i, x_j \in X-\{x_1\}$ with $i \neq j$ such that $u_1x_i, u_2x_j \notin E(G)$, then for any $x_k \in X-\{x_1,x_i,x_j\}$, $u_1x_iy_2x_ky_3x_ju_2u_1$ is a C_7 in $\overline{G}$, and hence we may assume $X-\{x_1,x_k\} \subseteq N(u_i)$ for some $x_k \in X-\{x_1\}$, where $i = 1,2$. By the symmetry of x_2 and x_{n-2}, we assume that $x_k \neq x_2$. If $x_k = x_{n-2}$, then $x_1v_0x_3u_1x_4u_2x_5\overrightarrow{C}x_{n-2}x_1$ is a C_n in $N(x_2)$ and if $x_k \neq x_{n-2}$, then noting that $G[X-\{x_1\}] = K_{n-3}$, we may assume that $x_k = x_3$, which implies that $x_1v_0x_3x_4u_1x_5u_2x_6\overrightarrow{C}x_{n-2}x_1$ is a C_n in $N(x_2)$. Therefore, G contains a W_n with the hub x_2, again a contradiction.

Case 2. $\delta(\overline{G}) \ge 6$.

We first show the following claims.

Claim 1. $\overline{G}$ contains no K_5.

Proof. Suppose to the contrary that $X = \{x_i \mid 1 \le i \le 5\}$ and $\overline{G}[X] = K_5$. Set $V(G)-X = Y$. If there exists some $y \in Y$ such that $d_X(y) \le 3$, say $yx_4, yx_5 \in E(\overline{G})$, then since $\delta(\overline{G}) \ge 6$ and $\overline{G}$ has no C_7, we can choose three distinct vertices $y_1, y_2, y_3 \in Y-\{y\}$ such that $y_ix_i \in E(\overline{G})$ and $d_X(y_i) = 4$ for $1 \le i \le 3$. If $d_X(y) \ge 4$ for each $y \in Y$, then such three vertices y_1, y_2, y_3 exist obviously. Now, let $Y_i = N_{\overline{G}}[y_i]\cap Y$ for $1 \le i \le 3$. Since $\delta(\overline{G}) \ge 6$, we have $|Y_i| \ge 6$ for $1 \le i \le 3$. Because $\overline{G}$ has no C_7, we see that $\{x_4\}, Y_1, Y_2$ and Y_3 are pairwise disjoint and $\overline{G}$ has no edges between any two of them, which implies that $G[\{x_4\}\cup Y_1\cup Y_2\cup Y_3]$ has a W_n with the hub x_4 for $8 \le n \le 10$, a contradiction. □

Claim 2. $\overline{G}$ contains no K_1+P_4.

Proof. If not, we assume that $x_1x_2x_3x_4$ is a P_4 in $\overline{G}$ and $x_0x_i \in E(\overline{G})$ for $1 \le i \le 4$. Set $X = \{x_i \mid 0 \le i \le 4\}$ and $Y = V(G)-X$. Since $\delta(\overline{G}) \ge 6$ and $\overline{G}$ has no C_7, we can choose three distinct vertices $y_0, y_1, y_2 \in Y$ such that $y_0x_0, y_1x_1, y_2x_4 \in E(\overline{G})$. Let $Y_i = N_{\overline{G}}[y_i]\cap Y$ for $0 \le i \le 2$. Because $\overline{G}$ has no C_7, we see that Y_0, Y_1, Y_2 are pairwise disjoint and $\overline{G}$ has no edges between any two of them. By Claim 1, $G[Y_i]$ has at least one edge if $|Y_i| \ge 5$ and two edges if $|Y_i| \ge 6$. Thus, if $|Y_i| \ge 5$ and $|Y_j| \ge 5$, or $|Y_i| \ge 4$ and $|Y_j| \ge 6$ for some $0 \le i < j \le 2$, then $G[Y_0\cup Y_1\cup Y_2]$ contains a W_n for $8 \le n \le 10$, a contradiction. We now show that $|Y_0| \ge 5$. If

$y_0x_1 \in E(\overline{G})$, then we have $x_0,x_1,x_2,x_3 \in N(y_2)$ and $x_2,x_4 \in N(y_1)$ for otherwise $\overline{G}$ contains a C_7, which implies that $|Y_1| \geq 4$ and $|Y_2| \geq 6$, a contradiction. Hence, by symmetry, we may assume that $x_1,x_4 \in N(y_0)$. If $y_0x_2, y_0x_3 \in E(\overline{G})$, then $d_X(y_1) = d_X(y_1) = 4$ for otherwise G has a C_7, which implies that $|Y_i| \geq 6$ for $i = 1,2$, also a contradiction. Therefore, $|Y_0| \geq 5$. If $y_1x_i \in E(\overline{G})$ for some $x_i \in \{x_0,x_2,x_3\}$, then since $\overline{G}$ has no C_7, we have $d_X(y_2) = 4$, which implies that $|Y_2| \geq 6$. If $\{x_0,x_2,x_3\} \subseteq N(y_1)$, then $|Y_1| \geq 5$. Thus, we have $|Y_1| \geq 5$ or $|Y_2| \geq 5$ which is a contradiction since $|Y_0| \geq 5$. □

Claim 3. $\overline{G}$ contains no B_2.

Proof. If $\overline{G}$ has a B_2, we assume that $x_1x_2x_3x_4x_1$ is a C_4 with diagonal x_2x_4 in $\overline{G}$. Set $Y = V(G) - \{x_1,x_2,x_3,x_4\}$. Since $\delta(\overline{G}) \geq 6$, we can choose three distinct vertices $y_1, y_2, y_3 \in Y$ such that $y_ix_i \in E(\overline{G})$ for $1 \leq i \leq 3$. Let $Y_i = N_{\overline{G}}(y_i) \cap Y$ for $1 \leq i \leq 3$. Since $\delta(\overline{G}) \geq 6$, we have $|Y_i| \geq 4$ by Claim 2. Because $\overline{G}$ has no C_7, Y_1, Y_2, Y_3 are pairwise disjoint. For the same reason, we see that if $\overline{G}$ has an edge between Y_i and Y_j, then the edge must be y_iy_j, which implies that $\overline{G}$ has at most one edge among the three sets. Let $Y_i \in \{Y_1,Y_2,Y_3\}$ such that Y_i has no edges to other two sets in $\overline{G}$. By Claim 2, $G[Y_i]$ has a P_3. Thus, noting that $G[\cup_{j=1}^{3} Y_j - Y_i]$ contains a $K_{4,4} - e$ as subgraph and $P_3 + (K_{4,4} - e)$ contains a W_n for $8 \leq n \leq 10$, we get a contradiction. □

By Theorem 2.1, $\overline{G}$ has a triangle $x_1x_2x_3$. Let $Y = V(G) - \{x_1,x_2,x_3\}$ and $Y_i = N_{\overline{G}}(x_i) \cap Y$ for $1 \leq i \leq 3$. Since $\delta(\overline{G}) \geq 6$, we have $|Y_i| \geq 4$. By Claim 3, Y_1, Y_2, Y_3 are pairwise disjoint. Let $y' \in Y_1$ and $y'' \in Y_2$. Set $Y_1' = Y_1 \cup \{x_2\} - \{y'\}$ and $Y_2' = Y_2 \cup \{x_1\} - \{y''\}$. If $\overline{G}$ has an edge between some Y_i and Y_j, say $y'y'' \in E(\overline{G})$, then $\overline{G}$ has no edges between Y_3 and $Y_1' \cup Y_2'$ for otherwise $\overline{G}$ contains a C_7. If $\overline{G}$ has no edge between any two of Y_1, Y_2 and Y_3, then $\overline{G}$ still has no edges between Y_3 and $Y_1' \cup Y_2'$. By Claim 3, $G[Y_3]$ contains a P_3 and $G[Y_1 - \{y'\}]$ contains at least one edge. Noting that $Y_1 \subseteq N(x_2)$, we see that $G[Y_1']$ has a P_4. Similarly, $G[Y_2']$ has a P_4. Thus, $G[Y_3 \cup Y_1' \cup Y_2']$ contains $P_3 + 2P_4$ as a subgraph and $P_3 + 2P_4$ has a W_n for $8 \leq n \leq 10$, a contradiction.

The proof of Theorem 2.5 is completed. □

2.4 Proof of Theorem 2.6

There are two facts that are required in the proof of Theorem 2.6.

Lemma 2.7. *Let G be a graph, $C = v_1v_2\dots v_nv_1$ a cycle with $n \geq 8$ and $U \subseteq V(G)-V(C)$ with $3 \leq |U| \leq 5$. If $d_U(v) \geq 3$ for any $v \in V(C)$, then G contains a C_{n+3}.*

Proof. Let $3 \leq l \leq 5$ and $U = \{u_1,u_2,\dots,u_l\}$ with $d_C(u_1) \geq d_C(u_2) \geq \cdots \geq d_C(u_l)$. Obviously, $d_C(u_3) \geq \lceil (3n-d_C(u_1)-d_C(u_2))/(l-2)\rceil \geq \lceil n/3\rceil \geq 3$. If $d_C(u_3) \geq n-1$, then it is easy to see that $G[V(C)\cup\{u_1,u_2,u_3\}]$ has a C_{n+3}. Thus we may assume that $d_C(u_3) \leq n-2$. If there exists $v_i, v_j \notin N_C(u_3)$ and $i<j$ such that $v_{i-1}, v_{j-1} \in N_C(u_3)$, the subscripts are taken modulo n, then since $d_U(v) \geq 3$ for any $v \in V(C)$, we can choose $u' \in U-\{u_3\}$ such that $u' \in N(v_i)\cap N(v_{i+1})$ and $u'' \in U-\{u_3,u'\}$ such that $u'' \in N(v_i)\cap N(v_j)$. This means that $u_3v_{j-1}\overleftarrow{C}v_{i+1}u'v_iu''v_j\overrightarrow{C}v_{i-1}u_3$ is a C_{n+3}. Therefore, we may assume that $N_C(u_3) = \{v_1,\dots,v_k\}$. For the same reason as above, we can choose $u' \in U-\{u_3\}$ such that $u' \in N(v_k)\cap N(v_{k+1})$ and $u'' \in U-\{u_3,u'\}$ such that $u'' \in N(v_{k+1})\cap N(v_{k+2})$, which implies that G contains a cycle $C_{n+3}: v_1\overrightarrow{C}v_{k-1}u_3v_ku'v_{k+1}u''v_{k+2}\overrightarrow{C}v_1$. □

The following lemma is due to Sun and Chen [126]. For the sake of completeness, we include its proof here.

Lemma 2.8 (Sun and Chen [126])**.** *Let $W = W_n$ be a wheel with $n \geq 4$ in a graph G, $V \subseteq V(G)-V(W)$ with $|V| = 4$. If $d_V(u) \geq 3$ for any $u \in V(W)$, then $G[V \cup V(W)]$ contains a W_{n+2}.*

Proof. Let $C = u_1u_2\dots u_nu_1$ be a cycle and $W = \{u_0\}+C$. Since $d_V(u) \geq 3$ for any $u \in W$, we may assume without loss of generality that $v_1,v_2,v_3 \in N(u_0)$, $v_1,v_2 \in N(u_1)$ and $v_1 \in N(u_2)$. For the same reason, there exists some vertex $v_i \in \{v_1,v_2,v_3\}$ such that $v_i \in N(u_3)\cap N(u_4)$. If $i \neq 1$, then $u_1v_1u_2u_3v_iu_4\overrightarrow{C}u_1$ is a C_{n+2} in $N(u_0)$, and hence $i=1$. If $u_2 \in N(v_2)$ or $u_3 \in N(v_2)$, then $u_1v_2u_2u_3v_1u_4\overrightarrow{C}u_1$ or $u_1v_2u_3u_2v_1u_4\overrightarrow{C}u_1$ is a C_{n+2} in $N(u_0)$. Thus $v_2 \notin N(u_2)\cup N(u_3)$. Since $d_V(u_i) \geq 3$ for $i = 2,3$, we have $v_3 \in N(u_2)\cap N(u_3)$, which implies $u_1v_1u_2v_3u_3\overrightarrow{C}u_1$ is a C_{n+2} in $N(u_0)$. □

Since $2K_n$ contains no W_n and its complement has no C_7, we have $R(W_n,C_7) \geq 2n+1$. In the following, we need only to show that $R(W_n,C_7) \leq 2n+1$. Let G be a graph of order $2n+1$ with $n \geq 8$.

Let G_1 and G_2 be two graphs of order 6 as shown in Figure 2.2.

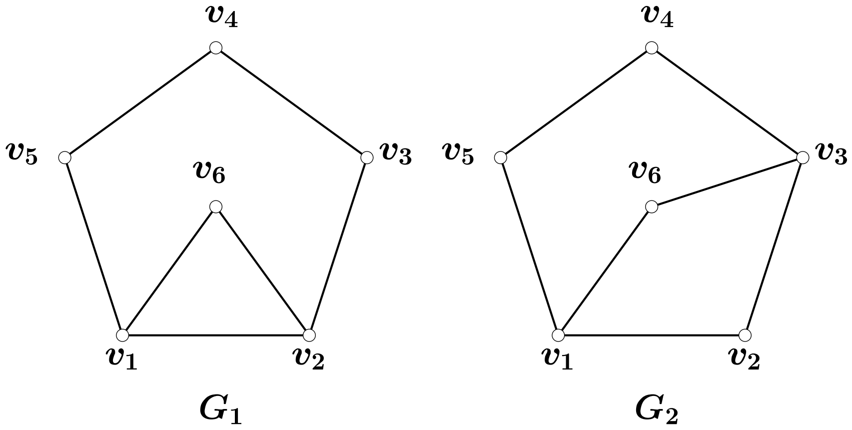

Figure 2.2: Two graphs G_1, G_2 of order 6

We will prove Theorem 2.6 by induction on n. If $n \le 10$, then the result holds by Theorem 2.5. Assume that the result is true for small values of n. Suppose to the contrary that neither G contains a W_n nor $\overline{G}$ has a C_7.

Assume that $\overline{G}$ contains a G_1. Set $V(G_1) = X$, then $|V(G) - X| = 2(n-3)+1$. Because $\overline{G} - X$ has no C_7, by induction hypothesis, $G - X$ contains a $W = W_{n-3} = \{u_0\} + C$, where $C = u_1 u_2 \ldots u_{n-3} u_1$. Since G_1 has a C_6 and $\overline{G}$ has no C_7, we have $d_X(u_i) \ge 3$ for any $u_i \in V(W)$.

Suppose that $d_X(u) = 3$ for some $u \in V(W)$. Since $\overline{G}$ contains no C_7, we have $N_X(u) = \{v_3, v_5, v_6\}$. Noting that $\overline{G}[X \cup \{u\}]$ has a (v', v'')-path of order 6 and 7 for any $v', v'' \in \{v_3, v_5, v_6\}$ and $\overline{G}$ has no C_7, we get that $v_3v_5, v_3v_6, v_5v_6 \in E(G)$ and each u_i has at least two neighbors in $\{v_3, v_5, v_6\}$. Set $U_1 = \{u_i \mid v_3, v_5, v_6 \in N(u_i)\}$ and $U_2 = V(W) - U_1$. If $u_0 \in U_1$, then let $\{v_3, v_5, v_6\} = \{v_i, v_j, v_k\}$ with $u_1 v_i, u_2 v_k \in E(G)$, we see that $u_1 v_i v_j v_k u_2 \overrightarrow{C} u_1$ is a C_n in $N(u_0)$, a contradiction. Hence, $u_0 \in U_2$. Since $\overline{G}[X \cup \{u\}]$ has a (v', v'')-path of order 6 for any $v' \in \{v_3, v_5, v_6\}$ and $v'' \in \{v_1, v_2, v_4\}$ and $\overline{G}$ has no C_7, we have $d_X(u_i) = 5$ for any $u_i \in U_2$. If $|U_2| \le 2$, we assume that $U_2 \subseteq \{u_0, u_1\}$. Let $\{v_3, v_5, v_6\} = \{v_i, v_j, v_k\}$ with $v_i \in N(u_0) \cap N(u_1)$. Then $u_0 u_1 u_2 v_j v_k u_3 \overrightarrow{C} u_{n-3} u_0$ is a C_n in $N(v_i)$, a contradiction. If $|U_2| \ge 3$, then since $U_1 \ne \emptyset$, we may assume that $u_1 \in U_1$ and $u_2, u_l \in U_2$ with $3 \le l \le n-3$. By the arguments above, we may assume that $v_i, v_j \in N(u_0) \cap \{v_3, v_5, v_6\}$ and $v_j \in N(u_{l-1})$. In such a case, $u_1 v_i v_j u_{l-1} \overleftarrow{C} u_2 v_1 u_l \overrightarrow{C} u_1$ is a C_n in $N(u_0)$, again a contradiction. Therefore, we have $d_{G_1}(u_i) \geq 4$ for any $u_i \in V(W)$.

If $d_X(u_0) \geq 5$, then since $d_X(u_i) \geq 4$ for any $u_i \in V(W)$, G contains a W_n by Lemma 2.7. Thus we have $d_X(u_0) = 4$. Since $\overline{G}$ contains no C_7, by symmetry, we need only to consider the following five cases: $v_1, v_3 \notin N(u_0)$, $v_1, v_4 \notin N(u_0)$, $v_4, v_6 \notin N(u_0)$, $v_3, v_5 \notin N(u_0)$ and $v_1, v_2 \notin N(u_0)$. For the first three cases, it is not difficult to check that $\overline{G}[X \cup \{u_0\}]$ contains a (v', v'')-path of order 6 for any $v', v'' \in N_{G_1}(u_0)$. Noting that $\overline{G}$ has no C_7, we see that each vertex of $V(C)$ has at least three neighbours in $N_X(u_0)$. By Lemma 2.7, G contains a W_n, a contradiction. Therefore, we are left to consider the cases $v_3, v_5 \notin N(u_0)$ and $v_1, v_2 \notin N(u_0)$.

If $v_3, v_5 \notin N(u_0)$, then $v_1v_4, v_2v_4, v_6v_4 \in E(G)$ for otherwise $\overline{G}[X \cup \{u_0\}]$ has a C_7. If $v_5v_6 \notin E(G)$, then $\overline{G}[X \cup \{u_0\}]$ has a (v_i, v_j)-path of order 6 for any $v_i, v_j \in N_X(u_0)$. Because $\overline{G}$ contains no C_7, we see that each u_i has at least three neighbours in $N_X(u_0)$. By Lemma 2.7, G contains a W_n, a contradiction. Thus, $v_5v_6 \in E(G)$. If $V(C) \subseteq N_{\overline{G}}(v_1)$, then we have $V(C) \subseteq N(v_i)$ for $4 \leq i \leq 6$ for otherwise $\overline{G}$ has a C_7. In this case, $u_0u_1v_4u_2v_5u_3\overrightarrow{C}u_{n-3}u_0$ is a C_n in $N(v_6)$, a contradiction. Assume that $u_1v_1 \in E(G)$. Noting that $u_2v_2 \in E(G)$ or $u_2v_6 \in E(G)$ for otherwise $\overline{G}$ has a C_7, we see that $u_1v_1v_4v_2u_2\overrightarrow{C}u_1$ or $u_1v_1v_4v_6u_2\overrightarrow{C}u_1$ is a C_n in $N(u_0)$, again a contradiction.

If $v_1, v_2 \notin N(u_0)$, then $v_3v_6, v_5v_6 \in E(G)$ for otherwise $\overline{G}[X \cup \{u_0\}]$ has a C_7. If $v_4v_2 \notin E(G)$ or $v_4v_6 \notin E(G)$, then $\overline{G}[X \cup \{u_0\}]$ has a (v_i, v_j)-path of order 6 for any $v_i, v_j \in N_X(u_0)$. Since $\overline{G}$ has no C_7, each u_i has at least three neighbours in $N_X(u_0)$, which implies that G contains a W_n by Lemma 2.7, and thus $v_4v_2, v_4v_6 \in E(G)$. If $V(C) \subseteq N_{\overline{G}}(v_3)$, then since $\overline{G}$ has no C_7, we have $V(C) \subseteq N(v_i)$ for $i = 2, 4, 6$, which implies that $u_0u_1v_2u_2v_6u_3\overrightarrow{C}u_{n-3}u_0$ is a C_n in $N(v_4)$, a contradiction. Assume that $u_1v_3 \in E(G)$. Since $u_2v_4 \in E(G)$ or $u_2v_5 \in E(G)$ for otherwise $\overline{G}$ contains a C_7, we see that $u_1v_3v_6v_4u_2\overrightarrow{C}u_1$ or $u_1v_3v_6v_5u_2\overrightarrow{C}u_1$ is a C_n in $N(u_0)$, also a contradiction. Therefore,

$$\overline{G} \text{ contains no } G_1. \tag{2.5}$$

If $\overline{G}$ contains a G_2, let $V(G_2) = X$. Obviously, $|V(G) - X| = 2(n-3) + 1$. Since $\overline{G} - X$ contains no C_7, by induction hypothesis, $G - X$ contains a $W = W_{n-3} = \{u_0\} + C$, where $C = u_1u_2 \ldots u_{n-3}u_1$. If $d_X(u) \leq 3$ for some $u \in V(W)$, then s-ince $v_1v_2v_3v_4v_5v_1$ is a C_5, u has at least three neighbours in $\{v_1, v_2, v_3, v_4, v_5\}$

by (2.5), which implies that $uv_6 \notin E(G)$. By symmetry, $uv_2 \notin E(G)$. Thus, $v_1v_2uv_6v_3v_4v_5v_1$ is a C_7 in $\overline{G}$, a contradiction. Hence, $d_X(u) \geq 4$ for any $u \in V(W)$. If $d_X(u_0) \geq 5$, then G contains a W_n by Lemma 2.7. If $d_X(u_0) = 4$, say $v_i, v_j \notin N_X(u_0)$, then $v_iv_j \notin E(G_2)$ by (2.5). Noting that $\overline{G}$ has no C_7, we have $v_2, v_6 \notin \{v_i, v_j\}$. Thus, by symmetry we may assume that $\{v_i, v_j\} = \{v_1, v_3\}$ or $\{v_1, v_4\}$. In both cases, $\overline{G}[X \cup \{u_0\}]$ has a (v', v'')-path of order 6 for any $v', v'' \in N_X(u_0)$ and $v'v'' \notin E(G_2)$. By (2.5), $\{v', v''\} \not\subseteq N_{\overline{G}}(u)$ for any $u \in V(W)$ if $v'v'' \in E(G_2)$. Thus, noting that $\overline{G}$ contains no C_7, we see that each vertex of $V(C)$ has at least three neighbours in $N_X(u_0)$. By Lemma 2.7, G has a W_n, a contradiction. Hence,

$$\overline{G} \text{ contains no } G_2. \tag{2.6}$$

If $\overline{G}$ contains an $H = C_5$, then by induction hypothesis, $G - V(H)$ has a $W = W_{n-3}$. By (2.5) and (2.6), $d_H(u) \geq 4$ for any $u \in V(W)$. By Lemma 2.7, G contains a W_n, a contradiction. Thus,

$$\overline{G} \text{ contains no } C_5. \tag{2.7}$$

Suppose that $\overline{G}$ has a $B = B_l$. Assume that $v_1v_2 \in E(B)$ and $v_i \in N_B(v_1) \cap N_B(v_2)$ for $i = 3, \ldots, l+2$. If $3 \leq l \leq 4$, then $|G - V(B)| = (2n+1) - (l+2) \geq 2(n-3)+1$. By induction hypothesis, $G - V(B)$ has a $W = W_{n-3}$. By (2.7), each vertex of $V(W)$ has at least l neighbours in $V(B) - \{v_1\}$. If $l = 4$, then $|V(B) - \{v_1\}| = 5$. By Lemma 2.7, G contains a W_n, and hence $\overline{G}$ has no B_4. If $l = 3$, then $\{v_1, v_2\} \not\subseteq N_{\overline{G}}(u)$ for any $u \in V(W)$ since otherwise $\overline{G}$ has a B_4. By the argument above, we see that $d_B(u) \geq 4$ for any $u \in V(W)$. Noting that $|B| = 5$, we see that G contains a W_n by Lemma 2.7. Thus, $\overline{G}$ contains no B_3. If $l = 2$, then induction hypothesis, $G - V(B)$ has a $W' = W_{n-2}$. Since $\overline{G}$ has no B_3 and C_5 by (2.7), we get that $d_B(u) \geq 3$ for any $u \in V(W')$. By Lemma 2.8, G contains a W_n, a contradiction. Hence,

$$\overline{G} \text{ contains no } B_2. \tag{2.8}$$

If $\overline{G}$ has an $F = F_2$, then by induction hypothesis, $G - V(F)$ contains a $W = W_{n-3}$. By (2.7) and (2.8), we have $d_F(u) \geq 4$ for any $u \in V(W)$. By Lemma 2.7, G contains a W_n, a contradiction. Hence,

$$\overline{G} \text{ contains no } F_2. \tag{2.9}$$

By Theorem 2.1, $\overline{G}$ contains a triangle $v_1v_2v_3$. By induction hypothesis, $G-\{v_1,v_2,v_3\}$ contains a $W=W_{n-2}$. If $\{v_1,v_2,v_3\}\subseteq N(u)$ for each vertex of $V(W)$, then it is obvious that G has a W_n. Thus we may assume that $v_4\in V(W)$ and $v_3v_4\notin E(G)$. Let $X=\{v_1,v_2,v_3,v_4\}$. By induction hypothesis, $G-X$ contains a $W'=W_{n-2}$. By (2.7), (2.8) and (2.9), we have $d_X(u)\geq 3$ for any $u\in V(W')$. By Lemma 2.8, G contains a W_n, a contradiction.

The proof of Theorem 2.6 is completed. □

Chapter 3

Small even cycles versus stars or wheels

3.1 Introduction

From the early 1970s, cycles and stars have been well-studied in graph Ramsey theory. The following well-known theorem on cycle-star Ramsey numbers is due to Lawrence [87] and dates back to 1973; a proof of this result can also be found in [108].

Theorem 3.1 (Lawrence [87]).

$$R(C_m, K_{1,n}) = \begin{cases} 2n+1 & \text{for odd } m \leq 2n-1, \\ m & \text{for } m \geq 2n. \end{cases}$$

In fact, the graphs $K_{n,n}$ and K_{m-1} establish the lower bounds on $R(C_m, K_{1,n})$ for odd $m \leq 2n-1$ and $m \geq 2n$, respectively. For the upper bounds, both cases may be viewed as a direct corollary of Theorem 1.3.

For even $m < 2n$, not many results on exact values of these Ramsey numbers are known. In fact, all generic results we know of deal with the case that $m = 4$, which we will provide a detailed introduction in the sequel.

In this chapter, we first study the Ramsey numbers $R(C_m, K_{1,n})$ for values of m that are even and not too small relative to n. In particular, we prove the following theorem in the next section.

Theorem 3.2.

$$R(C_m, K_{1,n}) = \begin{cases} 2n & \text{for even } m \text{ with } n < m \leq 2n, \\ 2m-1 & \text{for even } m \text{ with } 3n/4+1 \leq m \leq n. \end{cases}$$

Our proof techniques cannot be used to obtain exact values of $R(C_m, K_{1,n})$ for even m below $3n/4+1$, but we can give a lower bound on $R(C_m, K_{1,n})$ for even m in the interval $\lfloor n/2 \rfloor + 2 \leq m \leq 3(n+1)/4$. This bound is based on the following graphs.

Let $G_1 = G_2 = K_{\lceil n/2 \rceil}$, $G_3 = K_{\lfloor n/2 \rfloor + 1}$, and $v_i \in V(G_i)$ for $i = 1,2,3$. Consider the graph G obtained from $G_1 \cup G_2 \cup G_3$ by identifying v_1, v_2, v_3 (merging them into one vertex, while keeping the remaining parts of the graphs G_1, G_2, G_3 mutually disjoint). It is straightforward to check that G is a graph of order $\lceil 3n/2 \rceil - 1$, that G contains no cycle of length $m \geq \lfloor n/2 \rfloor + 2$, and that $\delta(G) = \lceil n/2 \rceil - 1$. This implies that $\Delta(\overline{G}) = n-1$, hence that $\overline{G}$ contains no $K_{1,n}$. Thus, for $\lfloor n/2 \rfloor + 2 \leq m \leq 3(n+1)/4$, $R(C_m, K_{1,n}) \geq \lceil 3n/2 \rceil$. In fact, we expect that equality holds in the latter inequality. This motivates the following conjecture.

Conjecture 2. $R(C_m, K_{1,n}) = \lceil 3n/2 \rceil$ for even m with $\lfloor n/2 \rfloor + 2 \leq m \leq 3(n+1)/4$.

Note that, for proving the statement in the above conjecture, by the above examples it suffices to show $R(C_m, K_{1,n}) \leq \lceil 3n/2 \rceil$ for these values of m. Since $R(C_4, K_{1,5}) = 8$ by Parsons [107], Conjecture 2 is true for $m = 4$ (and $n = 5$; this is the only value of n for which $m = 4$ lies in the specified interval).

We also confirm that Conjecture 2 holds for $m = 6$. From $\lfloor n/2 \rfloor + 2 \leq 6 \leq 3(n+1)/4$, it follows that $7 \leq n \leq 9$. We prove the following theorems in Sections 3.3 and 3.4, respectively.

Theorem 3.3. $R(C_6, K_{1,n}) = n+4$ *for* $n = 7, 8$.

Theorem 3.4. $R(C_6, K_{1,n}) = n+5$ *for* $n = 9, 10, 11$.

Now we can summarize the known $R(C_6, K_{1,n})$ as follows. The numbers for $4 \leq n \leq 11$ are obtained from the results in this chapter.

n	1	2	3	4	5	6	7	8	9	10	11
$R(C_6, K_{1,n})$	6	6	6	8	10	11	11	12	14	15	16

Table 1: Exact values of $R(C_6, K_{1,n})$ for $1 \leq n \leq 11$.

Luo et al. [95] reported that they calculated 11 exact values of the Ramsey numbers $R(C_m, W_n)$ by using an efficient algorithm called one-vertex extension method. The 11 values include $R(C_6, W_n)$ for $n = 6,7,8$. We can show that $R(C_6, W_n) = 16$ for $n = 3,5,7,9$ as an immediate corollary of Theorem 3.4, where $R(C_6, W_9) = 16$ is a new value.

Corollary 3.1. $R(C_6, W_n) = 16$ *for* $n = 3,5,7,9$.

Proof. Since $3K_5$ contains no C_6 and its complement contains no W_n for odd n, we have $R(C_6, W_n) \geq 16$ for $n = 3,5,7,9$. Let G be a graph of order 16 and suppose that G contains no C_6, we will show that $\overline{G}$ contains W_n. By Theorem 3.4, $\overline{G}$ contains $K_{1,11}$. In particular, this implies there exists a vertex v such that $d(v) \geq 11$ in $\overline{G}$. Since $G[N_{\overline{G}}(v)]$ contains no C_6, by Theorem 1.17, $\overline{G}[N_{\overline{G}}(v)]$ contains C_n, together with v forming a W_n in $\overline{G}$ for $n = 3,5,7,9$. This completes the proof. $\square$

Recall that to prove Conjecture 2 it suffices to show $R(C_m, K_{1,n}) \leq \lceil 3n/2 \rceil$ for the values of m stated in Conjecture 2. Notice that to prove $R(C_m, K_{1,n}) \leq \lceil 3n/2 \rceil$ one actually has to show that any graph G with $|V(G)| = \lceil 3n/2 \rceil$ and $\delta(G) \geq \lceil n/2 \rceil$ contains cycles of every even length between $\lfloor n/2 \rfloor + 2$ and $3(n+1)/4$. Allen [3] proved the following extension of Theorem 1.3: there exists a positive integer n_0 such that any graph G with $|V(G)| = n \geq n_0$ and $\delta(G) \geq n/3$ contains cycles of every even length between 4 and $\lceil n/2 \rceil$. It is not difficult to check that this implies that there exists a positive integer m_0, such that Conjecture 2 holds for all $m \geq m_0$.

We turn to the study of C_4. It is well-known that it is difficult to deal with some extremal problems involving C_4. We are interested in the relationship between two Ramsey numbers involving C_4, that is, $R(S_{n+1}, C_4)$ and $R(W_n, C_4)$. The former has been well-studied and the latter has received more attention recently.

Parsons [107] began to consider the Ramsey numbers $R(S_{n+1}, C_4)$ back in 1975. By using the existence of projective planes over Galois fields and the generalized friendship theorem, in [107] he established upper bounds for $R(S_{n+1}, C_4)$ and determined the exact values for several specific values of n, as expressed in the following two results.

Theorem 3.5 (Parsons [107]). *$R(S_{n+1}, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 2$ for all $n \ge 2$, and if $n = q^2 + 1$ and $q \ge 1$, then $R(S_{n+1}, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 1$.*

Theorem 3.6 (Parsons [107]). *If q is a prime power, then $R(S_{q^2+1}, C_4) = q^2 + q + 1$ and $R(S_{q^2+2}, C_4) = q^2 + q + 2$.*

Noting that if $n = q^2$, then $n + \lfloor\sqrt{n-1}\rfloor + 2 = q^2 + q + 1$, we see that the general bound for $R(S_{n+1}, C_4)$ in Theorem 3.5 is best possible.

Obviously, S_{n+1} is a (spanning) subgraph of W_n, so $R(W_n, C_4) \ge R(S_{n+1}, C_4)$. By using an exhaustive computer search, Tse [134] was able to calculate the value of $R(W_n, C_4)$ for $3 \le n \le 12$. An interesting question in this respect is: what is the best possible upper bound for $R(W_n, C_4)$? Surahmat et al. [132] showed that $R(W_n, C_4) \le n + \lceil n/3 \rceil + 1$ for $n \ge 6$. Clearly, this upper bound is not tight in general. Because $R(W_n, C_4) \ge R(S_{n+1}, C_4)$ showing that the best bound for $R(W_n, C_4)$ is at least $n + \lfloor\sqrt{n-1}\rfloor + 2$, one may ask whether $R(W_n, C_4) - R(S_{n+1}, C_4)$ is a constant or a function depending on n. Recently, by using Reiman's theorem [113] on the Turán number $t(n, C_4)$, Ore's theorem [104] on hamiltonicity, a result of Faudree and Schelp [58] on $R(C_n, C_4)$ and the Erdős-Rényi graph, Dybizbański and Dzido [41] established a general upper bound for $R(W_n, C_4)$ for $n \ge 10$ and determined some exact values of $R(W_n, C_4)$. We summarized some of their results in the following theorem.

Theorem 3.7 (Dybizbański and Dzido [41]). *$R(W_n, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 2$ for all $n \ge 10$, and if $q \ge 4$ is a prime power, then $R(W_{q^2}, C_4) = q^2 + q + 1$.*

In the same paper, with the help of computers, they determined the exact values of some Ramsey numbers for a small wheel versus a C_4.

Theorem 3.8 (Dybizbański and Dzido [41]). *$R(W_n, C_4) = n + 5$ for $13 \le n \le 16$.*

Clearly, Theorem 3.7 implies that Parsons' bound for $R(S_{n+1}, C_4)$ is also a best possible upper bound for $R(W_n, C_4)$ if $n \ge 10$. In an unpublished paper, Wu et al. [137] obtained nine new values for $R(W_n, C_4)$; as in the other cases their calculations have been performed with the aid of computer search.

Theorem 3.9 (Wu et al. [137]). *$R(W_n, C_4) = n+5$ for $17 \le n \le 20$; $R(W_{26}, C_4) = 32$; $R(W_n, C_4) = n+7$ for $34 \le n \le 36$; $R(W_{43}, C_4) = 51$.*

The exact values of the Ramsey numbers $R(S_{n+1}, C_4)$ for $n \le 6$ can be found in [110]. For the value of $R(S_8, C_4)$, we get $R(S_8, C_4) \le 11$ by Theorem 3.5. Since the Petersen graph contains no C_4 and its complement has no S_8, we get $R(S_8, C_4) \ge 11$ and so we obtain that $R(S_8, C_4) = 11$. Using Theorem 3.6, we can get the exact values of $R(S_{n+1}, C_4)$ for $n = 9, 10, 16, 17$. By considering Theorems 3.5, 3.6 and 3.7, and these known values of $R(S_n, C_4)$ and $R(W_n, C_4)$ for small $n \ge 6$, we observe that there is an infinite number of values of n for which $R(W_n, C_4) = R(S_{n+1}, C_4)$. Motivated by this observation, a natural question is whether this equality holds in general. Now we give an affirmative answer to this question. The result is as follows.

Theorem 3.10. *$R(W_n, C_4) = R(S_{n+1}, C_4)$ for $n \ge 6$.*

We postpone our proof of this result to Section 3.5.

By Theorem 3.10, we see that the two functions $R(W_n, C_4)$ and $R(S_{n+1}, C_4)$ are in fact the same when $n \ge 6$. Because the Ramsey numbers $R(S_{n+1}, C_4)$ are well-studied, we can use Theorem 3.10 and known results on $R(S_{n+1}, C_4)$ to establish new results on $R(W_n, C_4)$. Of course, we can do that in reverse as well. Up to now, most known values of $R(W_n, C_4)$ for small n are obtained with the help of computers. Because finding an S_{n+1} is much easier than finding a W_n in a graph using computers, we can focus our calculation on $R(S_{n+1}, C_4)$ by computers instead of $R(W_n, C_4)$ if we want to determine some values of $R(W_n, C_4)$ with the help of computers.

Combining Theorems 3.5, 3.6 and 3.10, we obtain the following.

Theorem 3.11. *$R(W_n, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 2$ for $n \ge 6$, and if $n = q^2+1$ and $q \ge 3$, then $R(W_n, C_4) \le n + \lfloor\sqrt{n-1}\rfloor + 1$. Furthermore, if $q \ge 3$ is a prime power, then we have $R(W_{q^2}, C_4) = q^2+q+1$ and $R(W_{q^2+1}, C_4) = q^2+q+2$.*

Clearly, Theorem 3.11 is stronger than Theorem 3.7. Furthermore, by Theorems 3.5-3.11 and some other known results on $R(S_{n+1}, C_4)$, we can summarize several exact values (see Table 2) for $R(W_n, C_4)$ and $R(S_{n+1}, C_4)$ when $n \ge 6$

is small. Here the numbers marked with $*$ are obtained from the results in this chapter, and the numbers marked with $\star$ can be obtained by Theorem 3.11 avoiding computer search.

n	6	7-8	9-10	11-15	16-17	18-20	25	26	34-36	43
$R(W_n,C_4)$	9	$n+4$	$n+4^\star$	$n+5$	$n+5^\star$	$n+5$	31	$32^\star$	$n+7$	51
$R(S_{n+1},C_4)$	9	$n+4$	$n+4$	$n+5^*$	$n+5$	$n+5^*$	31	32	$n+7^*$	51^*

Table 2: Exact values of $R(W_n,C_4)$ and $R(S_{n+1},C_4)$ for $6 \le n \le 43$.

As for the lower bounds of $R(S_{n+1},C_4)$, Burr et al. [22] showed that $R(S_{n+1},C_4) > n+\sqrt{n}-6n^{11/40}$. In the same paper, they proposed the following conjecture, for which Erdős, one of the authors, offered $100 for a proof or disproof.

Conjecture 3 (Burr et al. [22])**.** $R(S_{n+1},C_4) < n+\sqrt{n}-c$ holds infinitely often, where c is an arbitrary constant.

After an easy calculation, we find that all exact values of $R(S_{n+1},C_4)$ listed in Table 2 satisfy $R(S_{n+1},C_4) \ge n+\lceil\sqrt{n}\rceil$. Thus we pose the following intriguing problem.

Question. Is it true that $R(W_n,C_4) = R(S_{n+1},C_4) \ge n+\lceil\sqrt{n}\rceil$ for all $n \ge 6$?

For the Ramsey numbers $R(W_n,C_m)$ when $n > m > 4$ and m is even, quite a few Ramsey numbers are known. We just mentioned that with the help of computers, Luo et al. [95] calculated 11 Ramsey numbers $R(C_m,W_n)$, including $R(C_6,W_7) = 16$ and $R(C_6,W_8) = 13$. We have showed that $R(C_6,W_7) = 16$ (avoiding computer search) and $R(C_6,W_9) = 16$. These three numbers should be all the Ramsey numbers we have known for this part. Now we fill some of this gap by proving the following result in the final section.

Theorem 3.12. $R(W_n,C_m) = 3m-2$ *for n odd, m even and $m < n < 3m/2$.*

3.2 Proof of Theorem 3.2

In our proof of Theorem 3.2, we need Theorems 1.1, 1.3, 1.4 and the following lemma, which is a corollary of Theorem 1.5.

Lemma 3.1. *Let G be a graph with $\delta(G) \geq (|V(G)|+2)/3$. Then G contains cycles of every even length between 4 and $c(G)$.*

Proof. If G is a nonbipartite graph, the result follows directly from Theorem 1.5. If G is a bipartite graph, let X and Y denote the partition classes of G, and consider a vertex $x \in X$. Since the conditions imply that $\delta(G) \geq 2$, x has at least two neighbors x_1 and x_2, and obviously $x_1, x_2 \in Y$. Now construct a new graph G' from G by adding the edge x_1x_2. Then clearly G' is not bipartite. Since $\delta(G') \geq (|V(G)|+2)/3$ and $|V(G')| = |V(G)|$, by Theorem 1.5, G' contains cycles of every length between 4 and $c(G')$. Since $c(G') \geq c(G)$, it remains to prove that every even cycle in G' is also a cycle in G. If not, then G' has an even cycle C containing x_1x_2 as an edge, say, $C = x_1x_2x_3 \ldots x_{2k}x_1$. But then $x_2 \in Y$, $x_3 \in X, \ldots, x_{2k-1} \in X, x_{2k} \in Y$. Since $x_1 \in Y$, this implies that $G'[Y]$ contains $x_{2k}x_1$ as an edge, contradicting the fact that $G'[Y]$ contains exactly one edge x_1x_2. We conclude that every even cycle in G' is also a cycle in G. Therefore, G contains cycles of every even length between 4 and $c(G)$. □

Now we have all ingredients to prove Theorem 3.2. We prove the two statements separately.

For the purpose of proving the first statement of the theorem, let F be the graph obtained from $2K_n$ by identifying precisely one vertex of each K_n. It is easy to check that $|V(F)| = 2n-1$, $\delta(F) = n-1$, F contains no C_m for $m > n$, and $\overline{F}$ contains no $K_{1,n}$. Thus, $R(C_m, K_{1,n}) \geq 2n$ for $n < m \leq 2n$.

It is sufficient for proving the first statement to prove that $R(C_m, K_{1,n}) \leq 2n$ for even m with $n < m \leq 2n$. Let G be a graph of order $2n$. If $\overline{G}$ contains no $K_{1,n}$, then $\Delta(\overline{G}) \leq n-1$, implying that $\delta(G) \geq n$. By Theorem 1.3, G contains C_m for even m and $n < m \leq 2n$. This completes the proof of the first statement of Theorem 3.2.

We continue with the proof of the second statement. First observe that for even m and $3n/4+1 \leq m \leq n$, the inequalities imply that $n \geq 4$, and hence that $m \geq 4$. It is obvious that for these values of m and n, $2K_{m-1}$ contains no C_m and its complement contains no $K_{1,n}$. Thus, $R(C_m, K_{1,n}) \geq 2m-1$.

To prove that $R(C_m, K_{1,n}) \leq 2m-1$, let G be a graph of order $2m-1$. Suppose to the contrary that neither G contains a C_m nor $\overline{G}$ contains a $K_{1,n}$. Then $\Delta(\overline{G}) \leq n-1$, hence $\delta(G) \geq 2m-n-1$. Using $m \geq 3n/4+1$, we get that $\delta(G) \geq (|V(G)|+2)/3$. By Lemma 3.1, G contains cycles of every even length between 4 and $c(G)$. It remains to prove that $c(G) \geq m$. We complete the proof by proving three claims.

Claim 1. Suppose G_1 is a graph obtained from G by deleting at most two vertices. Then $\omega(G_1) \leq 2$.

Proof. If $\omega(G_1) \geq 3$, let G_2 be the smallest component of G_1. Then $\delta(G_1) \leq \delta(G_2) \leq |V(G_2)| - 1 \leq |V(G_1)|/3 - 1$. Thus, $\Delta(\overline{G_1}) \geq 2|V(G_1)|/3 \geq 2(2m-3)/3 \geq n - 2/3$, that is, $\Delta(\overline{G_1}) \geq n$. Since $\overline{G_1}$ is a subgraph of $\overline{G}$, then $\Delta(\overline{G}) \geq n$, which contradicts the fact that $\overline{G}$ contains no $K_{1,n}$. This proves our claim that $\omega(G_1) \leq 2$. □

Claim 2. Suppose H is a graph obtained from G by deleting at most one vertex. If $\omega(H) = 2$, then each component of H is a 2-connected (sub)graph.

Proof. Let H_1, H_2 be the two components of H. Then $\delta(H_i) \geq \delta(H) \geq \delta(G) - 1 \geq 2m - n - 2$ for $i = 1, 2$. Since $m \geq 3n/4 + 1$ and $n \geq 4$, this implies that $|V(H_i)| \geq \delta(H_i) + 1 \geq 3$ for $i = 1, 2$. If H_1 is not 2-connected, then there exists a vertex u such that $H_1 - u$ is disconnected. Hence, $H - u$ is a graph obtained from G by deleting at most two vertices, and $\omega(H - u) \geq 3$, contradicting Claim 1. We conclude that H_1 is 2-connected. For the same reason, H_2 is 2-connected, proving our claim. □

Claim 3. $c(G) \geq m$.

Proof. Recall that G is a graph of order $2m - 1$ and that $\delta(G) \geq (|V(G)| + 2)/3$. If G is 2-connected, by Theorem 1.1, $c(G) \geq \min\{2\delta(G), |V(G)|\} \geq m$.

Next assume that G is not 2-connected. Then there exists a vertex $v \in V(G)$ such that $G - v$ is disconnected. By Claim 1, $\omega(G - v) = 2$. Let H_1, H_2 be the two components of $G - v$. Then $\delta(H_i) \geq \delta(G) - 1 \geq 2m - n - 2$. By Claim 2, H_i is 2-connected for $i = 1, 2$. Assuming that $|V(H_1)| \geq |V(H_2)|$, we get that $|V(H_1)| \geq m - 1$. If $|V(H_1)| \geq m$, then, since $3n/4 + 1 \leq m \leq n$, using Theorem 1.1, we obtain that $c(G) \geq c(H_1) \geq \min\{2\delta(H_1), |V(H_1)|\} \geq \min\{2(2m - n - 2), m\} \geq m$.

Finally, assume that $|V(H_1)| = m - 1$. Then $|V(H_2)| = m - 1$. Since $d_G(v) \geq \delta(G) \geq 2m - n - 1 \geq 3$, then either $d_{H_1}(v) \geq 2$ or $d_{H_2}(v) \geq 2$. Without loss of generality, assume that $d_{H_1}(v) \geq 2$. Let $H_3 = G[V(H_1) \cup \{v\}]$. For any two vertices x, y of $V(H_1)$, $d_{H_3}(x) + d_{H_3}(y) \geq 2\delta(G) \geq 2(2m - n - 1) \geq m = |V(H_3)|$. It is easy to check that the closure of H_3 is a complete graph. By Theorem 1.4, H_3 contains a C_m.

This completes the proof of Claim 3 and of Theorem 3.2. □

3.3 Proof of Theorem 3.3

By the construction above Conjecture 2, we have seen that $R(C_6, K_{1,n}) \geq n+4$ for $n = 7, 8$. Let G be a graph of order $n+4$. Suppose to the contrary that G contains no C_6 and $\overline{G}$ contains no $K_{1,n}$, then $\delta(G) \geq 4$. By Theorem 3.5, G contains C_4. Let C be a longest cycle with $|V(C)| \leq 6$ in G, then $4 \leq |V(C)| \leq 5$. We distinguish two cases.

Case 1. $|V(C)| = 4$.

If G contains $K_{2,3}$, assume that both v_1 and v_2 are adjacent to each vertex of v_3, v_4, v_5. We see that $v_3v_4 \notin E(G)$, otherwise $v_1v_3v_4v_2v_5v_1$ is a C_5, a contradiction. For the same reason, $v_4v_5, v_3v_5 \notin E(G)$. Since $\delta(G) \geq 4$, for $i = 3, 4, 5$, each v_i has another adjacent vertex in $V(G) - \{v_1, v_2, v_3, v_4, v_5\}$, denoted by u_i. Since G contains no C_6, then u_3, u_4, u_5 are three distinct vertices. Let $V_1 = \{v_i, u_j \mid 1 \leq i \leq 5, 3 \leq j \leq 5\}$, then for $i = 3, 4, 5$, u_i is nonadjacent to $V_1 - \{v_i\}$, otherwise G contains C_5 or C_6, a contradiction. Since $\delta(G) \geq 4$, each of u_3, u_4, u_5 has at least three adjacent vertices in $V(G) - V_1$. Since $11 \leq |V(G)| \leq 12$, $3 \leq |V(G) - V_1| \leq 4$, it follows that u_3 and u_4 have a common adjacent vertex in $V(G) - V_1$, say, w. Hence, $wu_3v_3v_1v_4u_4w$ is a C_6, a contradiction. Thus,

G contains no $K_{2,3}$.

Since $|V(C)| = 4$, let $C = v_1v_2v_3v_4v_1$. Since $\delta(G) \geq 4$, each v_i has another adjacent vertex in $V(G) - \{v_1, v_2, v_3, v_4\}$, denoted by u_i, where $1 \leq i \leq 4$. We see that u_1, u_2, u_3, u_4 are four distinct vertices, otherwise G contains C_5 or $K_{2,3}$. Let $V_1 = \{v_i, u_i \mid 1 \leq i \leq 4\}$, then for $1 \leq i \leq 4$, u_i is nonadjacent to $V_1 - \{v_i\}$, otherwise G contains C_5 or C_6 or $K_{2,3}$, a contradiction. Since $\delta(G) \geq 4$, each u_i has at least three adjacent vertices in $V(G) - V_1$. Since $11 \leq |V(G)| \leq 12$, $3 \leq |V(G) - V_1| \leq 4$, it follows that u_1 and u_2 have a common adjacent vertex in $V(G) - V_1$, say, w. Hence, $wu_1v_1v_2u_2w$ is a C_5, a contradiction.

Case 2. $|V(C)| = 5$.

If G contains K_5, let $G' = G - V(K_5)$. Since $11 \leq |V(G)| \leq 12$, we have $6 \leq |V(G')| \leq 7$. For any $v \in V(G')$, v has at most one adjacent vertex in the K_5, otherwise G contains C_6. By $\delta(G) \geq 4$, we have $\delta(G') \geq 3$. If G' is not 2-connected, then there exists a vertex u in G' such that $G' - u$ is disconnected,

$5 \le |V(G'-u)| \le 6$ and $\delta(G'-u) \ge 2$. Thus, the subgraph $G'-u$ is a disjoint union of two triangles, denoted by $v_1v_2v_3v_1$ and $v_4v_5v_6v_4$. Since $\delta(G) \ge 4$, both v_1 and v_6 are adjacent to the K_5. By discussing whether v_1, v_6 are adjacent to the same vertex of K_5 or not, we can always find a C_6. If G' is 2-connected, by Theorem 1.1, $c(G') \ge 2\delta(G') \ge 6$. By Lemma 3.1, G' contains a C_6, a contradiction. Therefore,

G contains no K_5.

If G contains C_5+e, let $v_1v_2v_3v_4v_5v_1$ be a C_5 (with one chord). Since G contains C_5+e but no K_5, we can always find some i such that $v_iv_{i+2} \in E(G)$ and $v_{i+1}v_{i+3} \notin E(G)$, where the indices are taken modulo 5. Without loss of generality, we may assume that $v_1v_3 \in E(G)$ and $v_2v_4 \notin E(G)$. Since $\delta(G) \ge 4$, both v_2 and v_4 have another adjacent vertex in $V(G) - \{v_1, v_2, v_3, v_4, v_5\}$, denoted by u_2 and u_4, respectively. We see that u_2, u_4 are distinct, otherwise $u_2v_2v_3v_1v_5v_4u_2$ is a C_6. For the same reason, u_2 is nonadjacent to $\{v_1, v_3, v_4, v_5, u_4\}$; u_4 is nonadjacent to $\{v_2, v_3, v_5, u_2\}$. Let $V_1 = \{v_1, v_2, \ldots, v_5, u_2, u_4\}$. Since $\delta(G) \ge 4$, u_2 has at least three adjacent vertices in $V(G) - V_1$, three of which are denoted by $\{w_1, w_2, w_3\}$; u_4 has at least two adjacent vertices in $V(G) - V_1$, two of which are denoted by $\{w_4, w_5\}$. We deduce that w_1, w_2, w_3, w_4, w_5 are pairwise distinct, otherwise for some $w \in N(u_2) \cap N(u_4)$, $wu_2v_2v_3v_4u_4w$ is a C_6. Since $|V(G)| \le 12$, then $V(G) = \{v_i, w_i, u_j \mid 1 \le i \le 5, j = 2, 4\}$. Since $\delta(G) \ge 4$ and G contains no C_6, u_4 has to be adjacent to v_1. Moreover, w_4 is nonadjacent to $\{w_1, w_2, w_3\}$, otherwise, say, $w_1w_4 \in E(G)$, then $w_1u_2v_2v_1u_4w_4w_1$ is a C_6. For the same reason, w_4 is nonadjacent to $\{v_1, v_2, v_3, v_5, u_2\}$. Thus, $d(w_4) \le 3$, a contradiction. Hence,

G contains no C_5+e.

Now let $C = v_1v_2v_3v_4v_5v_1$. Since G contains no C_5+e and $\delta(G) \ge 4$, each v_i has at least two adjacent vertices in $V(G) - V(C)$. Because $|V(G) - V(C)| \le 7$, there exist v_i, v_j such that v_i and v_j have at least one common adjacent vertex in $V(G) - V(C)$. Without loss of generality, assume that u_1 is adjacent to v_1 and v_3, where $u_1 \in V(G) - V(C)$. We distinguish two subcases as below.

Subcase 2.1. $N(v_2) \cap (N(v_4) \cup N(v_5)) \not\subseteq V(C)$.

There exists $u_2 \in V(G) - V(C)$ such that $u_2 \in N(v_2)$ and $u_2 \in N(v_4) \cup N(v_5)$. By symmetry, we may assume that u_2 is adjacent to v_2 and v_4. It is obvious that u_1 and u_2 are distinct and $u_1u_2 \notin E(G)$, otherwise G contains C_6. Since $\delta(G) \geq 4$, u_1 has at least two adjacent vertices in $V(G)-V(C)-\{u_1,u_2\}$, two of which are denoted by w_1,w_2; u_2 has at least two adjacent vertices in $V(G)-V(C)-\{u_1,u_2\}$, two of which are denoted by w_3,w_4; v_5 has at least two adjacent vertices in $V(G)-V(C)-\{u_1,u_2\}$, two of which are denoted by w_5,w_6. We prove that $w_1,w_2,\ldots,w_6$ are six distinct vertices. If u_1 and v_5 have a common adjacent vertex in $V(G)-V(C)-\{u_1,u_2\}$, say, w, then $wu_1v_3v_2v_1v_5w$ is a C_6, a contradiction. By symmetry, u_2 and v_5 have no common adjacent vertex in $V(G)-V(C)-\{u_1,u_2\}$. If u_1 and u_2 have a common adjacent vertex in $V(G)-V(C)-\{u_1,u_2\}$, say, w', then $w'u_1v_1v_5v_4u_2w'$ is a C_6, a contradiction. Thus, $V(G)$ contains $\{v_i,u_j,w_k \mid 1\leq i\leq 5, 1\leq j\leq 2, 1\leq k\leq 6\}$ and hence $|V(G)|\geq 13$, which contradicts $|V(G)|\leq 12$.

Subcase 2.2. $N(v_2)\cap(N(v_4)\cup N(v_5))\subseteq V(C)$.

Since G contains no C_5+e and $\delta(G)\geq 4$, each of v_2,v_4,v_5 has at least two adjacent vertices in $V(G)-V(C)-\{u_1\}$. Let $w_1,w_2\in N(v_2)\cap(V(G)-V(C)-\{u_1\})$, $w_3,w_4\in N(v_4)\cap(V(G)-V(C)-\{u_1\})$ and $w_5,w_6\in N(v_5)\cap(V(G)-V(C)-\{u_1\})$. Since $N(v_2)\cap(N(v_4)\cup N(v_5))\subseteq V(C)$, then $w_1,w_2,\ldots,w_6$ are six distinct vertices. Because $|V(G)|\leq 12$, $V(G)=\{v_i,w_j,u_1 \mid 1\leq i\leq 5, 1\leq j\leq 6\}$. We see that w_1 is nonadjacent to v_4 or v_5, otherwise $N(v_2)\cap(N(v_4)\cup N(v_5))\nsubseteq V(C)$, which is Subcase 2.1. And w_1 is nonadjacent to w_3, otherwise $w_1w_3v_4v_5v_1v_2w_1$ is a C_6. For the same reason, w_1 is nonadjacent to w_4,w_5,w_6. Hence, $d(w_1)\leq 3$, which contradicts $\delta(G)\geq 4$.

Therefore, we have $R(C_6,K_{1,n})=n+4$ for $n=7,8$. □

3.4 Proof of Theorem 3.4

For $i=1,2,3$, let G_i be formed from $3K_5$, i of which are sharing exactly one vertex. Then $|V(G_i)|=16-i$, G_i contains no C_6 and $\delta(G_i)\geq 4$. Thus, $\overline{G_i}$ contains no $K_{1,12-i}$ for $i=1,2,3$. It follows that $R(C_6,K_{1,n})\geq n+5$ for $n=9,10,11$. Let G be a graph of order $n+5$. Suppose to the contrary that G contains no C_6 and $\overline{G}$ contains no $K_{1,n}$, then $\delta(G)\geq 5$. By Theorem 3.5, G contains C_4. Let C be a longest cycle with $|V(C)|\leq 6$ in G, then $4\leq |V(C)|\leq 5$. We distinguish two cases.

Case 1. $|V(C)|=4$.

If G contains $K_{2,3}$, assume that both v_1 and v_2 are adjacent to each vertex of v_3,v_4,v_5. We see that $v_3v_4 \notin E(G)$, otherwise $v_1v_3v_4v_2v_5v_1$ is a C_5, a contradiction. For the same reason, $v_4v_5, v_3v_5 \notin E(G)$. Since $\delta(G) \geq 5$, for $i = 3,4,5$, each v_i has another adjacent vertex in $V(G) - \{v_1,v_2,v_3,v_4,v_5\}$, denoted by u_i. Since G contains no C_6, then u_3,u_4,u_5 are three distinct vertices. Let $V_1 = \{v_i,u_j \mid 1 \leq i \leq 5, 3 \leq j \leq 5\}$, then for $i = 3,4,5$, u_i is nonadjacent to $V_1 - \{v_i\}$, otherwise G contains C_5 or C_6, a contradiction. Since $\delta(G) \geq 5$, each of u_3,u_4,u_5 has at least four adjacent vertices in $V(G)-V_1$. Since $14 \leq |V(G)| \leq 16$, $6 \leq |V(G) - V_1| \leq 8$, it follows that at least two vertices of u_3,u_4,u_5 have a common adjacent vertex in $V(G) - V_1$, say, both u_3 and u_4 are adjacent to $w \in V(G) - V_1$. Hence, $wu_3v_3v_1v_4u_4w$ is a C_6, a contradiction. Thus,

G contains no $K_{2,3}$.

Since $|V(C)| = 4$, let $C = v_1v_2v_3v_4v_1$. Since $\delta(G) \geq 5$, each v_i has another adjacent vertex in $V(G) - \{v_1,v_2,v_3,v_4\}$, denoted by u_i, where $1 \leq i \leq 4$. We see that u_1,u_2,u_3,u_4 are four distinct vertices, otherwise G contains C_5 or $K_{2,3}$. Let $V_1 = \{v_i,u_i \mid 1 \leq i \leq 4\}$, then for $1 \leq i \leq 4$, u_i is nonadjacent to $V_1 - \{v_i\}$, otherwise G contains C_5 or C_6 or $K_{2,3}$, a contradiction. Since $\delta(G) \geq 5$, each u_i has at least four adjacent vertices in $V(G) - V_1$. Since $14 \leq |V(G)| \leq 16$, $6 \leq |V(G) - V_1| \leq 8$, it follows that at least two vertices of u_1,u_2,u_3 have a common adjacent vertex in $V(G)-V_1$. If both u_1 and u_2 are adjacent to $w \in V(G)-V_1$, then $wu_1v_1v_2u_2w$ is a C_5, a contradiction. By symmetry, u_2 and u_3 have no common adjacent vertex in $V(G) - V_1$. If both u_1 and u_3 are adjacent to $w \in V(G) - V_1$, then $wu_1v_1v_2v_3u_3w$ is a C_6, also a contradiction.

Case 2. $|V(C)| = 5$.

If G contains K_5, let $V(K_5) = \{v_1,v_2,v_3,v_4,v_5\}$. Since $\delta(G) \geq 5$, each v_i has an adjacent vertex not in $V(K_5)$, denoted by u_i. It is easy to check that u_1,u_2,u_3,u_4,u_5 are pairwise distinct. Let $V_1 = \{v_i,u_i \mid 1 \leq i \leq 5\}$, then u_i has at most one adjacent vertex in V_1. In this way, u_i has at least four adjacent vertex in $V(G) - V_1$, denoted by w_{ij}, where $1 \leq i \leq 5$, $1 \leq j \leq 4$. Any two vertices of $\{u_i \mid 1 \leq i \leq 5\}$ have no common adjacent vertex in $V(G) - V_1$, otherwise G contains C_6. For this reason, $|V(G)| \geq 30$, which contradicts $|V(G)| \leq 16$. Therefore,

G contains no K_5.

If G contains C_5+e, let $v_1v_2v_3v_4v_5v_1$ be a C_5 (with one chord). Since G contains C_5+e but no K_5, we can always find some i such that $v_iv_{i+2}\in E(G)$ and $v_{i+1}v_{i+3}\notin E(G)$, where the indices are taken modulo 5. Without loss of generality, we may assume that $v_1v_3\in E(G)$ and $v_2v_4\notin E(G)$. Each of v_2,v_4,v_5 has another adjacent vertex in $V(G)-\{v_1,v_2,v_3,v_4,v_5\}$, denoted by u_2,u_4,u_5, respectively. We see that u_2,u_4 are distinct, otherwise $u_2v_2v_3v_1v_5v_4u_2$ is a C_6. For the same reason, u_2,u_4,u_5 are pairwise distinct. We assert that either $v_1u_4\notin E(G)$ or $v_3u_5\notin E(G)$, otherwise $v_1u_4v_4v_5u_5v_3v_1$ is a C_6. Without loss of generality, we assume that $v_1u_4\notin E(G)$. Let $V_1=\{v_1,v_2,\ldots,v_5,u_2,u_4,u_5\}$. It is easy to check that both u_2 and u_4 have at most one adjacent vertex in V_1. Since $\delta(G)\geq 5$, both u_2 and u_4 have at least four adjacent vertices in $V(G)-V_1$. Let $\{w_1,w_2,w_3,w_4\}$ be a subset of $N(u_2)\cap(V(G)-V_1)$ and $\{w_5,w_6,w_7,w_8\}$ be a subset of $N(u_4)\cap(V(G)-V_1)$. We deduce that $w_1,w_2,\ldots,w_8$ are pairwise distinct, otherwise for some $w\in N(u_2)\cap N(u_4)$, $wu_2v_2v_3v_4u_4w$ is a C_6. Since $|V(G)|\leq 16$, then $V(G)=\{v_i,u_j,w_k\mid 1\leq i\leq 5, 1\leq k\leq 8, j=2,4,5\}$. Since $\delta(G)\geq 5$, G contains no C_6 and $v_2v_4\notin E(G)$, it follows that v_2 has to be adjacent to $\{w_1,w_2,w_3,w_4\}$. Since $\delta(G)\geq 5$, G contains no C_6 and $v_1u_4\notin E(G)$, it follows that v_1 has to be adjacent to $\{w_1,w_2,w_3,w_4\}$. If v_1 and v_2 have a common adjacent vertex in $\{w_1,w_2,w_3,w_4\}$, say, w_1, then $w_1v_2v_3v_4v_5v_1w_1$ is a C_6. If v_1 and v_2 have no common adjacent vertex in $\{w_1,w_2,w_3,w_4\}$, say, $v_1w_1,v_2w_2\in E(G)$, then $v_1w_1u_2w_2v_2v_3v_1$ is a C_6. Hence,

G contains no C_5+e.

Now let $C=v_1v_2v_3v_4v_5v_1$. Since G contains no C_5+e and $\delta(G)\geq 5$, each v_i has at least three adjacent vertices in $V(G)-V(C)$. Because $|V(G)-V(C)|\leq 11$, there exist v_i,v_j such that v_i and v_j have at least one common adjacent vertex in $V(G)-V(C)$. Without loss of generality, assume that u_1 is adjacent to v_1 and v_3, where $u_1\in V(G)-V(C)$. We distinguish two subcases as below.

Subcase 2.1. $N(v_2)\cap(N(v_4)\cup N(v_5))\not\subseteq V(C)$.

There exists $u_2\in V(G)-V(C)$ such that $u_2\in N(v_2)$ and $u_2\in N(v_4)\cup N(v_5)$. By symmetry, we may assume that u_2 is adjacent to v_2 and v_4. It is obvious that u_1 and u_2 are distinct and $u_1u_2\notin E(G)$, otherwise G contains C_6. Since G contains no C_5+e and $\delta(G)\geq 5$, each of u_1,u_2,v_5 has at least three adjacent vertices in $V(G)-V(C)-\{u_1,u_2\}$, v_1 has at least two adjacent vertices in $V(G)-$

$V(C)-\{u_1,u_2\}$. It is not difficult to check that any two vertices of $\{u_1,u_2,v_1,v_5\}$ have no common adjacent vertex in $V(G)-V_1$, otherwise G contains C_6. For this reason, $|V(G)|\geq 18$, which contradicts $|V(G)|\leq 16$.

Subcase 2.2. $N(v_2)\cap(N(v_4)\cup N(v_5))\subseteq V(C)$.

We claim that either v_1,v_4 have no common adjacent vertex in $V(G)-V(C)-\{u_1\}$, or v_3,v_5 have no common adjacent vertex in $V(G)-V(C)-\{u_1\}$. If not, let $wv_1,wv_4,w'v_3w'v_5\in E(G)$, where $w,w'\in V(G)-V(C)-\{u_1\}$. Obviously, w,w' are distinct. Then $wv_1v_5w'v_3v_4w$ is a C_6, a contradiction. Without loss of generality, we assume that v_1,v_4 have no common adjacent vertex in $V(G)-V(C)-\{u_1\}$. Since G contains no C_5+e and $\delta(G)\geq 5$, each of v_2,v_4,v_5 has at least three adjacent vertices in $V(G)-V(C)-\{u_1\}$, v_1 has at least two adjacent vertices in $V(G)-V(C)-\{u_1\}$. It is not difficult to check that any two vertices of $\{v_1,v_2,v_4,v_5\}$ have no common adjacent vertex in $V(G)-V(C)-\{u_1\}$. For this reason, $|V(G)|\geq 17$, which contradicts $|V(G)|\leq 16$.

Therefore, we have $R(C_6,K_{1,n})=n+5$ for $n=9,10,11$. □

3.5 Proof of Theorem 3.10

In order to prove Theorem 3.10, we need the following four lemmas.

Lemma 3.2 (Faudree and Schelp [58]). $R(C_n,C_4)=n+1$ *for* $n\geq 6$.

Lemma 3.3 (Tse [134]). $R(W_6,C_4)=9$, $R(W_n,C_4)=n+4$ *for* $7\leq n\leq 10$.

Lemma 3.4 (Faudree et al. [57]). $R(S_7,C_4)=9$.

Lemma 3.5 (Zhang et al. [139]). *Let C be a longest cycle in a graph G and $u\in V(G)-V(C)$. Then $\alpha(G)\geq d_C(u)+1$.*

Now we have all ingredients to prove Theorem 3.10.

We first prove that $R(S_{n+1},C_4)\geq n+4$ for $n\geq 7$. Let $k=\lfloor (n+1)/4\rfloor$ and $C=x_1x_2\ldots x_{4k}x_1$ be a cycle of length $4k$. Set $X_1=\{x_1,x_2\}$, $X_2=\{x_3,x_4\}$, $X_3=\{x_i\mid i\equiv 1,2 \pmod 4 \text{ and } i\geq 5\}$ and $X_4=\{x_i\mid i\equiv 0,3 \pmod 4 \text{ and } i\geq 5\}$. We now construct a graph F of order $n+3$ from C as follows: $V(F) = V(C) \cup \{z_i \mid 1 \le i \le l\}$, where $4k + l = n+3$. If $n \equiv 3 \pmod 4$, then let $N(z_1) = X_1 \cup X_3$ and $N(z_2) = X_2 \cup X_4$; if $n \equiv 0 \pmod 4$, then let $N(z_1) = X_1 \cup \{z_2\}$, $N(z_2) = X_3 \cup \{z_1\}$ and $N(z_3) = X_2 \cup X_4$; if $n \equiv 1 \pmod 4$, then let $N(z_1) = X_1 \cup \{z_2\}$, $N(z_2) = X_3 \cup \{z_1\}$, $N(z_3) = X_2 \cup \{z_4\}$ and $N(z_4) = X_4 \cup \{z_2\}$; if $n \equiv 2 \pmod 4$, then let $N(z_1) = X_1 \cup \{z_2\}$, $N(z_2) = X_3 \cup \{z_1\}$, $N(z_3) = X_2 \cup \{z_4\}$, $N(z_4) = X_4 \cup \{z_2\}$ and $N(z_5) = \{z_1, z_2, z_3, z_4\}$. It is easy to check that F has no C_4 and $\delta(F) \ge 3$. Therefore, $R(S_{n+1}, C_4) \ge n+4$ for $n \ge 7$.

Since $S_{n+1} \subseteq W_n$, we have $R(W_n, C_4) \ge R(S_{n+1}, C_4)$. By Lemmas 3.3 and 3.4, we see that $R(W_6, C_4) = R(S_7, C_4)$ and $R(W_n, C_4) = n+4$ for $7 \le n \le 10$. Since $R(S_{n+1}, C_4) \ge n+4$ for $n \ge 7$, we get that $R(W_n, C_4) = R(S_{n+1}, C_4)$ for $7 \le n \le 10$. Now it remains to show that $R(W_n, C_4) \le R(S_{n+1}, C_4)$ for $n \ge 11$. Let G be a graph of order $N = R(S_{n+1}, C_4) \ge n+4$. Set $v \in V(G)$ with $d(v) = \Delta(G)$, $Z = V(G) - N[v]$. Suppose to the contrary that neither G contains a W_n nor $\overline{G}$ contains a C_4. Thus, noting that $N = R(S_{n+1}, C_4)$, we have $d(v) \ge n$. If $d(v) \ge n+1$, then by Lemma 3.2, $G[N(v)]$ contains a C_n, which together with v forms a W_n in G, a contradiction. Hence we have $d(v) = n$. By Theorem 3.5, $|Z| = N - (n+1) \le \lfloor \sqrt{n-1} \rfloor + 1$. Let C be a longest cycle in $G[N(v)]$. By Lemma 3.2, we have $|C| \ge n-1$, and so $|C| = n-1$. Set $u = N(v) - V(C)$. If $d_C(u) \ge 3$, then by Lemma 3.5, $\alpha(G[N(v)]) \ge 4$, which implies that $\overline{G}$ contains a C_4, and hence $d_C(u) \le 2$. If there exists some vertex $y \in V(G) - \{u\}$ such that y has two nonadjacent vertices $y_1, y_2 \in V(C) - N_C(u)$, then uy_1yy_2u is a C_4 in $\overline{G}$, and hence y has at most one nonadjacent vertex in $V(C) - N_C(u)$ for each $y \in V(G) - \{u\}$. Since $n \ge 11$, $|Z| \le \lfloor \sqrt{n-1} \rfloor + 1$ and $d_C(u) \le 2$, we have

$$\begin{aligned} |V(C) - N_C(u)| - |N_C(u) \cup Z| &= |C| - d_C(u) - |Z| - d_C(u) \\ &\ge (n-1) - 2 - (\lfloor \sqrt{n-1} \rfloor + 1) - 2 \ge 2. \end{aligned}$$

Because every vertex of $N_C(u) \cup Z$ has at least $|V(C) - N_C(u)| - 1$ adjacent vertices in $V(C) - N_C(u)$, by the Pigeonhole Principle, there exists some vertex $w \in V(C) - N_C(u)$ such that $N_C(u) \cup Z \subseteq N(w)$. Noting that w has at most one nonadjacent vertex in $V(C) - N_C(u)$ and $wv \in E(G)$, we have

$$d(w) \ge |V(C) - N_C(u)| - 2 + |N_C(u) \cup Z| + 1 = |C| + |Z| - 1 = N - 3 \ge n+1,$$

which contradicts the fact that $d(v) = \Delta(G) = n$.

This completes the proof of Theorem 3.10. □

3.6 Proof of Theorem 3.12

We need Theorems 1.1, 1.2, 1.5, 1.6 and the following lemmas.

Lemma 3.6 (Jackson [78]). *Let $G=(X,Y)$ be a bipartite graph with bipartition classes X and Y such that $d(x)\geq t$ for all $x\in X$, where $|X|\geq 2$ and $2\leq t\leq |Y|\leq 2t-2$. Then G contains all cycles on $2m$ vertices for $2\leq m\leq \min\{|X|,t\}$.*

The following lemma is in fact the same as Lemma 2.3, and the proof of which can be found in Chapter 2.

Lemma 3.7 (Zhang et al. [141]). *Let C be a longest cycle in a graph G and $v_1,v_2\in V(G)-V(C)$ with $t=|N_C(v_1)\cup N_C(v_2)|$. Then $t\leq \lfloor |C|/2\rfloor+1$ and if $v_1v_2\in E(G)$, then $t\leq \lfloor |C|/2\rfloor$.*

Lemma 3.8 (Jayawardene and Rousseau [79]). $R(W_5,C_4)=10$.

By Lemma 3.8, we may assume that $m\geq 6$. The lower bound $R(W_n,C_m)\geq 3m-2$ follows from the fact that a complete tripartite graph $K_{m-1,m-1,m-1}$ contains no W_n and its complement contains no C_m. To prove $R(W_n,C_m)\leq 3m-2$, let G be graph of order $3m-2$, and suppose that neither G contains W_n nor $\overline{G}$ contains C_m.

We first show that G contains no K_n. If G contains a K_n, then every other vertex in G has at most two neighbors in K_n; otherwise G contains a W_n. Since $n-2\geq m/2$, by Lemma 3.6, $\overline{G}$ contains a C_m, a contradiction. Thus, G contains no K_n.

We next show that $\delta(\overline{G})=m-1$. Let $v\in V(G)$ with $d(v)=\Delta(G)=d$, let $H=G[N(v)]$ and let $Z=V(G)-N[v]$. If $\overline{G}$ is a bipartite graph, say with $V(\overline{G})=(X,Y)$ and $|X|\geq|Y|$, then $|X|\geq 3m/2-1\geq n$, which implies that $G[X]$ contains a K_n, a contradiction. Thus, $\overline{G}$ is nonbipartite. If $\delta(\overline{G})\geq m$, then by Theorems 1.1 and 1.5, $\overline{G}$ contains C_m, a contradiction. If $\delta(\overline{G})\leq m-2$, then $d(v)\geq 2m-1$, that is, $|H|\geq 2m-1$. Since $\overline{H}$ has no C_m, H contains a C_n by Theorem 1.17, which together with v forms a W_n in G, a contradiction. Thus, we have $\delta(\overline{G})=m-1$ and $d=2m-2$.

We next show that H and $\overline{H}$ are both nonbipartite. First assume that H is a bipartite graph, say with $V(H)=(X,Y)$. Then, since $\overline{G}$ contains no C_m, we have

$|X| = |Y| = m-1$ and $e(X,Y) \geq |X||Y|-1$. Because $\delta(\overline{G}) = m-1 \geq 5$, there exist two distinct vertices $x_1, x_2 \in X$ such that $x_1z_1, x_2z_2 \in E(\overline{G})$, where $z_1, z_2 \in Z$. If $z_1 = z_2$, then $\overline{G}[X \cup \{z_1\}]$ contains a C_m; and if $z_1 \neq z_2$, then noting that $z_1, z_2 \in N_{\overline{G}}(v)$, we see that $\overline{G}[X \cup \{z_1, z_2, v\}]$ has a C_m. Hence, H is nonbipartite.

Now suppose $\overline{H}$ is a bipartite graph. Let $V(\overline{H}) = (X,Y)$ and $|X| \geq |Y|$. Since G has no K_n, we get $|X \cup \{v\}| \leq n-1$, hence $|X| \leq n-2$ and $|Y| \geq 2m-n \geq m/2+1$. If $\kappa(H) \geq 2$, then H has two independent edges between X and Y. Since both $G[X]$ and $G[Y]$ are complete graphs, H contains a C_n, a contradiction. Let now $\kappa(H) \leq 1$. Then there exists a vertex w such that $\overline{H}-w$ is a complete bipartite graph in which each partite set has at least $m/2$ vertices. Since m is even, $\overline{H}$ contains a C_m, a contradiction. Therefore, $\overline{H}$ is also nonbipartite.

If $|E(\overline{H})| \geq d(d-1)/4+1/2$, then by Theorems 1.6 and 1.2, $\overline{H}$ contains a C_m, a contradiction. Thus, $|E(\overline{H})| < d(d-1)/4+1/2$. Since m is even and $d = 2m-2$, we have $d \equiv 2 \pmod 4$. Thus, $|E(\overline{H})| \leq d(d-1)/4-1/2$, and hence $|E(H)| \geq d(d-1)/4+1/2$. By Theorems 1.6 and 1.2, H is weakly pancyclic with $g(H) = 3$ and $c(H) \geq m$. Let $C = x_1x_2\cdots x_rx_1$ be a longest cycle in H and $V(H)-V(C) = Y = \{y_1, y_2, \ldots, y_{d-r}\}$. Then $m \leq r \leq n-1$ and $d-r \geq 2m-n-1 \geq m/2$. By Lemma 3.7, $|N_C(y_i) \cup N_C(y_j)| \leq \lfloor |C|/2 \rfloor + 1$, and so $|N_{\overline{G}}(y_i) \cap N_{\overline{G}}(y_j) \cap V(C)| \geq \lceil |C|/2 \rceil - 1 \geq m/2-1$ for any two distinct vertices $y_i, y_j \in Y$. If $|N_{\overline{G}}(y_i) \cap N_{\overline{G}}(y_j) \cap V(C)| \geq m/2$ for two distinct vertices $y_i, y_j \in Y$, say $y_1, y_{m/2}$ are two such vertices. Then we can choose $m/2$ vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_{m/2}}$ from $V(C)$ such that $x_{i_j} \in N_{\overline{G}}(y_j) \cap N_{\overline{G}}(y_{j+1})$ for $1 \leq j \leq m/2-1$ and $x_{i_{m/2}} \in N_{\overline{G}}(y_{m/2}) \cap N_{\overline{G}}(y_1)$, implying that $y_1x_{i_1}y_2x_{i_2}y_3\cdots y_{m/2}x_{i_{m/2}}y_1$ is a C_m in $\overline{G}$, a contradiction. Thus, we have $|N_{\overline{G}}(y_i) \cap N_{\overline{G}}(y_j) \cap V(C)| = m/2-1$ for any two distinct vertices $y_i, y_j \in Y$. By Lemma 3.7, we have $r = m$, $d-r = m-2$ and $\overline{G}[Y] = K_{m-2}$. Let $x' \in N_{\overline{G}}(y_1) \cap N_{\overline{G}}(y_2) \cap V(C)$ and $x'' \in N_{\overline{G}}(y_2) \cap N_{\overline{G}}(y_3) \cap V(C) - \{x'\}$. Then $\overline{G}[Y \cup \{x', x''\}]$ contains a C_m, our final contradiction.

This completes the proof of Theorem 3.12. □

Chapter 4

Small wheels versus cycles

4.1 Introduction

In this chapter, we study Ramsey numbers for small (generalized) wheels versus cycles. That is, $R(C_m, W_n)$ and $R(C_m, W_{2,n})$ for $m \geq n$, where $W_{2,n} = K_2 + C_n$.

Wheels have enjoyed quite a lot of attention in the context of Ramsey numbers. We mentioned in Chapter 2 that, in the earliest contribution involving cycle-wheel Ramsey numbers, dating back to 1983, Burr and Erdős [20] determined the Ramsey numbers of a triangle versus wheels of arbitrarily large order. From then on, many papers have been published on cycle-wheel Ramsey numbers. But most of them are about small wheels versus large cycles. Note that even (odd) wheels correspond to odd (even) n.

For large cycles versus even wheels, Surahmat et al. [131] determined that $R(C_m, W_n) = 3m-2$ for odd $n \geq 5$ and $m > (5n-9)/2$. This result was improved by Shi [119] who showed that $R(C_m, W_n) = 3m-2$ for odd n and $m \geq 3n/2+1$. Then Zhang et al. [139] refined that result to $R(C_m, W_n) = 3m-2$ for odd n, $m \geq n$ and $m \geq 20$. Finally, Chen et al. gave a simpler proof that completely solves this case.

Theorem 4.1 (Chen et al. [27]). *$R(C_m, W_n) = 3m-2$ for odd n, $m \geq n \geq 3$ and $(m, n) \neq (3, 3)$.*

For large cycles versus odd wheels, Surahmat et al. [130] proved the first nontrivial result, which is $R(C_m, W_n) = 2m-1$ for even n and $m \geq 5n/2-1$.

Chen et al. [26] improved this result by reducing the lower bound on m from $m \geq 5n/2-1$ to $m \geq 3n/2+1$.

Theorem 4.2 (Chen et al. [26]). *$R(C_m, W_n) = 2m-1$ for even n and $m \geq 3n/2+1$.*

To completely solve this case, Surahmat et al. [130] proposed the following conjecture.

Conjecture 4 (Surahmat et al. [130]). $R(C_m, W_n) = 2m-1$ for even n, $m \geq n$ and $(m,n) \neq (4,4)$.

We will confirm the above conjecture for large m in this chapter by proving the following result. We postpone the proof to Section 4.2.

Theorem 4.3. *$R(C_m, W_n) = 2m-1$ for even n and $m \geq n+502$.*

For large cycles versus generalized even wheels, Surahmat et al. [133] showed that $R(C_m, W_{2,n}) = 3m-2$ for even $n \geq 4$ and $m \geq 9n/2+1$. Shi [119] improved this result by reducing the lower bound on m from $m \geq 9n/2+1$ to $m \geq \max\{3n/2+1, 71\}$. For large cycles versus generalized odd wheels, in Section 4.3 we prove the following result that has the same flavor.

Theorem 4.4. *$R(C_m, W_{2,n}) = 4m-3$ for n odd, $m \geq 9n/8+1$.*

Another result we have obtained is based on the definition of Ramsey unsaturated graph, which is a little different with the original definition given by Balister et al. [4].

Definition 5. A graph G_1 is Ramsey unsaturated with respect to G_2 if there exists an edge $e \in \overline{G_1}$ such that $R(G_1+e, G_2) = R(G_1, G_2)$. The graph G_1 is Ramsey saturated with respect to G_2 if $R(G_1+e, G_2) > R(G_1, G_2)$ for all $e \in \overline{G_1}$.

It has been proved by Ali and Surahmat [2] that C_m is Ramsey unsaturated with respect to W_n for $m \geq 5n/2+1$. We here consider all C_m with respect to W_n for $m \geq n+1$, and all W_n with respect to C_m for $n \geq m$. The proof will be given in the final section.

Theorem 4.5. (1) *C_m is Ramsey unsaturated with respect to W_n for $m \ge \max\{n+1, 6\}$.*

(2) *C_4 is Ramsey saturated with respect to W_n. C_5 is Ramsey saturated with respect to W_4, and Ramsey unsaturated with respect to W_3.*

(3) *W_n is Ramsey unsaturated with respect to C_m for $n \ge \max\{m, 5\}$.*

(4) *W_4 is Ramsey unsaturated with respect to C_3, and Ramsey saturated with respect to C_4.*

4.2 Proof of Theorem 4.3

We need Theorems 1.1, 1.3, 1.6, 1.17, 2.1 and three lemmas as below.

Lemma 4.1 (Brandt et al. [16]). *Let G be a 2-connected nonbipartite graph of order n with minimum degree $\delta(G) \ge n/4 + 250$. Then G is weakly pancyclic unless G has odd girth 7, in which case it has cycles of every length from 4 up to its circumference except a 5-cycle.*

Lemma 4.2 (Dirac [38]). *Let G be a graph of order n. If $\delta(G) \ge n/2+1$, then G is Hamilton-connected.*

Lemma 4.3 (Károlyi and Rosta [82]). *Suppose that G has a cycle $C = x_1x_2\cdots x_{2\ell}x_1$, but neither G nor $\overline{G}$ has a $C_{2\ell-1}$. Then $\overline{G}[\{x_1, x_3, \dots, x_{2\ell-1}\}] = \overline{G}[\{x_2, x_4, \dots, x_{2\ell}\}] = K_\ell$.*

Let G be a graph of order $2m-1$ with n even and $m \ge n+502$. Suppose to the contrary that neither G contains W_n nor $\overline{G}$ contains C_m.

Assume that $v \in V(G)$ with $d(v) = d = \Delta(G)$ and $H = G[N(v)]$. We are first going to show that $d \ge (3m+1)/2 - 252$. To the contrary, assume that $d \le 3m/2 - 252$. Then $\delta(\overline{G}) \ge 2m-2-(3m/2-252) = m/2+250$. By Theorem 2.1, we have $g(\overline{G}) = 3$, and so $\overline{G}$ is nonbipartite. If $\kappa(\overline{G}) \ge 2$, then $\overline{G}$ contains C_m by Theorem 1.1 and Lemma 4.1, a contradiction. So, we assume next that $\kappa(\overline{G}) \le 1$. Then there exists some $u \in V(G)$ such that $G-u$ contains a spanning complete bipartite graph with bipartite sets V_1, V_2 and $|V_1| \ge |V_2|$. Obviously, $|V_1| \ge m-1$ and $|V_2| \ge \delta(\overline{G})$. If $\Delta(G[V_1]) \ge n/2$, let $x \in V_1$ with $d(x) = \Delta(G[V_1])$. Then x together with $n/2$ neighbors in V_1 and $n/2$ neighbors in V_2 form a W_n with x as its hub, a contradiction. This implies that $\delta(\overline{G}[V_1]) \ge |V_1| - n/2 > |V_1|/2+1$.

If $|V_1| \geq m$, then by Theorem 1.3, $\overline{G}[V_1]$ contains C_m, a contradiction. Thus, we conclude that $|V_1| = |V_2| = m-1$. Since $\delta(\overline{G}) > 250$, we may assume that $x_1, x_2 \in N_{\overline{G}}(u) \cap V_1$. By Lemma 4.2, $\overline{G}[V_1]$ has a Hamilton (x_1,x_2)-path which together with u forms C_m in $\overline{G}$, our final contradiction. Therefore, henceforth we assume that $d \geq (3m+1)/2 - 252$.

By the assumptions, H has no C_n. If m is even, then since $m \geq n + 502$, we get that $R(C_n, C_m) \leq 3m/2 - 252$ by Theorem 1.17, a contradiction. Thus, m is odd. Next, we first prove the following claim.

Claim 1. *H contains no $2K_{(m+1)/2}$.*

Proof. If there exist two independent edges between V_1 and V_2, then H contains a C_n, a contradiction. If there is at least one edge between V_1 and V_2, then V_1 or V_2 contains a vertex w such that $E(V_1 - \{w\}, V_2 - \{w\}) = \emptyset$. For any $u \notin V_1 \cup V_2 - \{w\}$, we have $V_1 - \{w\} \subseteq N(u)$ or $V_2 - \{w\} \subseteq N(u)$, for otherwise $\overline{G}[V_1 \cup V_2 \cup \{u\}]$ contains C_m. Let $U_i = \{u \mid V_i - \{w\} \subseteq N(u), u \notin V_1 \cup V_2 - \{w\}\}$ for $i = 1, 2$. Assume that $|(V_1 - \{w\}) \cup U_1| \geq |(V_2 - \{w\}) \cup U_2|$. Then $|(V_1 - \{w\}) \cup U_1| \geq m \geq n + 502$. Taking $n+1$ vertices from $(V_1 - \{w\}) \cup U_1$ such that at least $n/2+1$ of them are in $V_1 - \{w\}$, we obtain W_n in $G[(V_1 - \{w\}) \cup U_1]$, a contradiction. If there is no edge between V_1 and V_2, we can derive a contradiction in a similar way. □

We next show that H is nonbipartite. To the contrary, suppose H is a bipartite graph with $V(H) = (X, Y)$ and $|X| \geq |Y|$. Since $\overline{G}$ contains no C_m, we have $|X| \leq m-1$ and $|Y| = d - |X| \geq (m+1)/2 - 251 > n/2$. If there exists two independent edges between X and Y in $\overline{G}$, then since both $\overline{G}[X]$ and $\overline{G}[Y]$ are complete graphs, $\overline{H}$ contains C_m, a contradiction. Thus, there exists some vertex $z \in X \cup Y$ such that $H - z$ with bipartition $(X - z, Y - z)$ is a complete bipartite graph. But then $H - z$ contains $K_{n/2,n/2}$, and so H contains C_n, a contradiction. Therefore, henceforth we assume that H is nonbipartite.

We now show that $\overline{H}$ is also nonbipartite. To the contrary, suppose $\overline{H}$ is a bipartite graph with $V(\overline{H}) = (X, Y)$ and $|X| \geq |Y|$. Since $G[X]$ contains no C_n, we have $|X| \leq n - 1 \leq m - 503$ and hence $|Y| = d - |X| \geq (m+1)/2 + 251$. Clearly, H contains $2K_{(m+1)/2}$, contradicting Claim 1. Thus, henceforth we assume that $\overline{H}$ is nonbipartite.

If $|E(\overline{H})| > (d-1)^2/4 + 1$, then by Theorem 1.6, $\overline{H}$ is weakly pancyclic with $g(\overline{H}) = 3$. By Theorem 1.17, $R(C_n, C_{m+1}) = m + n/2 \leq (3m+1)/2 - 252$ for m is odd and $m \geq n+502$. Since H contains no C_n, $\overline{H}$ contains C_{m+1}, which implies that $\overline{H}$ contains C_m, a contradiction. Thus, we have $|E(H)| \geq d(d-1)/2-(d-1)^2/4-1 > (d-1)^2/4+1$. By Theorem 1.6, H is weakly pancyclic with $g(H)=3$. Since H contains no C_n, we have $c(H) < n$ and so H has no C_m. Because $\overline{H}$ contains C_{m+1} and has no C_m, H contains $2K_{(m+1)/2}$ by Lemma 4.3, contradicting Claim 1.

Since $2K_{m-1}$ contains no C_m and its complement contains no W_n, we obtain that $R(C_m,W_n) \geq 2m-1$. By the above arguments, $R(C_m,W_n) \leq 2m-1$ for n even and $m \geq n+502$. This completes the proof of Theorem 4.3. □

4.3 Proof of Theorem 4.4

We need the following useful lemmas. Except for one (Lemma 4.9 below), all results are from literature and presented without proofs. Also, we need some Theorems from Chapter 1, including Theorems 1.1, 1.2, 1.3, 1.5, 1.6 and 1.17.

Lemma 4.4 (Bollobás et al. [11]). $R(C_n,K_5)=4n-3$ *for* $n \geq 5$.

Lemma 4.5 (Faudree et al. [55]). *Let G be a graph of order $n \geq 6$. Then* $\max\{c(G), c(\overline{G})\} \geq \lceil 2n/3 \rceil$.

The following lemma can be found as Proposition 9.4 in [14].

Lemma 4.6. *Let G be a k-connected graph, and let X and Y be subsets of $V(G)$ of cardinality at least k. Then there exists a family of k pairwise disjoint (X,Y) paths in G.*

The following two lemmas are in fact the same as Lemmas 2.3 and 2.4, respectively. We present them here for convenience.

Lemma 4.7 (Zhang et al. [141]). *Let C be a longest cycle in a graph G and $v_1,v_2 \in V(G)-V(C)$ with $t=|N_C(v_1) \cup N_C(v_2)|$. Then $t \leq \lfloor |C|/2 \rfloor +1$ and if $v_1v_2 \in E(G)$, then $t \leq \lfloor |C|/2 \rfloor$.*

Lemma 4.8 (Zhang et al. [141]). *For a graph G, let (X,Y) be a partition of $V(G)$. Suppose that for some odd $n \geq 5$, $|Y| \geq (n+1)/2$ and any two vertices of Y have at least $(n-1)/2$ common nonadjacent vertices in X. If $\overline{G}$ contains no C_n, then $G[Y]$ is a complete graph.*

Lemma 4.9. *Let $C = x_1x_2\cdots x_rx_1$ be a longest cycle in a graph G with $r \geq n$, and let $Y = \{y_1, y_2, \ldots, y_{d-r}\} = V(G) - V(C)$ with $d - r \geq (n+1)/2$. Suppose that $\overline{G}$ contains no C_n and $G[Y]$ is a complete graph. Then $\overline{G}$ is bipartite.*

Proof. If $E(V(C), Y) = \emptyset$, then $G[V(C)]$ is a complete graph by Lemma 4.8. Hence, in this case it is easy to deduce that $\overline{G}$ is bipartite. Now let $E(V(C), Y) \neq \emptyset$, and let $x_1y_1 \in E(G)$. Then, since C is a longest cycle in G and $G[Y]$ a complete graph, it follows that $E(X_1, Y - \{y_1\}) = \emptyset$, where $X_1 = \{x_2, x_3, \ldots, x_{d-r+1}\} \cup \{x_r, x_{r-1}, \ldots, x_{2r-d+1}\}$. Because $d - r - 1 \geq (n-1)/2$, using Lemma 4.8 again, we obtain that $G[X_1]$ is a complete graph. Let $H \subseteq G[V(C)]$ be a maximal complete graph containing x_2. We claim that $V(C) - \{x_1\} \subseteq V(H)$. If not, there is an $x_i \in V(C) - V(H)$ such that x_i is adjacent to some vertex of H on the cycle and nonadjacent to some vertex of H. Then $E(\{x_i\}, Y - \{y_1\}) = \emptyset$; otherwise there clearly is a longer cycle than C. Furthermore, $\overline{G}[V(H) \cup (Y - \{y_1\}) \cup \{x_i\}]$ contains a C_n, a contradiction. For the same reason, we have $V(C) - \{x_1\} \subseteq N(x_1)$ or $Y - \{y_1\} \subseteq N(x_1)$. Therefore, $\overline{G}$ is bipartite. □

Lemma 4.10 (Surahmat et al. [128]). $R(C_n, W_4) = 2n - 1$ *for* $n \geq 5$.

Since $W_{2,3} = K_5$, using Lemma 4.4 we see that Theorem 4.4 holds for $n = 3$, and so we may assume that $n \geq 5$. Because neither $4K_{m-1}$ contains a C_m nor its complement contains a $W_{2,n}$, we get that $R(C_m, W_{2,n}) \geq 4m - 3$. So it suffices to show $R(C_m, W_{2,n}) \leq 4m - 3$. Let G be a graph of order $4m - 3$ with $m \geq 9n/8 + 1$. Suppose that G contains no $W_{2,n}$ and that $\overline{G}$ contains no C_m.

We distinguish the following two cases.

Case 1. $\overline{G}$ contains no $2K_{\lceil m/2 \rceil}$.

If $\overline{G}$ is bipartite, then $\alpha(\overline{G}) \geq 2m - 1 \geq n + 2$, which implies that G contains $W_{2,n}$, a contradiction. So $\overline{G}$ is nonbipartite. If $\delta(\overline{G}) \geq (4m-1)/3$, then $\overline{G}$ contains C_m by Theorems 1.5 and 1.1, a contradiction. Hence, $\delta(\overline{G}) \leq (4m-2)/3$ and so $\Delta(G) \geq (8m - 10)/3$. Let u be a vertex with $d_G(u) = \Delta(G) = d$ and let $G_u = G[N(u)]$.

Since $|G_u| \geq 2n + 3$ and G has no $W_{2,n}$, $\overline{G_u}$ is nonbipartite. We first show that $\delta(\overline{G_u}) \leq (d+1)/3$. To the contrary, assume $\delta(\overline{G_u}) \geq (d+2)/3$. Then by Theorem 1.5, $\overline{G_u}$ is weakly pancyclic with $g(\overline{G_u}) \leq 4$. If $\kappa(\overline{G_u}) \geq 2$, then by Theorem 1.1, $c(\overline{G_u}) \geq 2\delta(\overline{G_u}) \geq m$, so then $\overline{G_u}$ contains a C_m, a contradiction. Assume

$\kappa(\overline{G_u}) \le 1$. Then for some $w \in V(G_u)$, $V(G_u)-w$ can be partitioned into two parts V', V'' such that $e_G(V', V'') = |V'||V''|$. Assume that $|V'| \ge |V''|$ and choose w under these restrictions such that $|V'|-|V''|$ is as large as possible. It is obvious that $|V''| \ge \delta(\overline{G_u}) \ge \lceil m/2 \rceil$. Noting that $m \ge n+2 \ge 7$ and $d \ge \lceil (8m-10)/3 \rceil$, we get that $d \ge 2m+2$, and hence $|V'| \ge m+1$. If $\delta(\overline{G}[V']) \ge (|V'|+1)/2$, then by Theorem 1.3, $\overline{G}[V']$ contains a C_m, a contradiction. Thus there exists some $u' \in V'$ such that $d_{V'}(u') = \Delta(G[V']) > (|V'|-3)/2 \ge n/2$. If $G[V'']$ has at least one edge, then G has a C_n in $N(u')$ which together with u, u' forms a $W_{2,n}$ in G. So $\overline{G}[V'']$ is a complete graph. In this case, $|V''| \le m-1$, since $\overline{G}$ has no C_m. Thus, $|V'| = d-1-|V''| \ge m+2$, $d_{V'}(u') > (|V'|-3)/2$ and $d_{V'}(u') \ge \lceil m/2 \rceil \ge n/2+1$. Since $\overline{G}$ has no $2K_{\lceil m/2 \rceil}$, G has at least one edge in $N_{V'}(u')$, and so G has also a C_n in $N(u')$ which together with u, u' forms a $W_{2,n}$ in G. Therefore, $\delta(\overline{G_u}) \le (d+1)/3$, implying that $\Delta(G_u) \ge (2d-4)/3$.

Let v be a vertex of G_u with $d_{G_u}(v) = \Delta(G_u) = h$, and set $H = G_u[N(v)]$. Then $h \ge 16(m-2)/9$. If H contains a C_n, then this C_n together with u, v forms a $W_{2,n}$ in G, a contradiction. Hence, H contains no C_n.

We are now going to show that H and $\overline{H}$ are both nonbipartite. First assume that H is bipartite, say with $V(H) = (X, Y)$ and $|X| \ge |Y|$. Then $|X| \le m-1$; otherwise $\overline{H}$ has a C_m. Thus, $|Y| \ge h-(m-1) \ge (7m-23)/9$. If $m \ge 8$, then we have $|Y| \ge \lceil m/2 \rceil$; if $m = 7$, then since $d \ge (8m-10)/3$ and $h \ge (2d-4)/3$, we have $h \ge 10$ and so $|Y| \ge 4 = \lceil m/2 \rceil$. Thus, $\overline{H}$ always contains $2K_{\lceil m/2 \rceil}$, a contradiction. Therefore, H is nonbipartite.

Next suppose that $\overline{H}$ is bipartite, say with $V(\overline{H}) = (X, Y)$ and $|X| \ge |Y|$. Since $h \ge 16(m-2)/9 \ge 16((9n/8+1)-2)/9 = 2n-16/9$, then $h \ge \lceil 2n-16/9 \rceil = 2n-1$. Hence, $|X| \ge n$ and $G[X]$ contains a C_n, a contradiction. Thus, $\overline{H}$ is also nonbipartite.

Noting that $m \ge 9n/8+1$, $n \ge 5$ and $h \ge 16(m-2)/9$, we have $\lfloor h/2 \rfloor + 1 \ge n$ and $h \ge 7$. If $|E(H)| \ge (h+1)(h-1)/4-1$, then $|E(H)| \ge (((\lfloor h/2 \rfloor+1)-1)(h-1)+1)/2$. Thus, by Theorems 1.6 and 1.2, H contains a C_n, a contradiction. This implies that $|E(\overline{H})| > (h-1)^2/4+1$. By Theorem 1.6, $\overline{H}$ is weakly pancyclic with $g(\overline{H}) = 3$. If $c(\overline{H}) \le \lfloor h/2 \rfloor$, then $|E(\overline{H})| < (h+1)(h-1)/4-1$ by Theorem 1.2, which implies that $|E(H)| > (h-1)^2/4+1$, and $c(H) \ge n$ by Lemma 4.5. Thus, H contains a C_n by Theorem 1.6, a contradiction. Therefore, $\overline{H}$ is weakly pancyclic with $g(\overline{H}) = 3$ and $c(\overline{H}) \ge \lfloor h/2 \rfloor + 1$.

Let $C = x_1x_2\cdots x_rx_1$ be a longest cycle in $\overline{H}$, and let $Y = \{y_1, y_2, \ldots, y_{h-r}\} = V(\overline{H}) - V(C)$. Then $\lfloor h/2 \rfloor + 1 \le r \le m-1$ and $h - r \ge h - m + 1$. If $n \ge 7$, then it is easy to check that $|Y| \ge (n+1)/2$; if $n = 5$, we have $m \ge 7$ and $|Y| \ge \lceil (7m-23)/9 \rceil \ge 3$, and we also get that $|Y| \ge (n+1)/2$. By Lemma 4.7, any two vertices of Y have at least $\lceil r/2 \rceil - 1$ common nonadjacent vertices in C. It is easy to check that $\lceil r/2 \rceil - 1 \ge r/2 - 1 \ge (\lfloor h/2 \rfloor + 1)/2 - 1 \ge h/4 - 3/4 > (n-3)/2$. Since n is odd, $\lceil r/2 \rceil - 1 \ge (n-1)/2$. By Lemmas 4.8 and 4.9, H is bipartite, a contradiction. This completes Case 1.

Case 2. $\overline{G}$ contains $2K_{\lceil m/2 \rceil}$.

We first deal with the subcase that $\kappa(\overline{G}) \le 2$. We assume that $\{u, w\}$ is a cut set and that $V(\overline{G}) - \{u, w\} = X \cup Y$ with $|X| \ge |Y|$ and $E_{\overline{G}}(X, Y) = \emptyset$. Obviously, $|X| \ge 2m-2$. If $G[X]$ contains W_n, then $G[X \cup \{y\}]$ contains $W_{2,n}$ for any $y \in Y$, and hence $|X| \le 3m-3$ by Theorem 4.1. Thus, we have $2m-2 \le |X| \le 3m-3$ and $|Y| \ge m-2$. If $|X| \ge 2m-1$, then $G[X]$ contains C_n by Theorem 1.17. If $y'y'' \in E(G[Y])$, then $G[X \cup \{y', y''\}]$ contains $W_{2,n}$, and so $\overline{G}[Y]$ is a complete graph. Since $\overline{G}$ has no C_m, $|Y| \le m-1$. If $|Y| = m-2$, then $\overline{G}[Y] = K_{m-2}$. Since $\overline{G}[Y \cup \{u, w\}]$ has no C_m, we may assume $uy \in E(G)$ for some $y \in Y$. By Theorem 4.1, $G[X \cup \{u\}]$ contains W_n, which implies that $G[X \cup \{u, y\}]$ has $W_{2,n}$, a contradiction. If $|Y| = m-1$, then $\overline{G}[Y] = K_{m-1}$. Since $\overline{G}[Y \cup \{u, w\}]$ contains no C_m, we can choose $y \in Y$ such that $uy, wy \in E(G)$. By Theorem 4.1, $G[X \cup \{u, w\}]$ contains W_n, which implies that $G[X \cup \{u, w, y\}]$ contains $W_{2,n}$, again a contradiction. Therefore, we conclude that $|X| = 2m-2$ and $|Y| = 2m-3$. By Lemma 4.10, $G[Y \cup \{u, w\}]$ contains W_4. Assuming that w is not the hub of the W_4, then $G[Y \cup \{u\}]$ has a triangle $vy'y''$, where $y', y'' \in Y$. If $G[X]$ contains C_n, then $G[X \cup \{y', y''\}]$ contains $W_{2,n}$, and hence $G[X]$ contains no C_n. By Theorem 1.17, $G[X \cup \{u\}]$ contains C_n, which implies that $G[X]$ contains P_{n-1} and any C_n in $G[X \cup \{u\}]$ contains u. If $v = u$, then any C_n in $G[X \cup \{u\}]$ together with y', y'' forms $W_{2,n}$, and if $v \ne u$, then any P_{n-1} in $G[X]$ together with v, y', y'' forms $W_{2,n}$ in G, a contradiction. Henceforth, we may assume that $\kappa(\overline{G}) \ge 3$.

We set A, B as the vertex sets of the $2K_{\lceil m/2 \rceil}$. Since $m \ge 9n/8 + 1$, then $m \ge n+2$ and $\lceil m/2 \rceil \ge \lceil (n+2)/2 \rceil = (n+3)/2 \ge 4$. By Lemma 4.6, $\overline{G}$ contains three disjoint paths joining A and B, denoted by $Q_i = a_ic_{i1}c_{i2}\ldots c_{ip_i}b_i$, where $1 \le i \le 3$, $a_i \in A$, $b_i \in B$ and $c_{ij} \notin A \cup B$ for $1 \le j \le p_i$. It is obvious that $p_i \ge 0$ and $p_i + 2$ is the order of Q_i for $1 \le i \le 3$. We choose three such disjoint paths Q_1, Q_2, Q_3

from $\overline{G}$ in such a way that $p_1+p_2+p_3$ is as small as possible. Without loss of generality, we may assume $p_1 \geq p_2 \geq p_3$.

If $p_2+p_3 \leq m-4$, then it is easy to check that $\overline{G}$ contains a C_m, a contradiction, implying that $p_2+p_3 \geq m-3 \geq 4$ and $p_2 \geq 2$. If $p_1 \geq 7$, then $c_{12}c_{14} \in E(G)$; otherwise $Q'_1 = a_1c_{11}c_{12}c_{14}\ldots c_{1p_1}b_1$ is a path shorter than Q_1 in $\overline{G}$ and Q'_1, Q_2, Q_3 are also three disjoint paths joining A and B, contradicting the choice of Q_1, Q_2, Q_3. For the same reason, to avoid a path Q'_1 which is shorter than Q_1 and together with Q_2, Q_3 forms three disjoint paths joining A and B in $\overline{G}$, G contains a complete multipartite graph with five partite sets: $\{c_{12}\}, \{c_{14}\}, \{c_{16}\}, A-\{a_2,a_3\}, B-\{b_2,b_3\}$. Since both $|A-\{a_2,a_3\}|$ and $|B-\{b_2,b_3\}|$ are at least $(n-1)/2$, then G contains a $W_{2,n}$, a contradiction. This implies that $p_1 \leq 6$.

By the choice of Q_1, Q_2, Q_3, we see that every vertex of $A-\{a_2,a_3\}$ is adjacent to every vertex of $B-\{b_2,b_3\}$. Furthermore, if $p_i \geq 1$, then a_i is adjacent to every vertex of $V(Q_i)-\{a_i,c_{i1}\}$, c_{i1} is adjacent to every vertex of $V(Q_i)-\{a_i,c_{i1},c_{i2}\}$, c_{i2} is adjacent to every vertex of $V(Q_i)-\{c_{i1},c_{i2},c_{i3}\}$, ..., b_i is adjacent to every vertex of $V(Q_i)-\{c_{ip_i},b_i\}$, where $1 \leq i \leq 3$. If $p_i \geq 2$, then for $j \geq 2$, c_{ij} is adjacent to every vertex of $A-\{a_1,a_2,a_3\}$; for $j \leq p_i - 1$, c_{ij} is adjacent to every vertex of $B-\{b_1,b_2,b_3\}$. For $1 \leq i < s \leq 3$, if $\lfloor m/2 \rfloor \leq j+t \leq m-2$, then c_{ij} is adjacent to c_{st}. This is because, if $c_{ij}c_{st} \in E(\overline{G})$, then $a_ic_{i1}\ldots c_{ij}c_{st}\ldots c_{s1}a_s$ is a path which together with $m-2-j-t$ vertices of $A-\{a_i,a_s\}$ forms a C_m in $\overline{G}$, a contradiction. For $1 \leq i < s \leq 3$, if $\lfloor m/2 \rfloor \leq (p_i-j+1)+(p_s-t+1) \leq m-2$, then c_{ij} is adjacent to c_{st}. This is because, if $c_{ij}c_{st} \in E(\overline{G})$, then $b_ic_{ip_i}\ldots c_{ij}c_{st}\ldots c_{sp_s}b_s$ is a path which together with $m+j+t-p_i-p_s-4$ vertices of $B-\{b_i,b_s\}$ forms a C_m in $\overline{G}$, a contradiction. We can also determine whether c_{ij} is adjacent to a_s or b_s under similar conditions. In the following, through a tedious but straightforward case distinction, we will always find a $W_{2,n}$ in G, which is a contradiction and confirms our claim. Unless specifically mentioned, the existence of the edges of the $W_{2,n}$ that we will find each time is validated by the above arguments.

Set $A-\{a_1,a_2,a_3\} = \{a_4,a_5,\ldots,a_{\lceil m/2 \rceil}\}$ and $B-\{b_1,b_2,b_3\} = \{b_4,b_5,\ldots,b_{\lceil m/2 \rceil}\}$. If $(p_1,p_2) = (6,6),(6,5)$, then $7 \leq m \leq p_2+p_3+3 \leq 2p_2+3 \leq 15$. We see that G contains a $W_{2,n} = \{c_{12}\}+\{c_{14}\}+C_n$, where $C_n = a_4c_xb_4a_5b_5\ldots a_{(n+3)/2}b_{(n+3)/2}a_1b_1a_4$, where $c_x = c_{23}$ for $10 \leq m \leq 11$, and $c_x = c_{22}$ for $8 \leq m \leq 9$ or $12 \leq m \leq 15$. For $m=7$, either $a_1c_{21} \in E(G)$ or $c_{14}c_{22} \in E(G)$; otherwise $\overline{G}$ contains a $C_7 = a_1c_{11}c_{12}c_{13}c_{14}c_{22}c_{21}a_1$, a contradiction. Thus, G contains a $W_{2,5} = \{c_{12}\}+\{c_{14}\}+$

C_5, where $C_5 = a_1c_{21}b_4a_4b_1a_1$ if $a_1c_{21} \in E(G)$, and $C_5 = a_4c_{22}b_4a_1b_1a_4$ if $c_{14}c_{22} \in E(G)$. If $(p_1,p_2) = (6,4),(6,3),(5,5),(5,4),(5,3)$, then $7 \le m \le 2p_2+3 \le 13$. We see that G contains a $W_{2,n} = \{c_{12}\}+\{c_{14}\}+C_n$, where $C_n = a_4c_{22}b_4a_5b_5\ldots a_{(n+3)/2}b_{(n+3)/2}a_1b_1a_4$ for $8 \le m \le 13$. For $m = 7$, we can obtain the same $W_{2,5}$ as in the previous case. If $(p_1,p_2) = (6,2),(5,2)$, then $m = 7$ and $p_3 = 2$. In this case, G contains a $W_{2,5} = \{c_{12}\}+\{c_{14}\}+C_5$, where $C_5 = a_4c_{22}c_{32}c_{21}b_4a_4$. For the remainder we may assume that $p_1 \le 4$.

If $(p_1,p_2,p_3) = (4,4,4)$ and $m \ne 8$, or if $(p_1,p_2,p_3) = (4,4,3)$, then $m \le 11$, and G contains a $W_{2,n} = \{c_{12}\}+\{c_{22}\}+C_n$, where $C_n = a_4c_{32}c_{14}c_{31}b_4a_5b_5\ldots a_{(n+3)/2}b_{(n+3)/2}a_4$. If $(p_1,p_2,p_3) = (4,4,4)$ and $m = 8$, then $n = 5$ and G contains a $W_{2,5} = \{a_4\}+\{b_4\}+C_5$, where $C_5 = c_{12}c_{22}c_{32}c_{23}c_{33}c_{12}$. If $(p_1,p_2,p_3) = (4,4,2)$, then $m \le 9$. We see that G contains a $W_{2,n} = \{c_{12}\}+\{c_{22}\}+C_n$, where $C_n = a_4c_{32}c_{14}c_{31}b_4a_5b_5\ldots a_{(n+3)/2}b_{(n+3)/2}a_4$ for the cases $m = 8,9$ or the case that $m = 7$ and $c_{14}c_{32} \in E(G)$. If $m = 7$ and $c_{24}c_{32} \in E(G)$, since $c_{14}c_{32}$ and $c_{24}c_{32}$ are symmetrical, we can also obtain a $W_{2,n}$ in G. Thus, $c_{14}c_{32}, c_{24}c_{32} \in E(\overline{G})$, and then $\overline{G}$ contains a $C_7 = b_1c_{14}c_{32}c_{24}b_2b_3b_4b_1$, a contradiction. If $(p_1,p_2,p_3) = (4,4,1)$, then $m \le 8$ and $n = 5$. For $m = 7$, G contains a $W_{2,5} = \{c_{21}\}+\{c_{31}\}+C_5$, where $C_5 = c_{14}c_{23}c_{13}c_{24}c_{12}c_{14}$. For $m = 8$, G contains a $W_{2,5} = \{a_3\}+\{b_3\}+C_5$, where $C_5 = c_{13}c_{22}c_{12}c_{23}c_{11}c_{13}$. If $(p_1,p_2,p_3) = (4,4,0)$, then $m = 7$ and G contains a $W_{2,5} = \{c_{21}\}+\{b_3\}+C_5$, where $C_5 = c_{14}c_{23}c_{13}c_{24}c_{12}c_{14}$. If $(p_1,p_2,p_3) = (4,3,3),(3,3,3)$, then $m \le 9$ and G contains a $W_{2,n} = \{c_{12}\}+\{c_{22}\}+C_n$, where $C_n = a_4c_{33}c_{31}b_4a_5b_5\ldots a_{(n+3)/2}b_{(n+3)/2}c_{32}a_4$. If $(p_1,p_2,p_3) = (4,3,2)$, then $m \le 8$ and $n = 5$. For the case $m = 8$ or the case $m = 7$ and $c_{11}c_{31} \in E(G)$, G contains a $W_{2,5} = \{c_{13}\}+\{c_{22}\}+C_5$, where $C_5 = c_{31}c_{11}c_{32}a_4b_4c_{31}$. For $m = 7$ and $c_{11}c_{31} \in E(\overline{G})$, we have $b_3c_{14} \in E(G)$; otherwise $c_{11}c_{12}c_{13}c_{14}b_3c_{32}c_{31}c_{11}$ is a C_7 in $\overline{G}$, a contradiction. Then G contains a $W_{2,5} = \{c_{12}\}+\{c_{22}\}+C_5$, where $C_5 = b_4c_{31}c_{14}b_3a_4b_4$. If $(p_1,p_2,p_3) = (4,3,1)$, then $m = 7$. If $a_3c_{11}, b_3c_{14} \in E(\overline{G})$, then $c_{11}c_{12}c_{13}c_{14}b_3c_{31}a_3c_{11}$ is a C_7 in $\overline{G}$, a contradiction. By symmetry, we may assume that $a_3c_{11} \in E(G)$, and G contains a $W_{2,5} = \{c_{13}\}+\{c_{22}\}+C_5$, where $C_5 = a_3c_{11}b_3a_4b_4a_3$. If $(p_1,p_2,p_3) = (4,2,2)$, then $m = 7$. If $a_3c_{12}, a_4c_{21} \in E(\overline{G})$, then $a_1c_{11}c_{12}a_3a_4c_{21}a_2a_1$ is a C_7 in $\overline{G}$, a contradiction. Hence, either $a_3a_{12} \in E(G)$ or $a_4a_{21} \in E(G)$. Thus, G contains a $W_{2,5} = \{c_{12}\}+\{c_{21}\}+C_5$, where $C_5 = a_xc_{32}b_2c_{31}b_3a_x$, $a_x = a_3$ if $a_3a_{12} \in E(G)$, and $a_x = a_4$ if $a_4a_{21} \in E(G)$. If $(p_1,p_2,p_3) = (3,3,2)$, then $m \le 8$ and $n = 5$. For the case $m = 8$ or the case $m = 7$ and $a_3c_{21} \in E(G)$, G contains a $W_{2,5} = \{c_{12}\}+\{c_{21}\}+C_5$, where $C_5 =$

$a_3c_{32}c_{23}c_{31}b_3a_3$. For $m = 7$ and $a_3c_{21} \in E(\overline{G})$, we have $b_3c_{23} \in E(G)$; otherwise $a_3c_{31}c_{32}b_3c_{23}c_{22}c_{21}a_3$ is a C_7 in $\overline{G}$, a contradiction. Since a_3c_{21} and b_3c_{23} are symmetrical, we can also obtain a $W_{2,5}$ if $b_3c_{23} \in E(G)$. If $(p_1,p_2,p_3) = (3,2,2)$, then $m = 7$ and G contains a $W_{2,5} = \{c_{12}\}+\{c_{21}\}+C_5$, where $C_5 = a_3c_{32}b_2c_{31}b_3a_3$. If $(p_1,p_2,p_3) = (2,2,2)$, then $m = 7$ and G contains a $W_{2,5} = \{c_{11}\} + \{c_{21}\} + C_5$, where $C_5 = a_3c_{32}b_2c_{31}b_3a_3$. Since $p_2 + p_3 \geq 4$, we have considered all the possible combinations of values for (p_1,p_2,p_3), and each time we derived a contradiction. This completes the proof of Case 2 and of Theorem 4.4. □

4.4 Proof of Theorem 4.5

To confirm Theorem 4.5, the following results are needed: Theorem 1.3 and Theorem 1.17 from Chapter 1; Theorem 2.1 from Chapter 2; Theorem 3.11 from Chapter 3; and the following two lemmas.

Lemma 4.11 (Chvátal and Erdős [33]). *Let G be a graph with at least three vertices. If $\alpha(G) \leq \kappa(G)$, then G is hamiltonian. If $\alpha(G) \leq \kappa(G) - 1$, then G is Hamilton-connected.*

The second lemma consists of some known small Ramsey numbers that we need. We cite them from eleven different papers.

Lemma 4.12 ([36,37,56,57,67,76,79–81,127,134]).
(i) $R(W_n,C_3) = 2n+3$ *for* $n = 3,4$; $R(K_5 - e,C_3) = 11$; $R(W_5 + e,C_3) = 11$;
(ii) $R(W_n,C_4) = 10$ *for* $n = 3,5$; $R(W_n,C_4) = 9$ *for* $n = 4,6$; $R(W_n,C_4) = n+4$ *for* $7 \leq n \leq 10$; $R(K_{1,6},C_4) = 9$; $R(W_5 + e,C_4) = 10$;
(iii) $R(C_5,W_4) = 9$; $R(C_6,W_4) = 11$; $R(P_3,W_4) = 5$;
(iv) $R(K_4 - e,K_4) = 11$; $R(C_5 + e,K_4) = 13$; $R(W_5 + e,C_5) = 13$.

(i) Let G be any graph of order $R(C_m,W_n)$. Assume that $\overline{G}$ contains no W_n, then G contains a C_m. Suppose to the contrary that G contains no $C_m + e$ for all $e \in \overline{C_m}$, then $\overline{G}$ contains $K_m - E(C_m)$, denoted by H. For any vertex $v \in H$, $\overline{G}[N_H(v)] = K_{m-3} - E(P_{m-3})$, denoted by H_1. For $m \geq 10$, $\delta(H_1) \geq m-6 > (m-3)/2$, by Theorem 1.3, H_1 is pancyclic. For $m = 8,9$, it is easy to check that H_1 is also pancyclic. Thus, for $m \geq 8$, H contains W_n for $3 \leq n \leq m-3$, a contradiction. Therefore, C_m is Ramsey unsaturated with respect to W_n for $m \geq \max\{8, n+3\}$.

Set $Y = V(G) - C_m$ and $C_m = x_1x_2 \ldots x_m x_1$, indices being taken modulo m and the same below. If there exist $x_i \in C_m$ and $y \in Y$ such that $x_i y, x_{i+2}y \in E(G)$, then $xy \notin E(G)$ for any $x \in C_m - \{x_i, x_{i+1}, x_{i+2}\}$, since otherwise $x_i y x_{i+2} \overrightarrow{C_m} x_i$ is a cycle of length m, which together with xy forms a $C_m + e$, a contradiction. In this case, $d_{C_m}(y) \leq 3$. Thus, if $d_{C_m}(y) \geq 4$ and $x_i y \in E(G)$, then $x_{i+2}y \notin E(G)$. That is, $d_{C_m}(y) \leq m/2$ for $m \geq 6$. Moreover, if $d_{C_m}(y) \geq 4$, then for any sequence $x_i, x_{i+1}, x_{i+2}, x_{i+3}$, y is adjacent to at most two of them.

We claim that any two vertices y_1, y_2 of Y would have at least one common nonadjacent vertex in C_m. If not, every vertex of C_m is either adjacent to y_1, or adjacent to y_2. For $m \geq 9$, by the above analysis, $d_{C_m}(y_i) \geq 4$ for $i = 1, 2$. There exists x_i such that $x_i y_1, x_{i+1} y_2 \in E(G)$, then $x_{i-1} y_1, x_{i+2} y_2 \in E(G)$. And then $x_{i+3} y_1, x_{i+4} y_1 \in E(G)$. Repeating this procedure, we have $m \equiv 0 \pmod 4$, y_1 is adjacent to x_j for $j \equiv i \pmod 4$ or $j \equiv i-1 \pmod 4$, y_2 is adjacent to x_k for $k \equiv i+1 \pmod 4$ or $k \equiv i+2 \pmod 4$. Thus $x_{i-1} y_1 x_i x_{i+1} x_{i+2} y_2 x_{i+5} \overrightarrow{C_m} x_{i-1}$ is a cycle of length m, which together with $x_{i+1} y_2$ forms a $C_m + e$, a contradiction. This proves our claim that any two vertices of Y would have at least one common nonadjacent vertex in C_m.

We also claim that $G[Y]$ is not a complete graph. Since $2K_{m-1}$ contains no C_m and its complement contains no W_n, we have $R(C_m, W_n) \geq 2m-1$, which implies $|Y| \geq m-1$. Suppose to the contrary that $G[Y]$ is a complete graph, then $|Y| = m-1$. If there is a $x \in C_m$ such that $d_Y(x) \geq 2$, then $G[Y \cup \{x\}]$ contains a $C_m + e$, a contradiction. If there are two independent edges $x_i y_i, x_j y_j$ for $x_i, x_j \in C_m$ and $y_i, y_j \in Y$, then we can also get a $C_m + e$ in G. Thus, there exists one vertex $y \in Y$ such that $E(C_m, Y - \{y\}) = \emptyset$. Then $\overline{G}$ contains $(m-2)K_1 + H$. Since $\delta(H) > m/2$ for $m \geq 7$, by Theorem 1.3, H is pancyclic, which together with any vertex of $Y - \{y\}$ forms a W_n for $n \leq m-1$. This contradiction proves that $G[Y]$ is not a complete graph.

Set $y_1, y_2 \in Y$ and $y_1 y_2 \notin E(G)$. For convenience, let x_{m-1} be a common nonadjacent vertex of y_1, y_2, $X = \{x_1, x_2, \ldots, x_{m-3}\}$, $H_1 = \overline{G}[X]$ and $H_2 = \overline{G}[X \cup \{y_1, y_2\}]$. In the following, in order to prove $\overline{G}$ contains W_{m-1} and W_{m-2} for $m \geq 9$, we need only to prove that $\overline{G}[X \cup \{y_1, y_2\}]$ contains C_{m-1} and C_{m-2}. Since $\alpha(H_1) = 2$, $\kappa(H_1) \geq m-6$, by Lemma 4.11, H_1 is Hamilton-connected for $m \geq 9$. Note that if $d_{C_m}(y) \geq 4$, then for any sequence $x_i, x_{i+1}, x_{i+2}, x_{i+3}$, y is adjacent to at most two of them. Then for $m \geq 9$ and $i = 1, 2$, $d_X(y_i) \geq \min\{4, m-7\}$. Thus, for $m \geq 9$, $\overline{G}[X \cup \{y_1\}]$ contains a C_{m-2}. For $m \geq 10$, $\alpha(H_2) \leq 3$ and $\kappa(H_2) \geq 3$, by Lemma 4.11, H_2 contains a C_{m-1}. For $m = 9$, if $\kappa(H_2) \geq 3$, then H_2 contains a C_{m-1} for the same reason. If $\kappa(H_2) \leq 2$, then $d_X(y_1) = d_X(y_2) = 2$ in $\overline{G}$, and $N_X(y_1) = N_X(y_2) = \{x_1, x_2, x_5, x_6\}$. Thus, $x_1 y_1 x_2 y_2 x_5 \overrightarrow{C_m} x_1$ is a cycle of length m, with together with $x_1 y_2$ forms a $C_m + e$ in G, a contradiction. Therefore, C_m is Ramsey unsaturated with respect to W_n for $m \geq \max\{9, n+1\}$.

In the following, we need only to consider the remaining entries, that is, C_m with respect to W_n for $m = 6,7$ and $3 \leq n \leq m-1$, and C_8 with respect to W_6, W_7.

Now we show that C_6 is Ramsey unsaturated with respect to W_n, where $n = 3,5$. By Theorem 4.1, $R(C_6, W_n) = 16$ for $n = 3,5$. Let G be a graph of order 16 and assume that $\overline{G}$ contains no W_n, then G contains a $C_6 = x_1 x_2 \ldots x_6 x_1$. Suppose to the contrary that G contains no $C_6 + e$. By Lemma 4.12, $R(C_4, W_n) = 10$ for $n = 3,5$. Since $|V(G)| - 6 = 10$, then $G - C_6$ contains a $C_4 = y_1 y_2 y_3 y_4 y_1$. Each y_i has to be adjacent to at least two vertices of C_6, otherwise $\overline{G}[N_{\overline{C_6}}(y_i)]$ contains a C_n, which together with y_i forms a W_n, a contradiction. By symmetry, we need only to consider three subcases: $x_1, x_2 \subseteq N(y_1)$, $x_1, x_3 \subseteq N(y_1)$, $x_1, x_4 \subseteq N(y_1)$. If y_1 is adjacent to x_i, then y_2 is nonadjacent to x_{i-1}, x_{i+1}, since otherwise $y_1 y_4 y_3 y_2 x_{i-1} x_i y_1$ or $y_1 y_4 y_3 y_2 x_{i+1} x_i y_1$ is a C_6 with $y_1 y_2$ as its chord, a contradiction. If y_1 is adjacent to x_1, x_2, then y_2 has to be adjacent to x_4, x_5, and then $x_1 x_2 x_3 x_4 y_2 y_1 x_1$ is a C_6 with $x_2 y_1$ as its chord. If y_1 is adjacent to x_1, x_3, then y_2 has to be adjacent to at least two of x_1, x_3, x_5. For the same reason, y_3 has to be adjacent to at least two of x_1, x_3, x_5, that is, at least one of x_1, x_3. Then $y_1 y_2 y_3 x_i x_2 x_j y_1$ is a C_6 with $x_i y_1$ as its chord, where $(i,j) = (1,3)$ or $(3,1)$. If y_1 is adjacent to x_1, x_4, then y_2 has to be adjacent to x_1, x_4, and then $x_1 x_2 x_3 x_4 y_1 y_2 x_1$ is a C_6 with $x_1 y_1$ as its chord. Since all these three subcases lead to a contradiction, C_6 is Ramsey unsaturated with respect to W_n, where $n = 3,5$.

To prove C_6 is Ramsey unsaturated with respect to W_4, we set G be a graph of order $R(C_6, W_4) = 11$ by Lemma 4.12. Assume that $\overline{G}$ contains no W_4, then G contains a $C_6 = x_1 x_2 \ldots x_6 x_1$. Suppose to the contrary that G contains no $C_6 + e$. Since $R(P_3, W_4) = 5$ by Lemma 4.12 and $|V(G)| - 6 = 5$, then $G - C_6$ contains a $P_3 = y_1 y_2 y_3$. Each y_i has to be adjacent to at least two vertices of C_6 and not exactly two vertices x_i, x_{i+3}, otherwise $\overline{G}[N_{\overline{C_6}}(y_i)]$ contains a C_4, which together with y_i forms a W_4, a contradiction. By symmetry, we need only to consider two subcases: $x_1, x_3 \subseteq N(y_2)$, $x_1, x_2 \subseteq N(y_2)$. If y_2 is adjacent to x_1, x_3, then both y_1 and y_3 are nonadjacent to x_4, x_6, otherwise there is a $C_6 + e$ in G. If

$x_1 \notin N(y_1) \cup N(y_3)$, then $x_1y_1x_6y_3x_1$ is a C_4, which together with x_4 forms a W_4 in $\overline{G}$, a contradiction. Hence, $x_1 \in N(y_1) \cup N(y_3)$. By symmetry, $x_3 \in N(y_1) \cup N(y_3)$. Furthermore, we have $\{x_1, x_3\} \cap N(y_1) \neq \emptyset$, otherwise $x_1x_3x_6x_4x_1$ is a C_4, which together with y_1 forms a W_4 in $\overline{G}$. For the same reason, we have $\{x_1, x_3\} \cap N(y_3) \neq \emptyset$. By Hall's Theorem, there is a matching between $\{x_1, x_3\}$ and $\{y_1, y_3\}$ which covers $\{y_1, y_3\}$. Thus, there exists a cycle of length six $x_1x_2x_3y_3y_2y_1x_1$ or $x_1x_2x_3y_1y_2y_3x_1$ with x_1y_2 as its chord, a contradiction. If y_2 is adjacent to x_1, x_2, then both y_1 and y_3 are nonadjacent to x_4, x_5, otherwise there is a $C_6 + e$ in G. If $y_1y_3 \notin E(G)$, then $x_1y_1 \in E(G)$, otherwise $x_1x_4y_3x_5x_1$ is a C_4, which together with y_1 forms a W_4. For the same reason, $x_2y_1, x_1y_3, x_2y_3 \in E(G)$. To avoid a $C_6 + e$, $x_3, x_6 \notin N(y_1)$ and $x_3, x_6 \notin N(y_3)$. And then $\overline{G}[\{y_1, y_3, x_3, x_4, x_6\}]$ contains a W_4. This contradiction proves that $y_1y_3 \in E(G)$. In this way, if $x_3y_1 \in E(G)$, then $x_1x_2x_3y_1y_3y_2x_1$ is a C_6 with x_2y_2 as its chord. Thus, $x_3y_1 \notin E(G)$. For the same reason, $x_6 \notin N(y_1)$ and $x_3, x_6 \notin N(y_3)$. Again, we have $\overline{G}[\{y_1, y_3, x_3, x_4, x_6\}]$ contains a W_4. This final contradiction proves that C_6 is Ramsey unsaturated with respect to W_4.

Now we show that C_7 is Ramsey unsaturated with respect to W_n, where $n = 3, 5$. By Theorem 4.1, $R(C_7, W_n) = 19$ for $n = 3, 5$. Let G be a graph of order 19 and assume that $\overline{G}$ contains no W_n, then G contains a $C_7 = x_1x_2 \ldots x_7x_1$. Suppose to the contrary that G contains no $C_7 + e$. By Lemma 4.12, $R(C_4, W_n) = 10$ for $n = 3, 5$. Since $|V(G)| - 7 = 12$, then $G - C_6$ contains a $C_4 = y_1y_2y_3y_4y_1$. Each y_i has to be adjacent to at least three vertices of C_7, otherwise $\overline{G}[N_{\overline{C_7}}(y_i)]$ contains a C_n, which together with y_i forms a W_n, a contradiction. By symmetry and to avoid a $C_7 + e$, we need only to consider two subcases: $x_1, x_2, x_3 \subseteq N(y_1)$, $x_1, x_2, x_5 \subseteq N(y_1)$. If y_1 is adjacent to x_1, x_2, x_3, since y_2 has to be adjacent to at least three vertices of C_7, we can always get a $C_7 + e$ in G, a contradiction. If y_1 is adjacent to x_1, x_2, x_5, then y_2 is adjacent to at most one vertex in each set of $\{x_i, x_{i+1}\}$ for any $1 \leq i \leq 6$ and $\{x_1, x_7\}$, otherwise G contains a $C_7 + e$. And then y_2 can not be adjacent to more than two vertices of C_7, a final contradiction.

To prove C_7 is Ramsey unsaturated with respect to W_4, we set G be a graph of order $R(C_7, W_4)$. Assume that $\overline{G}$ contains no W_4, then G contains a $C_7 = x_1x_2 \ldots x_7x_1$. Suppose to the contrary that G contains no $C_7 + e$. For any $y \in V(G) - C_7$, y has to be adjacent to at least three vertices of C_7, otherwise $\overline{G}$ contains a W_4. By symmetry and to avoid a $C_7 + e$, we need only to consider two subcases: $x_1, x_2, x_3 \subseteq N(y)$, $x_1, x_2, x_5 \subseteq N(y)$. If y is adjacent to x_1, x_2, x_3, then $\overline{G}$

contains a $C_4 = x_2x_4yx_5x_2$, which together with x_7 forms a W_4. If y is adjacent to x_1,x_2,x_5, then $\overline{G}$ contains a $C_4 = x_3x_6x_4x_7x_3$, which together with y forms a W_4. Thus, C_7 is Ramsey unsaturated with respect to W_4.

To prove C_7 is Ramsey unsaturated with respect to W_6, we set G be a graph of order $R(C_7,W_6)$. Since $2K_6$ contains no C_7 and its complement contains no W_6, we have $|V(G)| \geq 13$. Assume that $\overline{G}$ contains no W_6, then G contains a $C_7 = x_1x_2\ldots x_7x_1$. Suppose to the contrary that G contains no C_7+e. We distinguish two cases.

First case. There are three vertices $y_i \in V(G)-C_7$, such that $d_{C_7}(y_i) \leq 2$ for $1 \leq i \leq 3$. Each y_i has to be adjacent to at least two vertices of C_7, otherwise $\overline{G}$ contains a W_6. Thus, $d_{C_7}(y_i) = 2$ for $1 \leq i \leq 3$. By symmetry, we need only to consider three subcases: $N_{C_7}(y_1) = \{x_1,x_2\}$, $N_{C_7}(y_1) = \{x_1,x_3\}$ and $N_{C_7}(y_1) = \{x_1,x_4\}$. For the first subcase, $N_{C_7}(y_1) = \{x_1,x_2\}$, then y_2 has to be adjacent to at least one vertex of $\{x_2,x_4,x_7\}$, otherwise $x_2y_2x_4y_1x_3x_5x_2$ is a C_6 in $\overline{G}$, which together with x_7 forms a W_6, a contradiction. For the same reason, y_2 has to be adjacent to at least one vertex of each set $\{x_2,x_5,x_7\}$, $\{x_3,x_4,x_7\}$, $\{x_1,x_3,x_5\}$, $\{x_1,x_3,x_6\}$, $\{x_3,x_6,x_7\}$, $\{x_1,x_2,x_5\}$. Since $d_{C_7}(y_2) = 2$, we have $N_{C_7}(y_2) = \{x_2,x_3\}$ or $\{x_1,x_7\}$. For the same reason, $N_{C_7}(y_3) = \{x_2,x_3\}$ or $\{x_1,x_7\}$. But we may use the same analysis of getting $N_{C_7}(y_2)$ from $N_{C_7}(y_1)$, to obtain $N_{C_7}(y_3)$ from $N_{C_7}(y_2)$. That is, $N_{C_7}(y_3) = \{x_1,x_2\}$ or $\{x_3,x_4\}$, or $\{x_6,x_7\}$, which contradicts $N_{C_7}(y_3) = \{x_2,x_3\}$ or $\{x_1,x_7\}$. For the second subcase, $N_{C_7}(y_1) = \{x_1,x_3\}$. Let $A = \{x_4,x_5,x_6,x_7\}$, then x_2 is nonadjacent to A and $\{y_1\}$. For any two vertices of A, there is a path of length four joining them in $\overline{G}[A \cup \{y_1\}]$. Hence, if $x_2 \notin N(y_2)$, to avoid a W_6 in $\overline{G}$, y_2 has to be adjacent to at least three vertices of A, a contradiction. Thus, $x_2 \in N(y_2)$. If y_2 is nonadjacent to x_4,x_7, then y_2 has to be adjacent to x_3, since otherwise $x_3x_5y_1x_2x_4y_2x_3$ is a C_6 in $\overline{G}$, which together with x_7 forms a W_6. By symmetry, if $x_4,x_7 \notin N(y_2)$, then $x_1 \in N(y_2)$. In this way, $d_{C_7}(y_2) \geq 3$, a contradiction. Thus, $N(y_2) = \{x_2,x_4\}$ or $\{x_2,x_7\}$. For the same reason, $N_{C_7}(y_3) = \{x_2,x_4\}$ or $\{x_2,x_7\}$. But we may use the same analysis of getting $N_{C_7}(y_2)$ from $N_{C_7}(y_1)$, to obtain $N_{C_7}(y_3)$ from $N_{C_7}(y_2)$. That is, $N_{C_7}(y_3) = \{x_1,x_3\}$ or $\{x_1,x_6\}$, or $\{x_3,x_5\}$, which contradicts $N_{C_7}(y_3) = \{x_2,x_4\}$ or $\{x_2,x_7\}$. For the third subcase, $N_{C_7}(y_1) = \{x_1,x_4\}$. If y_2 is nonadjacent to x_2,x_4, since y_2 has to be nonadjacent to one vertex of x_5,x_6,x_7, then $\overline{G}[\{x_2,x_4,x_5,x_6,x_7,y_1,y_2\}]$ contains a W_6 with x_2 as its hub. This contradiction implies that y_2 has to be adjacent to at least one vertex of x_2,x_4. For the same reason, y_2 has to be adjacent to at least one vertex of x_2,x_5. If $x_2 \notin N(y_2)$, then $N_{C_7}(y_2) = \{x_4, x_5\}$, which has been considered in the first subcase. Thus, $x_2 \in N(y_2)$. By symmetry, $x_3 \in N(y_2)$. Since $N_{C_7}(y_2) = \{x_2, x_3\}$, this has also been considered in the first subcase.

Second case. There are four vertices $y_i \in V(G) - C_7$, such that $d_{C_7}(y_i) \geq 3$ for $1 \leq i \leq 4$. Since G contains no $C_7 + e$, then for any $1 \leq i \leq 4$, $N_{C_7}(y_i) = \{x_j, x_{j+1}, x_{j+2}\}$ or $\{x_j, x_{j+1}, x_{j+4}\}$, where $1 \leq j \leq 7$ and being taken modulo 7. We claim that $N_{C_7}(y_i) \neq \{x_j, x_{j+1}, x_{j+4}\}$ for any $1 \leq i \leq 4$. If not, say, $N_{C_7}(y_1) = \{x_1, x_2, x_5\}$. Since $\{x_k, x_{k+1}\} \subseteq N_{C_7}(y_2)$ for some $1 \leq k \leq 7$, then for $1 \leq k \leq 4$, $x_1 \ldots x_k y_2 x_{k+1} \ldots x_5 y_1 x_1$ is a C_7 with $x_2 y_1$ as its chord, a contradiction; for $5 \leq k \leq 7$, $x_1 x_2 y_1 x_5 \ldots x_k y_2 x_{k+1} \ldots x_1$ is a C_7 with $x_1 y_1$ as its chord, also a contradiction. Thus, for any $1 \leq i \leq 4$, $N_{C_7}(y_i) = \{x_j, x_{j+1}, x_{j+2}\}$ for some $1 \leq j \leq 7$. For $1 \leq i \leq j \leq 4$, $N_{C_7}(y_i) \cap N_{C_7}(y_j) \neq \emptyset$. If not, say, $N_{C_7}(y_1) = \{x_1, x_2, x_3\}$ and $N_{C_7}(y_2) = \{x_4, x_5, x_6\}$, then $x_2 y_2 x_3 x_5 y_1 x_4 x_2$ is a C_6 in $\overline{G}$, which together with x_7 forms a W_6. We claim that there is no P_4 in $G[\{y_1, y_2, y_3, y_4\}]$. If not, say, $P_4 = y_1 y_2 y_3 y_4$. Without loss of generality, assume $N_{C_7}(y_1) = \{x_1, x_2, x_3\}$. Then y_4 is nonadjacent to x_4, otherwise $x_2 x_3 x_4 y_4 y_3 y_2 y_1 x_2$ is a C_7 with $x_3 y_1$ as its chord. For the same reason, y_4 is nonadjacent to x_1, x_3, x_7. That is, $N_{C_7}(y_2) = \{x_2, x_5, x_6\}$, which is also a contradiction. This proves our claim that there is no P_4 in $G[\{y_1, y_2, y_3, y_4\}]$. Then it is easy to check that $\overline{G}[\{y_1, y_2, y_3, y_4\}]$ contains a P_3. Without loss of generality, assume $P_3 = y_1 y_2 y_3$ in $\overline{G}$. Suppose that $N_{C_7}(y_1) = \{x_1, x_2, x_3\}$, by symmetry, we need only to consider three subcases: $N_{C_7}(y_3) = \{x_1, x_2, x_3\}$, $N_{C_7}(y_3) = \{x_2, x_3, x_4\}$ and $N_{C_7}(y_3) = \{x_3, x_4, x_5\}$. For the first subcase, $N_{C_7}(y_1) = N_{C_7}(y_3) = \{x_1, x_2, x_3\}$. Since y_2 is nonadjacent to x_4 or x_7, say, x_7, then $\overline{G}$ contains $W_6 = x_7 + C_6$, where $C_6 = y_1 y_2 y_3 x_5 x_2 x_4 y_1$. For the second subcase, $N_{C_7}(y_1) = \{x_1, x_2, x_3\}$ and $N_{C_7}(y_3) = \{x_2, x_3, x_4\}$. Since y_2 is nonadjacent to x_5 or x_7, say, x_7, then $\overline{G}$ contains the same W_6 as the first subcase. For the third subcase, $N_{C_7}(y_1) = \{x_1, x_2, x_3\}$ and $N_{C_7}(y_3) = \{x_3, x_4, x_5\}$. Then $y_1 y_3 \notin E(G)$, otherwise $y_3 y_1 x_1 x_2 x_3 x_4 x_5 y_3$ is a C_7 with $x_2 y_1$ as its chord. Since $N_{C_7}(y_2) = \{x_1, x_2, x_3\}$, or $\{x_2, x_3, x_4\}$, or $\{x_3, x_4, x_5\}$, this subcase can be transferred to the former subcases.

To prove C_8 is Ramsey unsaturated with respect to W_6, we set G be a graph of order $R(C_8, W_6)$. Assume that $\overline{G}$ contains no W_6, then G contains a $C_8 = x_1 x_2 \ldots x_8 x_1$. Suppose to the contrary that G contains no $C_8 + e$. For any $y \in V(G) - C_8$, y has to be adjacent to at least three vertices of C_8, otherwise $\overline{G}$ contains a W_6. By symmetry and to avoid a $C_8 + e$, we need only to consider three subcases: $N_{C_8}(y) = \{x_1,x_2,x_3\}$, $N_{C_8}(y) = \{x_1,x_2,x_5\}$ and $N_{C_8}(y) = \{x_1,x_2,x_5,x_6\}$. For the first two subcases, $\overline{G}$ contains $W_6 = x_8 + C_6$, where $C_6 = x_4y_1x_6x_3x_5x_2x_4$. For the last subcase, $\overline{G}$ contains $W_6 = x_8 + C_6$, where $C_6 = x_3y_1x_4x_6x_2x_5x_3$.

In the following, we show that C_8 is Ramsey unsaturated with respect to W_7. Let G be a graph of order $R(C_8,W_7)$, which is 22 by Theorem 4.1. Assume that $\overline{G}$ contains no W_7, then G contains a $C_8 = x_1x_2\dots x_8x_1$. Suppose to the contrary that G contains no C_8+e. We choose y_1,y_2 from $V(G)-C_8$ such that $y_1y_2 \notin E(G)$. If y_1,y_2 has no common nonadjacent vertex in C_8, since G contains no C_8+e, then $d_{C_8}(y_1) = d_{C_8}(y_2) = 4$, $N_{C_8}(y_1) = \{x_i,x_{i+1},x_{i+4},x_{i+5}\}$ and $N_{C_8}(y_2) = \{x_{i+2},x_{i+3},x_{i+6},x_{i+7}\}$. In this way, $x_ix_{i+1}x_{i+2}y_2x_{i+3}x_{i+4}x_{i+5}y_1x_i$ is a C_8 with $x_{i+1}y_1$ as its chord, a contradiction. Thus, y_1,y_2 has at least one common nonadjacent vertex in C_8, say, x_7. Set $X = \{x_1,x_2,x_3,x_4,x_5\}$, then both y_1 and y_2 has at least two nonadjacent vertices in X. If y_1 or y_2 is nonadjacent to x_3, since $\overline{G}[X]$ contains a path of length four from x_3 to any vertex of $X-\{x_3\}$, then $\overline{G}[X\cup\{y_1,y_2\}]$ contains a C_7, together with x_7 forming a W_7. Thus, both y_1 and y_2 are adjacent to x_3. Since $\overline{G}[X]$ also contains a path of length four from x_2 to any vertex of $\{x_4,x_5\}$, and from x_4 to any vertex of $\{x_1,x_2\}$, then one of $\{x_2,x_4\}$ should be adjacent to both y_1 and y_2, say, x_2. Then both y_1 and y_2 are nonadjacent to at least two of $\{x_1,x_4,x_5\}$. To avoid a C_7 in $\overline{G}[X\cup\{y_1,y_2\}]$, we have $N_X(y_1) = N_X(y_2) = \{x_1,x_5\}$ in $\overline{G}$, or $N_X(y_1) = N_X(y_2) = \{x_4,x_5\}$ in $\overline{G}$. Thus, we have $N_X(y_1) = N_X(y_2) = \{x_i,x_{i+1},x_{i+2}\}$ for $i = 1$ or 2 in G. If $i = 1$, $\overline{G}$ contains a W_7 with x_8 as its hub; and if $i = 2$, $\overline{G}$ contains a W_7 with x_1 as its hub, a final contradiction.

(ii) For any $e \in \overline{C_4}$, $C_4+e = K_4-e$. By Lemma 4.12, $R(C_4,W_3) = 10$ and $R(K_4-e,W_3) = R(K_4-e,K_4) = 11$, we have C_4 is Ramsey saturated with respect to W_3. By Lemma 4.12 and Theorem 2.1, $R(C_4+e,W_n) \ge R(C_3,W_n) > R(C_4,W_n)$ for $n = 4,5$. That is, C_4 is Ramsey saturated with respect to W_n for $n = 4,5$. For $n \ge 6$, by Theorem 1.17, $R(C_4+e,W_n) \ge R(C_3,W_n) = 2n+1$; and by Theorem 3.11, $R(C_4,W_n) \le n+\lceil\sqrt{n}\rceil+1$. Since $n+\lceil\sqrt{n}\rceil+1 < 2n+1$ for $n \ge 6$, we have C_4 is Ramsey saturated with respect to W_n for $n \ge 6$.

For any $e \in \overline{C_5}$, $R(C_5+e,W_4) \ge R(C_3,W_4)$. By Lemma 4.12, $R(C_3,W_4) > R(C_5,W_4)$, we have C_5 is Ramsey saturated with respect to W_4. By Lemma 4.12, $R(C_5+e,W_3) = 13$, and by Theorem 4.1, $R(C_5,W_3) = 13$, we have C_5 is Ramsey unsaturated with respect to W_3.

(iii) Let G be any graph of order $R(W_n, C_m)$. Assume that $\overline{G}$ contains no C_m, then G contains W_n. If G contains no $W_n + e$ for all $e \in \overline{W_n}$, then $\overline{G}$ contains a $K_n - C_n$ as its subgraph. For $n \geq 6$, $d(K_n - C_n) = n - 3 \geq n/2$ and $K_n - C_n$ contains C_3. By Theorem 1.3, $\overline{G}$ contains a C_m, which contradicts our assumption. Then G contains $W_n + e$ for some $e \in \overline{W_n}$. Thus, W_n is Ramsey unsaturated with respect to C_m for $n \geq \max\{m, 6\}$.

For $n = 5$, by Lemma 4.12, $R(W_5 + e, C_m) = R(W_5, C_m)$ for $m = 3, 4, 5$, we have W_5 is Ramsey unsaturated with respect to C_m for $m = 3, 4, 5$.

(iv) For any $e \in \overline{W_4}$, $W_4 + e = K_5 - e$. By Lemma 4.12, $R(W_4, C_3) = 11$ and $R(K_5 - e, C_3) = 11$, we have W_4 is Ramsey unsaturated with respect to C_3. By Lemma 4.12, $R(W_4, C_4) = 9$ and $R(W_4 + e, C_4) \geq R(W_3, C_4) = 10$. That is, W_4 is Ramsey saturated with respect to C_4. □

Chapter 5

Trees versus fans or wheels

5.1 Introduction

We study tree-fan Ramsey numbers and tree-wheel Ramsey numbers in this chapter.

By Caylay's Formula, the number of labelled trees on n vertices is n^{n-2}. Up to isomorphism, the value is still large for the number of trees. This may be one of the reasons that there are only a few Ramsey numbers involving trees. Researchers always study special trees like paths and stars. We here define some special trees, which will appear in the sequel. A double-star is the union of two stars with one edge joining the centers. A spider is a tree with at most one vertex of degree more than 2, called the center of the spider. A leg of a spider is a path from the center to a vertex of degree 1. A caterpillar is a tree in which a single path is incident to (or contains) every edge.

For an arbitrary tree, the following is a celebrated early result due to Chvátal, which has appeared in Chapter 1.

Theorem 5.1 (Chvátal [32]). $R(T_n, K_m) = (n-1)(m-1)+1$ *for all positive integers m and n.*

By Theorem 5.1, it is easy to see that T_n is K_m-good. This raises the natural questions whether and when T_n is G-good if G consists of ℓ complete graphs K_m sharing exactly one vertex. A special case of the question is whether T_n is F_ℓ-good. Another natural question is for what graphs G, T_n is G-good.

In 1982, Burr et al. determined the Ramsey numbers of sufficiently large trees versus odd cycles, by showing that T_n is C_m-good for odd $m \geq 3$ and $n \geq 756m^{10}$.

Theorem 5.2 (Burr et al. [21]). $R(T_n, C_m) = 2n-1$ *for odd* $m \geq 3$ *and* $n \geq 756m^{10}$.

In 1988, Erdős et al. confirmed the Ramsey numbers of relatively large trees versus books, by showing that T_n is B_m-good for $n \geq 3m-3$, a result that we will use in our proof of Lemma 5.2 in the next section.

Theorem 5.3 (Erdős et al. [49]). $R(T_n, B_m) = 2n-1$ *for* $n \geq 3m-3$.

Other results on Ramsey numbers concerning trees can be found in [6,7,28, 29,31,69], see [110] for a survey. In this paper, we first show that S_n is F_m-good for all integers $n \geq \max\{m(m-1)+1, 6(m-1)\}$, by proving the following result.

Theorem 5.4. $R(S_n, F_m) = 2n-1$ *for* $n \geq m(m-1)+1$ *and* $m \neq 3,4,5$, *and the lower bound* $n \geq m(m-1)+1$ *is best possible.* $R(S_n, F_m) = 2n-1$ *for* $n \geq 6(m-1)$ *and* $m = 3,4,5$.

We postpone the proof of Theorem 5.4 to Section 5.2. Next we show that T_n is F_m-good for positive integers $n \geq 3m^2-2m-1$, which is the main theorem of this chapter.

Theorem 5.5. $R(T_n, F_m) = 2n-1$ *for all integers* $n \geq 3m^2-2m-1$.

We also postpone the proof of Theorem 5.5 to Section 5.3. Now we prove that the following more general result can be obtained from Theorem 5.5 by induction.

Corollary 5.1. $R(T_n, K_{\ell-1}+mK_2) = \ell(n-1)+1$ *for* $\ell \geq 2$ *and* $n \geq 3m^2-2m-1$.

Proof. By Theorem 5.5, the statement is valid for $\ell = 2$. Assume that $k \geq 3$ and that the statement holds for all integers ℓ with $2 \leq \ell < k$. We prove that it also holds for $\ell = k$.

Since kK_{n-1} contains no T_n and its complement contains no K_{k+1}, hence no $K_{k-1}+mK_2$, we have $R(T_n,K_{k-1}+mK_2) \geq k(n-1)+1$. Let G be a graph of order $k(n-1)+1$. If $\delta(G) \geq n-1$, then by the following folklore lemma that is straightforward to prove using a greedy approach, G contains T_n and the proof is complete. We present the lemma in a more specific form since we will use it in this form in the sequel.

Lemma 5.1. *Let G be a graph with $\delta(G) \geq k$, and let $u \in V(G)$. Let T be a tree of order $k+1$ with $v \in V(T)$. Then T can be embedded into G in such a way that v is mapped to u.*

Let us now assume that $\delta(G) \leq n-2$. Then $\Delta(\overline{G}) \geq (k-1)(n-1)+1$. Let v be a vertex with $d_{\overline{G}}(v) = \Delta(\overline{G})$. Then, by the induction hypothesis either $G[N_{\overline{G}}(v)]$ contains a T_n, or $\overline{G}[N_{\overline{G}}(v)]$ contains a $K_{k-2}+mK_2$, which together with v forms a $K_{k-1}+mK_2$ in $\overline{G}$. This completes the proof of Corollary 5.1. □

We finish the study of $R(T_n,F_m)$ by posing a conjecture on the best possible lower bound for n for which T_n is F_m-good.

Conjecture 5. $R(T_n,F_m) = 2n-1$ for $n \geq m^2 - m + 1$.

Let G be any given graph. It is believed that $R(T_n,G) \leq R(S_n,G)$ in general, and all known results point in this direction. Based on this and Theorem 5.4, we believe that the above conjecture holds, at least for $m \geq 6$.

Now we turn to study trees versus cycles and wheels. For Ramsey numbers of trees versus odd cycles, the best known result is Theorem 5.2.

As a tribute to Erdős and Sós, a tree T of order n is called an ES-tree if every graph $G = (V,E)$ with $|E(G)| > |V(G)|(n-2)/2$ contains T as a subgraph. Let $\mathscr{T}$ be the set of all ES-trees. In 1963, Erdős and Sós conjectured that every tree is an ES-tree. Even though this conjecture is still open, it has been shown that many trees are indeed ES-trees. For instance, stars (as an immediate consequence), paths (by Erdős and Gallai [50]), trees of order n that have a vertex with at least $n/2-1$ degree 1 neighbors (by Sidorenko [120]), trees of order n that have a vertex with at least $n/2-2$ degree 1 neighbors (by Eaton and Tiner [42]), double-stars (as an immediate corollary of Sidorenko's result), spiders of diameter at most 4 (by Woźniak [136]), trees of diameter at most 4 (by McLennan [99]), and spiders with three legs and spiders with no leg of length more than 4 (by Fan and Sun [53]). Two additional results were announced without being published: caterpillars are ES-trees (by Perles in 1990, as mentioned in [100]), and sufficiently large trees are ES-trees (by Ajtai et al. [1]).

In the following, we first present a result on Ramsey numbers of ES-trees versus odd cycles. This turns out to be easy to prove, as is clear from the proof in Section 5.4. Using this result, we establish a theorem on Ramsey numbers of ES-trees versus wheels of even order. By this result we generalize (to some extent) former results on Ramsey numbers of wheels versus special trees, including stars, paths and star-like trees. The proof is postponed to Section 5.5.

Theorem 5.6. *$R(T_n,C_m) = 2n-1$ for $T_n \in \mathscr{T}$, odd $m \geq 3$ and $n \geq m-1$.*

Theorem 5.7. *$R(T_n,W_m) = 3n-2$ for $T_n \in \mathscr{T}$, odd $m \geq 3$ and $n \geq m-2$.*

From Theorem 5.7 a more general result can be obtained by induction.

Theorem 5.8. *$R(T_n,K_\ell + C_m) = (\ell+2)(n-1)+1$ for $T_n \in \mathscr{T}$, odd $m \geq 3$, $\ell \geq 1$ and $n \geq m-2$.*

5.2 Proof of Theorem 5.4

We use the following lemma in our proof. It is the special case of the statement of Theorem 5.4 when $m = 2$.

Lemma 5.2. *$R(S_n,F_2) = 2n-1$ for $n \geq 3$.*

Proof. The lower bound $R(S_n,F_2) \geq 2n-1$ is implied by the fact that $2K_{n-1}$ contains no S_n and its complement contains no triangle, hence no F_2. It remains to prove that $R(S_n,F_2) \leq 2n-1$ for $n \geq 3$.

Let G be a graph of order $2n-1$. Suppose that G contains no F_2 and $\overline{G}$ has no S_n. Then $\Delta(\overline{G}) \leq n-2$ and so $\delta(G) \geq n$. By Theorem 5.3, G contains B_2. Let $x_1x_2x_3x_4$ be a C_4 with diagonal x_2x_4 in G. Set $X = \{x_1,x_2,x_3,x_4\}$ and $Y = V(G)-X$. If $n = 3$, then $|Y| = 1$ and the vertex in Y has at least three neighbors in X, and so G has F_2, a contradiction. Hence, $n \geq 4$. If $x_1x_3 \in E(G)$, then $N_Y(x_i) \cap N_Y(x_j) = \emptyset$ for $1 \leq i < j \leq 4$, otherwise G contains F_2. Thus, we have $4(n-2) \leq \sum_{k=1}^{4} d_Y(x_k)+4 \leq 2n-1$, which implies that $n \leq 3$, a contradiction.

If $x_1x_3 \notin E(G)$, then since G has no F_2, we get that $N_Y(x_1) \cap N_Y(x_i) = \emptyset$ for $i = 2,4$ and $N_Y(x_1)$ is an independent set of cardinality at least $n-2$. In this case, we have $d(y) \le n-1$ for any $y \in N_Y(x_1)$, which contradicts that $\delta(G) \ge n$. □

The result is easy to prove for $m = 1$ and in this case follows also from Theorem 5.1, and it holds for $m = 2$ by Lemma 5.2, thus we may assume that $m \ge 3$.

We are first going to show that if $n \le m(m-1)$, then $R(S_n, F_m) \ge 2n$, showing that the lower bound $n \ge m(m-1)+1$ is in some sense best possible. Since K_{2m-1} contains no F_m and its complement contains no S_n, we have $R(S_n, F_m) \ge 2m$, so we only need to consider the case that $n \ge m+1$. There exist positive integers p, q such that $n = pm + q$ and $1 \le q \le m$. Let $H = pS_m \cup S_q$ if $q \ne 1$, and $H = (p-1)S_m \cup S_{m-1} \cup S_2$ if $q = 1$. Since $n \le m(m-1)$, then $p \le m-2$. It is easy to check that H is a graph of order n with $\delta(H) \ge 1$, and that H contains neither S_{m+1} nor mK_2. Let $H' = K_{n-1} \cup \overline{H}$. Then H' contains no S_n and $\overline{H'}$ contains no F_m. Thus, if $n \le m(m-1)$, then $R(S_n, F_m) \ge 2n$.

It remains to show that $R(S_n, F_m) = 2n-1$ for $n \ge \max\{m(m-1)+1, 6(m-1)\}$ and $m \ge 3$. First we note that since $2K_{n-1}$ contains no S_n and its complement contains no F_m, we conclude that $R(S_n, F_m) \ge 2n-1$. To prove $R(S_n, F_m) \le 2n-1$, let G be a graph of order $2n-1$ and suppose to the contrary that G contains no F_m and $\overline{G}$ contains no S_n. Then $\Delta(\overline{G}) \le n-2$ and $\delta(G) \ge n$. For any vertex u of $V(G)$, let $M_u \subseteq E(G)$ be a maximum matching in $G[N(u)]$ and $X_u = N(u) - V(M_u)$. Then, obviously $G[X_u]$ contains no edges, and $|M_u| \le m-1$; otherwise $G[N[u]]$ contains an F_m, a contradiction. Moreover, by the maximality of M_u, for $xy \in M_u$, if $d_{X_u}(x) \ge 2$, then $d_{X_u}(y) = 0$; and if $d_{X_u}(x) = d_{X_u}(y) = 1$, then x and y are adjacent to the same vertex in X_u. Let $Y_u \subseteq V(M_u)$ be the set of vertices that have at least two neighbors in X_u, and let $Z_u = N(u) - Y_u$. It is obvious that $|Y_u| \le m-1$ and $|Z_u| \ge n-m+1$.

Since $X_u \subseteq Z_u$ and $|X_u| \ge n-2(m-1) \ge m$, there exists a vertex $v \in X_u$ with $d_{Z_u}(v) = 0$. We define M_v, X_v, Y_v, Z_v in a completely analogous way. Since $d_{Z_u}(v) = 0$ and $Z_v \subseteq N(v)$, we get that $Z_u \cap Z_v = \emptyset$. Hence, $X_u \cap X_v = \emptyset$. We first prove the following two claims.

Claim 1. Let $V_1 = \{w \mid |X_w \cap X_u| \ge |X_u| - 2m + 3, \text{and } X_w \cap X_v = \emptyset\}$, $V_2 = \{w \mid |X_w \cap X_v| \ge |X_v| - 2m + 3, \text{and } X_w \cap X_u = \emptyset\}$. Then for any vertex w of $V(G)$, either $w \in V_1$ or $w \in V_2$. Moreover, $Z_v \subseteq V_1$, $Z_u \subseteq V_2$.

Proof. For any vertex w of $V(G)$, if $X_w \cap X_u = \emptyset$ and $X_w \cap X_v = \emptyset$, then $2n-1 \geq |X_u| + |X_v| + |X_w| \geq 3(n-2(m-1))$, and hence $n \leq 6(m-1)-1$, a contradiction. Thus, either $X_w \cap X_u \neq \emptyset$ or $X_w \cap X_v \neq \emptyset$. If $X_w \cap X_u \neq \emptyset$, since both $G[X_w]$ and $G[X_u]$ are edgeless graphs, then for any vertex z in $X_w \cap X_u$, we have $d(z) \geq |X_w| + |X_u| - |X_w \cap X_u| - 1$ in $\overline{G}$. Since $d(z) \leq \Delta(\overline{G}) \leq n-2$, we obtain $|X_w \cap X_u| \geq |X_w| + |X_u| - 1 - (n-2)$. Hence, $|X_w \cap X_u| \geq |X_u| - 2m + 3$ and $|X_w \cap X_u| \geq |X_w| - 2m + 3$. For the same reason, if $X_w \cap X_v \neq \emptyset$, then $|X_w \cap X_v| \geq |X_v| - 2m + 3$ and $|X_w \cap X_v| \geq |X_w| - 2m + 3$. If both $X_w \cap X_u \neq \emptyset$ and $X_w \cap X_v \neq \emptyset$, then $|X_w| \geq |X_w \cap X_u| + |X_w \cap X_v| \geq 2(|X_w| - 2m + 3)$, and hence $|X_w| \leq 4m - 6$, which contradicts $|X_w| \geq n - 2(m-1) \geq 4m - 4$. Therefore, for any vertex w of $V(G)$, either $w \in V_1$ or $w \in V_2$.

Any vertex w of Z_v has at most one adjacent vertex in X_v, hence $w \in V_1$. Thus, $Z_v \subseteq V_1$. By symmetry, $Z_u \subseteq V_2$. □

Claim 2. For any two vertices $w_1, w_2 \in V_1$, $|X_{w_1} \cap X_{w_2}| \geq 2m - 1$. For any two vertices $w_3, w_4 \in V_2$, $|X_{w_3} \cap X_{w_4}| \geq 2m - 1$.

Proof. It is sufficient to prove the first statement. For any two vertices $w_1, w_2 \in V_1$, since $|X_{w_i} \cap X_u| \geq |X_u| - 2m + 3$ for $i = 1, 2$, we get that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1} \cap X_u| + |X_{w_2} \cap X_u| - |X_u| \geq 2$. Since both $G[X_{w_1}]$ and $G[X_{w_2}]$ are edgeless graphs, for any vertex z in $X_{w_1} \cap X_{w_2}$, we have $d(z) \geq |X_{w_1}| + |X_{w_2}| - |X_{w_1} \cap X_{w_2}| - 1$ in $\overline{G}$. Since $d(z) \leq \Delta(\overline{G}) \leq n-2$, we obtain that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1}| + |X_{w_2}| - 1 - (n-2)$. Hence, $|X_{w_1} \cap X_{w_2}| \geq 2m - 1$. □

By Claim 1, every vertex belongs to either V_1 or V_2, but not both. Since $|V(G)| = 2n - 1$, we have $|V_1| \geq n$ or $|V_2| \geq n$. Without loss of generality, we may assume that $|V_1| \geq n$. For any vertex z of V_1, if $d_{V_1}(z) \geq m$, we choose m adjacent vertices of z from V_1, denoted by $z_1, \ldots, z_m$. By Claim 2, for $1 \leq i \leq m$, z_i has at least m adjacent vertices in $X_z - \{z_1, \ldots, z_m\}$. Thus, we may find a matching of m edges in $G[N(z)]$, which together with z forms an F_m, a contradiction. Therefore, for any vertex z of V_1, we have $d_{V_1}(z) \leq m - 1$. If $|Z_v| \geq n$, since $X_v \subseteq Z_v$ and $|X_v| \geq n - 2(m-1) \geq m$, there exists a vertex of degree 0 in $G[Z_v]$, that is, $\overline{G}[Z_v]$ contains a vertex of degree at least $n-1$, a contradiction. This implies that $|Z_v| \leq n - 1$. Since $Z_v \subseteq V_1$, we choose a subset of V_1 containing Z_v and any $n - |Z_v|$ vertices of $V_1 - Z_v$. For simplicity, this subset of V_1 is also denoted by V_1 in the sequel. Thus, $|V_1| = n$.

In the remainder, we prove that there exists a vertex z_0 of V_1 such that $d_{V_1}(z_0) = 0$ in G, and then $d_{V_1}(z_0) = n-1$ in $\overline{G}$, which is a contradiction. Since $|Z_v| \geq n-m+1$, we distinguish three cases: $|Z_v| \geq n-m+3$, $|Z_v| = n-m+1$ and $|Z_v| = n-m+2$, separately.

If $|Z_v| \geq n-m+3$, X_v contains at most $m-1$ vertices which are adjacent to $Z_v - X_v$, and every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of X_v. Since $|X_v| \geq n-2(m-1)$, $|V_1 - Z_v| \leq m-3$ and $n-2(m-1)-(m-1)-(m-3)(m-1) \geq 1$, so we may find the required z_0 in X_v, that is, with $d_{V_1}(z_0) = 0$ in G.

If $|Z_v| = n-m+1$, then $|Y_v| \geq n - |Z_v| = m-1$, and by the maximality of M_v, $G[Z_v]$ is an edgeless graph. Since every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of Z_v, $|V_1 - Z_v| = m-1$ and $n-m+1-(m-1)^2 \geq 1$, so we may find the required z_0 in Z_v, that is, with $d_{V_1}(z_0) = 0$ in G.

If $|Z_v| = n-m+2$, then $|Y_v| \geq n - |Z_v| = m-2$, and by the maximality of M_v, $G[Z_v]$ contains at most one edge of M_v. Let x_1y_1 be the possible edge both in $G[Z_v]$ and M_v, and suppose that $d_{V_1}(x_1) \geq d_{V_1}(y_1)$. If there is at most one vertex in $Z_v - \{x_1, y_1\}$ which is adjacent to x_1 or y_1, then, since every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of Z_v, $|V_1 - Z_v| = m-2$ and $n-m+2-3-(m-2)(m-1) \geq 1$, so we may find the required z_0 in Z_v, that is, with $d_{V_1}(z_0) = 0$ in G. If there are at least two vertices in $Z_v - \{x_1, y_1\}$ which are adjacent to x_1 or y_1, then, by the maximality of M_v, they are all adjacent to x_1. Since $d_{Z_v}(x_1) \leq m-1$, every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of Z_v, $|V_1 - Z_v| = m-2$ and $n-m+2-m-(m-2)(m-1) \geq 1$, so we may find the required z_0 in Z_v, that is, with $d_{V_1}(z_0) = 0$ in G. This completes the proof.

We recall that we have shown that $R(S_n, F_m) \geq 2n$ for $n \leq m(m-1)$, so the lower bound $n \geq m(m-1)+1$ is best possible for $m \geq 6$. □

5.3 Proof of Theorem 5.5

We use the following lemma in our proof. It deals with Ramsey numbers of trees versus mK_2 instead of F_m and might be of some interest by itself.

Lemma 5.3. $R(T_n, mK_2) = n+m-1$ *for* $n \geq 4(m-1)$.

Proof. The result is trivial for $m = 1$, thus we assume that $m \geq 2$. Since $K_{n-1} \cup$

$(m-1)K_1$ contains no T_n and its complement contains no mK_2, we conclude that $R(T_n,mK_2) \geq n+m-1$. It remains to prove that $R(T_n,mK_2) \leq n+m-1$ for $n \geq 4(m-1)$.

Let G be a graph of order $n+m-1$, and suppose to the contrary that neither G contains a T_n nor $\overline{G}$ contains mK_2. Let $M = \{x_1y_1,\ldots,x_ty_t\} \subseteq E(\overline{G})$ be a maximum matching in $\overline{G}$ and $X = V(G)-V(M)$. Then, obviously $t \leq m-1$ since $\overline{G}$ contains no mK_2, and $G[X]$ is a complete graph by the maximality of M. Assume without loss of generality that $d_X(x_i) \leq d_X(y_i)$ for $1 \leq i \leq t$ in $\overline{G}$. By the maximality of M, $d_X(x_i) \leq 1$ for $1 \leq i \leq t$ in $\overline{G}$. Let Y be the subset of X containing all adjacent vertices of $\{x_1,\ldots,x_t\}$ in $\overline{G}$. Then, by the previous arguments $|Y| \leq t \leq m-1$. Since T_n is a bipartite graph, we may assume without loss of generality that $V(T_n) = (X',Y')$ with $|X'| \leq |Y'|$. Since $n \geq 4(m-1)$, we get that $|Y'| \geq n/2 \geq 2(m-1) \geq |Y|+t$. Now we can embed T_n into G using the following procedure. First map $|Y|+t$ vertices of Y' to $Y \cup \{x_1,\ldots,x_t\}$ arbitrarily, and then map the remaining vertices of T_n to $X-Y$ arbitrarily. This is possible because $|X|+t = n+m-1-2t+t = n+m-(t+1) \geq n$ and every vertex of $X-Y$ is adjacent to every vertex of $X \cup \{x_1,\ldots,x_t\}$ except itself. Thus, G contains T_n, a contradiction. This completes the proof of Lemma 5.3. □

Recall that we want to prove that $R(T_n,F_m) = 2n-1$ for all integers $n \geq 3m^2-2m-1$. The lower bound $R(T_n,F_m) \geq 2n-1$ is implied by the fact that $2K_{n-1}$ contains no T_n while its complement contains no F_m. Now we prove the upper bound.

We may assume that $m \geq 2$ since the result is easy to prove for $m=1$ and in this case follows also from Theorem 5.1. Let G be a graph of order $2n-1$ with $n \geq 3m^2-2m-1$ and $m \geq 2$. Suppose to the contrary that G contains no F_m and its complement contains no T_n. We first claim that $\Delta(G) \leq n+m-2$. If not, let v be a vertex with $d(v) = \Delta(G) \geq n+m-1$. Since $n \geq 3m^2-2m-1$ and $m \geq 2$, this implies that $n \geq 4(m-1)$. By Lemma 5.3, either $G[N(v)]$ contains mK_2, which together with v forms an F_m, or $\overline{G}[N(v)]$ contains a T_n. Therefore, we have $\Delta(G) \leq n+m-2$.

Next we prove that Theorem 5.5 holds when $\Delta(T_n) \geq 13n/24$. Let u be a vertex of largest degree in T_n, let A denote the set of vertices of T_n that are adjacent to u and have degree one in T_n, and let B denote the set of vertices of T_n that are adjacent to u and have degree at least two in T_n. Then, obviously since T_n is a tree, $|V(T_n)| \geq 1+|A|+2|B|$ and $\Delta(T_n) = |A|+|B|$. Since $|V(T_n)| = n$

and we assume that $\Delta(T_n) \geq 13n/24$, we obtain that $|A| + n = 1 + 2|A| + 2|B| = 1 + 2\Delta(T_n) \geq 1 + 13n/12$, hence $|A| \geq n/12 + 1$. Then $T_n - A$ is a tree of order at most $11n/12 - 1$. We want to apply Lemma 5.1 to embed $T_n - A$ in $\overline{G}$ such that u is mapped to the vertex of degree $n-1$ of an S_n. Since $|V(T_n - A)| \leq 11n/12 - 1$, it is sufficient to show that $\delta(\overline{G}) \geq 11n/12 - 2$ and that $\overline{G}$ contains an S_n.

Since $\Delta(G) \leq n+m-2$, we get that $\delta(\overline{G}) \geq (2n-1)-1-(n+m-2) = n-m$. Using $m \geq 2$, it is easy to check that $3m^2 - 2m - 1 \geq 12m - 24$. By the condition of the theorem, $n \geq 3m^2 - 2m - 1 \geq 12m - 24$, so $n/12 \geq m-2$, and hence $n-m \geq 11n/12-2$. Furthermore, again using $m \geq 2$, $3m^2 - 2m - 1 \geq \max\{m(m-1)+1, 6(m-1)\}$. By Theorem 5.4, $\overline{G}$ contains an S_n. By Lemma 5.1, $T_n - A$ can be embedded in $\overline{G}$ such that u is mapped to the vertex with degree $n-1$ of the S_n. Because u now has at least $n-1$ adjacent vertices in $\overline{G}$, the embedding of $T_n - A$ can easily be extended to T_n in $\overline{G}$. This contradicts the assumption that $\overline{G}$ contains no T_n, completing this case. So, in the remainder of the proof we assume that $\Delta(T_n) < 13n/24$.

By Lemma 5.1, $\delta(\overline{G}) \leq |V(T_n)| - 2 = n-2$; otherwise we can embed T_n in $\overline{G}$. So we obtain that $\Delta(G) \geq n$. Let x be a vertex with $d(x) = \Delta(G) \geq n$, let $M = \{x_1y_1, \ldots, x_ty_t\} \subseteq E(G[N(x)])$ be a maximum matching in $G[N(x)]$, and let $U = V(G[N(x)]) - V(M)$. Then $G[U]$ is an edgeless graph, and $t \leq m-1$; otherwise $G[N(x)]$ contains mK_2, which together with x forms an F_m, a contradiction. Without loss of generality, suppose that $d_U(x_i) \leq d_U(y_i)$ for $1 \leq i \leq t$, and suppose that k and the order of vertices is chosen such that $d_U(y_i) \leq 1$ for $1 \leq i \leq k$, and $d_U(y_i) \geq 2$ for $k+1 \leq i \leq t$ (We assume that the degenerate cases that all $d_U(y_i) \leq 1$ or all $d_U(y_i) \geq 2$ do not occur, but these can be dealt with similarly). By the maximality of M, $d_U(x_i) = 0$ for $k+1 \leq i \leq t$, $d_U(x_i) \leq 1$ for $1 \leq i \leq k$, and if $d_U(x_i) = d_U(y_i) = 1$, then x_i and y_i are adjacent to the same vertex of U. Let Y consist of the set $V(M) - \{y_{k+1}, \ldots, y_t\}$ and its adjacent vertex set in U, and let $X = U - Y$. It is easy to check that $|X| \geq n - 2t - k$, $|Y| \leq t + 2k$ and $|X| + |Y| \geq n - t + k$. Next we prove the following claim.

Claim 1. Let T' be an arbitrary tree of order $|X| + |Y|$ with $w_1 \in V(T')$, and let w_2 be a vertex of X. Then T' can be embedded in $\overline{G}[X \cup Y]$ such that w_1 is mapped to w_2.

Proof. Since T' is a bipartite graph, we may assume $V(T') = (X_1, Y_1)$ with $|X_1| \leq |Y_1|$. Since $n \geq 3m^2 - 2m - 1$, $m \geq 2$ and $k \leq t \leq m-1$, it is not difficult to check that $|Y_1|-1 \geq \lceil(|X|+|Y|)/2\rceil-1 \geq |Y|$. Now we can embed T' in $\overline{G}[X\cup Y]$ through the following procedure. First map w_1 to w_2; then map $|Y|$ vertices of Y_1 to Y arbitrarily. Finally, map the remaining vertices of T' to X arbitrarily. Because in $\overline{G}$ every vertex of X is adjacent to every vertex of $X\cup Y$ except itself, the embedding can succeed. □

If $|X|+|Y| \geq n$, then by Claim 1, $\overline{G}$ contains T_n, a contradiction. So we may assume $|X|+|Y| \leq n-1$. Let T' be a largest subtree of T_n that can be embedded in $\overline{G}$. Then T' is a proper subgraph of T_n. This implies there exists a vertex in T', say x', such that x' is adjacent to every vertex of $V(G)-V(T')$ in G. Hence, $d_{G-V(T')}(x') \geq n$.

In $G[N(x')]-(X\cup Y)$, we define M',U',t',k',X',Y' in a completely analogous way as we have defined M,U,t,k,X,Y in $G[N(x)]$. Now we distinguish two cases.

Case 1. In $\overline{G}$, $d_X(z) \geq m/2$ for some $z \in X'$, or $d_{X'}(z) \geq m/2$ for some $z \in X$.

By symmetry, we may assume that $d_X(z) \geq m/2$ for some $z \in X'$ in $\overline{G}$. For $v \in V(T_n)$, let $H_1,\ldots,H_\ell$ be all the components of $T_n - v$ with at most $m-1$ vertices, and ordered in such a way that $m-1 \geq |V(H_1)| \geq \ldots \geq |V(H_\ell)|$. We distinguish two subcases.

Subcase 1.1. There exists a vertex v of T_n such that $\sum_{i=1}^{p}|V(H_i)| \geq m-1$, where $p = \min\{\lceil m/2\rceil,\ell\}$.

We give an embedding of T_n in $\overline{G}$. First we map v to z. Let v_i be the vertex of H_i adjacent to v in T_n. Since $d_X(z) \geq m/2$ and $p \leq \lceil m/2\rceil$, we map $v_1,\ldots,v_p$ sequentially to the adjacent vertices of z in $\overline{G}[X]$. Since $|X| \geq n-2t-k$, $n \geq 3m^2-2m-1$ and $k \leq t \leq m-1$, we have $|X| \geq \lceil m/2\rceil(m-1) \geq \sum_{i=1}^{p}|V(H_i)|$. Since $\overline{G}[X]$ is a complete graph, $H_1,\ldots,H_p$ can be embedded in $\overline{G}[X]$ easily. Since $\sum_{i=1}^{p}|V(H_i)| \geq m-1$, $T_n - \bigcup_{i=1}^{p} V(H_i)$ is a tree of order at most $n-m+1$. Since $|X|+|Y| \geq n-m+1$, we have $|X'|+|Y'| \geq n-m+1$ by symmetry. By Claim 1 and the symmetry of $\overline{G}[X\cup Y]$ and $\overline{G}[X'\cup Y']$, $\overline{G}[X'\cup Y']$ contains $T_n - \bigcup_{i=1}^{p} V(H_i)$ such that v is mapped to z. Therefore, $\overline{G}$ contains T_n, a contradiction.

Subcase 1.2. For any vertex v of T_n, $\sum_{i=1}^{p}|V(H_i)| < m-1$, where $p = \min\{\lceil m/2\rceil,\ell\}$.

We first show that we may assume that for any vertex $v \in V(T_n)$, the largest component of $T_n - v$ is of order at least m. If not, each component of $T_n - v$ is of order at most $m-1$. Since Subcase 1.1 does not occur and each nontrivial component is of order at least two, then the number of nontrivial components is at most $m/2-1$, and the total order of the nontrivial components is at most $m-2$. Thus, the total order of the trivial components is at least $n-m+1$. This implies that $d(v) \geq n-m+1$. Using $n \geq 3m^2-2m-1$ and $m \geq 2$, we easily obtain that $d(v) \geq 13n/24$, but we have already shown that Theorem 5.5 holds when $\Delta(T_n) \geq 13n/24$. Thus, henceforth we may assume that for any vertex $v \in V(T_n)$, the largest component of $T_n - v$ is of order at least m.

Choose a vertex v from T_n such that the order of the largest component of $T_n - v$ is as small as possible. Let H_0 be a largest component of $T_n - v$ with $v_0 \in V(H_0)$ being adjacent to v in T_n. Then we claim that $|V(H_0)| \leq n-m+1$. Suppose to the contrary that $|V(H_0)| \geq n-m+2$. By the choice of v, the largest component of $T_n - v_0$ has at least $|V(H_0)| \geq n-m+2$ vertices, so this is the component of $T_n - v_0$ containing v. In that case, $|V(H_0)| \leq m-2$, a contradiction to our assumption. Therefore, there exists a vertex v such that $m \leq |V(H_0)| \leq n-m+1$, where H_0 is the largest component of $T_n - v$.

Let $zz' \in E(\overline{G})$ with $z \in X'$ and $z' \in X$. By symmetry and by Claim 1, we may embed H_0 in $\overline{G}[X \cup Y]$ such that v_0 is mapped to z', and $T_n - V(H_0)$ in $\overline{G}[X' \cup Y']$ such that v is mapped to z. Thus, $\overline{G}$ contains T_n, a contradiction. This completes Case 1.

Case 2. In $\overline{G}$, $d_X(z) < m/2$ for every $z \in X'$, and $d_{X'}(z) < m/2$ for every $z \in X$.

First consider an arbitrary vertex $v \in V(G)-(X \cup Y \cup X' \cup Y')$. Suppose $d_X(v) \geq \lceil 3m/2 \rceil - 1$ and $d_{X'}(v) \geq \lceil 3m/2 \rceil - 1$. Then, since every vertex of $N_X(v)$ has at most $\lceil m/2 \rceil - 1$ adjacent vertices of X' in $\overline{G}$, every vertex of $N_X(v)$ has at least m adjacent vertices of $N_{X'}(v)$ in G. Thus, in that case we may find a matching of m edges in $N_{X \cup X'}(v)$, which together with v forms an F_m, a contradiction. Therefore, for every vertex $v \in V(G)-(X \cup Y \cup X' \cup Y')$, if $d_X(v) \leq \lceil 3m/2 \rceil - 2$, then put $v \in Z$; if this is not the case, then put $v \in Z'$. Now (X,Y,Z,X',Y',Z') is a partition of G. Since $|V(G)| = 2n-1$, either $|X|+|Y|+|Z| \geq n$, or $|X'|+|Y'|+|Z'| \geq n$. Without loss of generality, assume that $|X|+|Y|+|Z| \geq n$. Let Z'' be a subset of Z with exactly $n-|X|-|Y| \leq t-k$ vertices. Then every vertex of Z'' has at most $\lceil 3m/2 \rceil - 2$ adjacent vertices in X. Since $n \geq 3m^2-2m-1$, $|X|-(\lceil 3m/2 \rceil -2)|Z''| \geq (n-2t-k)-(\lceil 3m/2 \rceil -2)(t-k) \geq n - \lceil 3m/2 \rceil t \geq n - \lceil 3m/2 \rceil (m-1) \geq n/2$. Since T_n is a bipartite graph, we may assume $V(T_n) = (X_2, Y_2)$ and $|X_2| \leq |Y_2|$. Now we can embed T_n in $\overline{G}[X \cup Y \cup Z'']$ through the following procedure. First map $|Y|+|Z''|+|N_X(Z'')|$ vertices of Y_2 to $Y \cup Z'' \cup N_X(Z'')$ arbitrarily; then map the remaining vertices of T_n to $X - N_X(Z'')$ arbitrarily. Because in $\overline{G}$, every vertex of $X - N_X(Z'')$ is adjacent to every vertex of $X \cup Y \cup Z''$ except itself, and $|X - N_X(Z'')| \geq n/2$, the embedding can succeed. Thus, $\overline{G}$ contains T_n, our final contradiction. □

5.4 Proof of Theorem 5.6

Since $2K_{n-1}$ contains no T_n and its complement contains no C_m for odd m, $R(T_n, C_m) \geq 2n-1$. It remains to show that $R(T_n, C_m) \leq 2n-1$ for $T_n \in \mathcal{T}$, odd $m \geq 3$ and $n \geq m-1$.

For this, let G be a graph of order $2n-1$ with odd $m \geq 3$ and $n \geq m-1$.

If $e(G) > (2n-1)(n-2)/2$, then by the definition of ES-trees, G contains T_n for any $T_n \in \mathcal{T}$. Thus, we may assume $e(G) \leq (2n-1)(n-2)/2$.

If $e(\overline{G}) < 1/2(n(2n-2)+1)$, then it is easy to calculate that $e(G)+e(\overline{G}) < (2n-1)(2n-2)/2$, a clear contradiction.

So $e(\overline{G}) \geq 1/2(n(2n-2)+1)$. Then, by Theorem 1.2, $c(\overline{G}) \geq n+1 \geq m$. We may assume that $\overline{G}$ is a nonbipartite graph; otherwise, if $\overline{G}$ is a bipartite graph, say $V(\overline{G}) = (X, Y)$ with $|X| \geq |Y|$, then $G[X]$ is a complete graph of order at least n, clearly containing T_n as a subgraph. Using $e(\overline{G}) \geq 1/2(n(2n-2)+1)$, it is easy to check that $e(\overline{G}) \geq (2n-2)^2/4+1$, and by Theorem 1.6, $\overline{G}$ is weakly pancyclic with $g(\overline{G}) = 3$. Thus, $\overline{G}$ contains C_m. This completes the proof of Theorem 5.6. □

5.5 Proof of Theorem 5.7

For proof of this theorem, we need the following preliminary results.

Lemma 5.4 (Erdős and Gallai [50]). *Let G be a graph of order n. If $e(G) > n(k-2)/2$, then G contains a path on k vertices.*

Lemma 5.5 (Hasmawati et al. [74]). *If m is odd and $2n-1 \geq m \geq 3$, then $R(K_{1,n-1}, W_m) = 3n-2$.*

Lemma 5.6 (Baskoro et al. [7]). *For odd $n \geq 3$, let G be a graph of order n which is obtained from K_n by removing $\lfloor n/2 \rfloor$ independent edges. Then G contains all trees on n vertices.*

It is clear that $3K_{n-1}$ contains no T_n and its complement contains no W_m for odd m. Thus, $R(T_n, W_m) \geq 3n-2$. It remains to prove that $R(T_n, W_m) \leq 3n-2$ for $T_n \in \mathscr{T}$, odd $m \geq 3$ and $n \geq m-2$. Since this is trivial for $n = 1,2$, we may assume that $n \geq 3$.

For this, let G be a graph of order $3n-2$ and suppose to the contrary that G does not contain some $T_n \in \mathscr{T}$ and $\overline{G}$ contains no W_m.

If $\delta(G) \geq n-1$, then by Lemma 5.1, G contains T_n, a contradiction. Thus, $\delta(G) \leq n-2$ and $\Delta(\overline{G}) \geq 2n-1$. Let v be a vertex with $d_{\overline{G}}(v) = \Delta(\overline{G}) \geq 2n-1$. If $n \geq m-1$, then, by Theorem 5.6 either $G[N_{\overline{G}}(v)]$ contains a T_n, which is a contradiction; or $\overline{G}[N_{\overline{G}}(v)]$ contains a C_m, which together with v forms a W_m, also a contradiction. Thus, for the remainder it is sufficient to consider the case when $n = m-2$.

If there exists a vertex u such that $d_{\overline{G}}(u) \geq 2n+1$, then, by Theorem 5.6 either $G[N_{\overline{G}}(u)]$ contains a T_{n+1}, which contains T_n as a subgraph, a contradiction; or $\overline{G}[N_{\overline{G}}(u)]$ contains a C_m, which together with u forms a W_m, also a contradiction. Thus, $\Delta(\overline{G}) \leq 2n$ and $\delta(G) \geq n-3$. By Lemma 5.5, G contains a $K_{1,n-1}$. Let w be the vertex of $K_{1,n-1}$ with degree $n-1$. If $n = 3$, then $T_3 = K_{1,2}$ is contained in G, a contradiction. Since $n = m-2$ and m is odd, we have $n = m-2 \geq 5$.

Suppose that T_n contains a vertex x with at least two degree 1 neighbors. Then we may get a tree T' of order $n-2$ from T_n by deleting exactly two degree 1 neighbors of x. By Lemma 5.1, T' can be embedded in G such that x is mapped to w. There are at least two adjacent vertices of w in G that are not in the embedding of T', showing that T_n can be embedded in G, a contradiction.

Hence, we may assume that every vertex of T_n has at most one degree 1 neighbor. This implies that T_n contains a vertex y with degree 2 which has exactly one degree 1 neighbor in T_n. Let y' be the degree 1 neighbor of y, and let z be the other neighbor of y in T_n. Then we may get a tree T'' of order $n-2$ from T_n by deleting y and y'.

We claim that T'' can be embedded in G such that z is mapped to w and there are at least three vertices of $N(w)$ which are not in the embedding of T''. If $d_G(w) \geq n$, then this follows directly from Lemma 5.1. Thus, we may assume that $d_G(w) = n-1$. We first show there is a vertex z' such that the distance between z and z' in T'' is two. Supposing this is not the case, every vertex of $T''-z$ is adjacent to z in T''. Since $n \geq 5$, then $|T''| \geq 3$ and z has at least two degree 1 neighbors, contrary to our assumptions.

Now let z'' be the common neighbor of z and z' in T''. Suppose first that there is an edge $w''w'$ such that $w'' \in N(w)$ and $w' \in V(G)-N[w]$. Then we may first map z to w, z'' to w'' and z' to w'. After that, since $\delta(G) \geq n-3$, T'' can be easily embedded in G by using a greedy algorithm. Since at most $n-3$ vertices of $N[w]$ are in the embedding of T'', at least three vertices of $N(w)$ are not in the embedding of T''. This proves our claim in this subcase.

Suppose next that there is no edge between $G[N[w]]$ and $G-N[w]$. We are going to reach a contradiction in this subcase. Since $|N[w]| = n$, we have $|V(G)-N[w]| = 2n-2$. We also have $\Delta(\overline{G}[N[w]]) \geq 2$; otherwise $G[N[w]]$ contains T_n by Lemma 5.6, a contradiction. Let $x_1x_2x_3$ be a path in $\overline{G}[N[w]]$. Since $|V(G)-N[w]| = 2n-2$, if $e(G-N[w]) > (2n-2)(n-2)/2$, then by the definition of ES-trees, $G-N[w]$ contains T_n, where $T_n \in \mathscr{T}$, a contradiction. Thus, $e(G-N[w]) \leq (2n-2)(n-2)/2$ and $e(\overline{G}-N[w]) \geq (2n-2)(n-1)/2$. By Lemma 5.4, $\overline{G}-N[w]$ contains a P_n. Set $P_n = y_1y_2\ldots y_n$. Since there is no edge between $G[N[w]]$ and $G-N[w]$, then $\overline{G}$ contains a $C_{n+2} = x_1y_1y_2\ldots y_{n-1}x_3y_nx_1$, which together with x_2 forms a $W_{n+2} = W_m$ in $\overline{G}$, a contradiction. This completes the proof of our claim that T'' can be embedded in G such that z is mapped to w and there are at least three vertices of $N(w)$ which are not in the embedding of T''.

Let z_1, z_2, z_3 be three vertices of $N(w)$ which are not in the embedding of T''. If z_1 has a neighbor that is not in the embedding of T'', say s, then we may map y to z_1, and y' to s. Then G contains T_n, a contradiction. This implies that all vertices of $N(z_1)$ are in the embedding of T''. For the same reason, all vertices of $N(z_i)$ are in the embedding of T'' for $i = 2,3$. Let H be the graph obtained from G by deleting the embedding of T'' and z_1, z_2, z_3. Then $|H| = 2n-3$. If $e(H) > (2n-3)(n-2)/2$, then by the definition of ES-trees, H contains T_n, where $T_n \in \mathscr{T}$, a contradiction. Thus, $e(H) \leq (2n-3)(n-2)/2$ and $e(\overline{H}) \geq (2n-3)(n-2)/2$. By Lemma 5.4, $\overline{H}$ contains a P_{n-1}. Set $P_{n-1} = s_1s_2\ldots s_{n-1}$, and let s_n be a vertex of H not in P_{n-1}. Then $\overline{G}$ contains a $C_{n+2} = z_1s_1s_2\ldots s_{n-1}z_2s_nz_1$, which together with z_3 forms a $W_{n+2} = W_m$ in $\overline{G}$, our final contradiction. This completes the proof of Theorem 5.7. $\square$

5.6 Proof of Theorem 5.8

By Theorem 5.7, the result holds for $\ell = 1$. Assume that $k \geq 2$ and that the result holds for all ℓ with $1 \leq \ell < k$. It suffices to prove that it also holds for $\ell = k$.

Since $(k+2)K_{n-1}$ contains no T_n and its complement contains no $K_k + C_m$ for odd m, we have $R(T_n, K_k + C_m) \geq (k+2)(n-1)+1$. Let G be a graph of order $(k+2)(n-1)+1$. If $\delta(G) \geq n-1$, then, by Lemma 5.1 G contains T_n and the proof is done. Let now $\delta(G) \leq n-2$. Then $\Delta(\overline{G}) \geq (k+1)(n-1)+1$. Let v be a vertex such that $d_{\overline{G}}(v) = \Delta(\overline{G})$. By the induction hypothesis, either $G[N_{\overline{G}}[v]]$ contains a T_n, or $\overline{G}[N_{\overline{G}}[v]]$ contains a $K_{k-1} + C_m$, which together with v forms a $K_k + C_m$ in $\overline{G}$. This completes the proof of Theorem 5.8. $\square$

Chapter 6

Fans versus fans, wheels or complete graphs

6.1 Introduction

In this chapter we deal with three main results: Ramsey numbers of large fans versus small fans, small even wheels and a complete graph of order five. Recall that a fan $F_n = K_1 + nK_2$ consists of n triangles sharing exactly one common vertex.

On the Ramsey numbers for fans, Li and Rousseau showed that F_n is F_1-good for $n \geq 2$ and obtained lower and upper bounds for $R(F_n, F_m)$ in terms of n and m.

Theorem 6.1 (Li and Rousseau [89]). *$R(F_n, F_1) = 4n+1$ for $n \geq 2$; and $4n+1 \leq R(F_n, F_m) \leq 4n+4m-2$.*

Lately, Lin and Li proved that F_n is F_2-good for $n \geq 2$ and improved the upper bound for $R(F_n, F_m)$ in Theorem 6.1.

Theorem 6.2 (Lin and Li [91]). *$R(F_n, F_2) = 4n+1$ for $n \geq 2$; and $R(F_n, F_m) \leq 4n+2m$ for $n \geq m \geq 2$.*

Obviously, Theorems 6.1 and 6.2 say that any F_n with $n \geq 2$ is both F_1-good and F_2-good. For a given $m \geq 3$, when F_n is F_m-good? Lin et al. established an approximate result by using Erdős-Simonovits Theorem.

Theorem 6.3 (Lin et al. [92]). $R(F_n,F_m) = 4n+1$ *for sufficiently large n.*

It is not difficult to see that F_n is not always F_m-good for $n \geq m \geq 2$. In fact, we can prove that $R(F_n,F_m) \geq 4n+2$ for $m \leq n < m(m-1)/2$. Since $m(m-1)/2 > m$, then $m \geq 4$. There exist positive integers p,q such that $2n+1 = pm+q$ and $1 \leq q \leq m$. Let $H = pS_m \cup S_q$ if $q \neq 1$, and $H = (p-1)S_m \cup S_{m-1} \cup S_2$ if $q = 1$. Since $n < m(m-1)/2$, then $2n+1 \leq m(m-1)$ and $p \leq m-2$. It is easy to check that H is a graph of order $2n+1$ with $\delta(H) \geq 1$, and that H contains neither S_{m+1} nor mK_2. Let $H' = K_{2n} \cup \overline{H}$. Then H' contains no F_n and $\overline{H'}$ contains no F_m. Thus, if $m \leq n < m(m-1)/2$, then $R(F_n,F_m) \geq 4n+2$.

The first result of this chapter is that, F_n is F_m-good for $n \geq \max\{m^2 - m/2, 11m/2-4\}$. We give the proof in Section 6.2.

Theorem 6.4. $R(F_n,F_m) = 4n+1$ *for* $n \geq \max\{m^2 - m/2, 11m/2-4\}$.

Remark: Since F_n is not F_m-good for $m \leq n < m(m-1)/2$, we wonder whether there is scope for some improvement in Theorem 6.4: can we reduce the lower bound from $n \geq \max\{m^2 - m/2, 11m/2-4\}$ to $n \geq m(m-1)/2$ and $m \geq 3$? If so, we have seen that $n \geq m(m-1)/2$ is best possible.

Now we establish new exact values of Ramsey numbers for fans versus even wheels, which are W_m with m odd.

In order to do so, we first establish the following two auxiliary theorems, that might be of some interest by themselves.

Theorem 6.5. $R(nK_2,W_m) = \max\{2n+\lceil m/2 \rceil, n+m\}$.

Theorem 6.6. $R(nK_2,C_m) = \max\{2n+\lceil m/2 \rceil - 1, n+m-1\}$.

Remark: Even though Theorem 6.6 is not an immediate consequence of Theorem 6.5, the proofs of both theorems are very similar and use basically the same method. We therefore omit the proof of Theorem 6.6. The proof of Theorem 6.5 is postponed to Section 6.3.

For Ramsey numbers of fans versus even wheels, Surahmat et al. proved that F_n is W_3-good for $n \geq 3$, and obtained the following result.

Theorem 6.7 (Surahmat et al. [129]). $R(F_n, W_3) = 6n + 1$ *for* $n \geq 3$.

We generalize the above result by showing that F_n is W_m-good for all odd $m \geq 3$ and $n \geq (5m+3)/4$.

Theorem 6.8. $R(F_n, W_m) = 6n + 1$ *for odd* $m \geq 3$ *and* $n \geq (5m+3)/4$.

The proof of Theorem 6.8 is postponed to Section 6.3.

For the pair F_l and K_n, noting that $\chi(K_n) = n$ and $s(K_n) = 1$, we have $R(F_l, K_n) \geq 2l(n-1)+1$ by Burr's lower bound. Gupta et al. showed the equality holds for $n = 3$ and established the following.

Theorem 6.9 (Gupta et al. [70]). $R(F_l, K_3) = 4l + 1$ *for* $l \geq 2$.

Since W_3 is in fact a complete graph of order four, by Theorem 6.7, Surahmat et al. proved the equality also holds for $n = 4$.

Maybe motivated by Theorems 6.9 and 6.7, Surahmat et al. conjectured that the equality holds in a more general case in the same paper, and posed the following.

Conjecture 6 (Surahmat et al. [129]). $R(F_l, K_n) = 2l(n-1) + 1$ for $l \geq n \geq 5$.

Other results on Ramsey numbers of fans versus complete graphs can be found in the dynamic survey [110]. We will confirm Conjecture 6 for $n = 5$, the proof of which will be posed in the last section.

Theorem 6.10. $R(F_l, K_5) = 8l + 1$ *for* $l \geq 5$.

6.2 Proof of Theorem 6.4

We need two preliminary lemmas before the proof.

Lemma 6.1 (Lin and Li [91]). $R(F_t, sK_2) = \max\{s,t\} + s + t$.

Lemma 6.2 (Hall [72]). *A bipartite graph* $G = (X, Y)$ *has a matching which covers every vertex in* X *if and only if* $|N(S)| \geq |S|$ *for all* $S \subseteq X$, *where* $N(S) = \bigcup_{v \in S} N_Y(v)$.

The lower bound $R(F_n,F_m) \geq 4n+1$ is implied by the fact that $2K_{2n}$ contains no F_n and its complement contains no triangle, hence no F_m. It remains to prove that $R(F_n,F_m) \leq 4n+1$ for $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$.

Let G be a graph of order $4n+1$ with $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$, and suppose to the contrary that neither G contains an F_n nor $\overline{G}$ contains an F_m. If $\Delta(G) \geq 2n+m$, let x be a vertex with $d(x) = \Delta(G)$ and $H = G[N(x)]$. By Lemma 6.1, either H contains nK_2, which together with x forms an F_n, or $\overline{H}$ contains an F_m, also a contradiction. Thus, we have $\Delta(G) \leq 2n+m-1$ and $\delta(\overline{G}) \geq 2n-m+1$.

Claim 1. For any vertex v of $V(G)$, $G - N_G[v]$ contains a subgraph H_v which satisfies one of the following conditions:

(1) $H_v = K_{2n-2m+2}$;
(2) $\overline{H_v} = K_3 \cup (2n-2m)K_1$;
(3) H_v is a graph of order $2n-m-l+1$ and at most $3m-2l-3$ vertices in $\overline{H_v}$ are of positive degree, where $0 \leq l \leq m-3$.

Moreover, there exists $X_v \subseteq V(H_v)$ such that $G[X_v] = K_{2n-3m+3}$ and $d_{X_v}(u) \geq 2n-3m+2$ for any $u \in V(H_v)$.

Proof. Since $\delta(\overline{G}) \geq 2n-m+1$, we have $|V(G) - N_G[v]| \geq 2n-m+1$. Let H_1 be an induced subgraph of $G - N_G[v]$ on $2n-m+1$ vertices, $M = \{x_1y_1, \ldots, x_ty_t\}$ a maximum matching of $\overline{H_1}$ and $H_2 = H_1 - V(M)$. We deduce that $t \leq m-1$, otherwise M together with v forms an F_m in $\overline{G}$, a contradiction. Since M is a maximum matching in $\overline{H_1}$, then $H_2 = K_{2n-m+1-2t}$. By the maximality of M, we can see that if $|N_{\overline{G}}(x_i) \cap V(H_2)| \geq 2$, then $|N_{\overline{G}}(y_i) \cap V(H_2)| = 0$ and vice versa. Assume without loss of generality that $x_1, x_2, \ldots, x_s$ are all the vertices of $V(M)$ such that $|N_{\overline{G}}(x_i) \cap V(H_2)| \geq 2$, where $s \leq t$. If $y_py_q \in E(\overline{G})$ for $1 \leq p < q \leq m-1$, then since $|N_{\overline{G}}(x_p) \cap V(H_2)| \geq 2$ and $|N_{\overline{G}}(x_q) \cap V(H_2)| \geq 2$, we can find an M-augmenting path in $\overline{H_1}$, which contradicts the maximality of M. Thus, $y_py_q \in E(G)$ for all $1 \leq p < q \leq s$.

Set $H_3 = H_1 - \{x_1, x_2, \ldots, x_s\}$. We first show that H_3 contains an H_v as required. By the assumption, $|N_{\overline{G}}(y_i) \cap V(H_2)| = 0$ for all $1 \leq i \leq s$. Noting that $y_py_q \in E(G)$ for all $1 \leq p < q \leq s$, we can see that $G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-m+1+s-2t}$.

If $s = m-1$, then $t = m-1$, and so $H_3 = G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-2m+2}$. Let $H_v = H_3$, then H_v is the subgraph as required.

If $s = m-2$ and $t = m-2$, then $H_3 = G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-2m+3}$, and hence H_3 contains a subgraph $H_v = K_{2n-2m+2}$. If $s = m-2$ and $t = m-1$, then $G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-2m+1}$. If $V(H_2) \subseteq N_G(x_{m-1})$ or $V(H_2) \subseteq$

$N_G(y_{m-1})$, then clearly H_3 contains an $H_v = K_{2n-2m+2}$. If not, then by the maximality of M, we have $N_{\overline{G}}(x_{m-1}) \cap V(H_2) = N_{\overline{G}}(y_{m-1}) \cap V(H_2)$ and $|N_{\overline{G}}(x_{m-1}) \cap V(H_2)| = |N_{\overline{G}}(y_{m-1}) \cap V(H_2)| = 1$, which implies that $H_3 = K_{2n-2m+3} - \{x_{m-1}y_{m-1}, x_{m-1}u, y_{m-1}u\}$ for some $u \in V(H_2)$. Taking $H_v = H_3$, then H_v is the subgraph as required.

If $s \le m-3$, we let $l = s$ and $H_v = H_3$. Obviously, $|H_v| = 2n - m - l + 1$. By the assumption, $|N_{\overline{G}}(x_i) \cap V(H_2)| \le 1$ and $|N_{\overline{G}}(y_i) \cap V(H_2)| \le 1$ for $s+1 \le i \le t$. By the maximality of M, we have $|(N_{\overline{G}}(x_i) \cup N_{\overline{G}}(y_i)) \cap V(H_2)| \le 1$ for $s+1 \le i \le t$. Thus, H_v contains at most $l + 3(t-l) \le 3m - 2l - 3$ vertices of positive degree in $\overline{H_v}$, where $0 \le l \le m-3$.

Since $|V(H_2)| = 2n - m + 1 - 2t \ge 2n - 3m + 3$, we may let $X_v \subseteq V(H_2)$ with $|X_v| = 2n - 3m + 3$. Because H_2 is a complete graph, we have $G[X_v] = K_{2n-3m+3}$. Noting that each vertex of $V(H_v) - V(H_2)$ has at most one nonadjacent vertex in $V(H_2)$, we have $d_{X_v}(u) \ge 2n - 3m + 2$ for any $u \in V(H_v)$. □

Let $v \in V(G)$ be given. By Claim 1, there exist H_v and X_v attached to v. Since $n \ge \max\{m^2 - m/2, 11m/2 - 4\}$, we have $2n - 2m \ge 1$ and $2n - m - l + 1 - (3m - 2l - 3) \ge 1$, it follows that $V(H_v)$ contains a vertex u such that $V(H_v) \subseteq N_G[u]$. By Claim 1, there exist H_u and X_u attached to u. Noting that $V(H_v) \subseteq N_G[u]$ and $V(H_u) \subseteq V(G) - N_G[u]$, we have $V(H_v) \cap V(H_u) = \emptyset$.

Set $V_1 = \{w \mid |X_w \cap X_u| \ge 2n - 7m + 6 \text{ and } X_w \cap X_v = \emptyset\}$ and $V_2 = \{w \mid |X_w \cap X_v| \ge 2n - 7m + 6 \text{ and } X_w \cap X_u = \emptyset\}$.

Claim 2. (V_1, V_2) is a partition of $V(G)$ with $V(H_v) \subseteq V_1$ and $V(H_u) \subseteq V_2$.

Proof. For any vertex w of $V(G)$, if $X_w \cap X_u = X_w \cap X_v = \emptyset$, then $4n + 1 \ge |X_u| + |X_v| + |X_w| \ge 3(2n - 3m + 3)$, and hence $n \le 9m/2 - 4$, a contradiction. Thus, either $X_w \cap X_u \ne \emptyset$ or $X_w \cap X_v \ne \emptyset$. If $X_w \cap X_u \ne \emptyset$, then since both $G[X_w]$ and $G[X_u]$ are complete graphs, we have $d(z) \ge |X_w| + |X_u| - |X_w \cap X_u| - 1$ for any vertex z in $X_w \cap X_u$. Because $d(z) \le \Delta(G) \le 2n + m - 1$, we obtain $|X_w \cap X_u| \ge |X_w| + |X_u| - 2n - m = 2n - 7m + 6$. Similarly, if $X_w \cap X_v \ne \emptyset$, then $|X_w \cap X_v| \ge 2n - 7m + 6$. If both $X_w \cap X_u \ne \emptyset$ and $X_w \cap X_v \ne \emptyset$, then $|X_w| \ge |X_w \cap X_u| + |X_w \cap X_v| \ge 2(2n - 7m + 6)$, and hence $n \le (11m - 9)/2$, which contradicts $n \ge (11m - 8)/2$. Therefore, for any vertex w of $V(G)$, either $w \in V_1$ or $w \in V_2$, but not in both, that is, (V_1, V_2) is a partition of $V(G)$.

By Claim 1, for any $w \in V(H_v)$, w is nonadjacent to at most one vertex of X_v, and $X_w \subseteq V(G) - N_G[w]$, hence $|X_w \cap X_v| \le 1$. Thus, $w \in V_1$ and $V(H_v) \subseteq V_1$. By symmetry, $V(H_u) \subseteq V_2$. □

Claim 3. For any two vertices $w_1, w_2 \in V_i$, $i = 1,2$, we have $|X_{w_1} \cap X_{w_2}| \geq 4m - 2$.

Proof. By symmetry, it is sufficient to assume that $w_1, w_2 \in V_1$. Since $|X_{w_j} \cap X_u| \geq 2n - 7m + 6$ for $j = 1,2$, we get that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1} \cap X_u| + |X_{w_2} \cap X_u| - |X_u| \geq 1$. Since both $G[X_{w_1}]$ and $G[X_{w_2}]$ are complete graphs, we have $d(z) \geq |X_{w_1}| + |X_{w_2}| - |X_{w_1} \cap X_{w_2}| - 1$ for any vertex z in $X_{w_1} \cap X_{w_2}$. Noting that $\Delta(G) \leq 2n + m - 1$ and $n \geq 11m/2 - 4$, we get that $|X_{w_1} \cap X_{w_2}| \geq 4m - 2$. □

Assume that $|V_1| \geq |V_2|$. By Claim 2, $|V_1| \geq \lceil (4n+1)/2 \rceil \geq 2n + 1$. For any vertex z of V_1, if $d_{V_1}(z) \geq m$ in $\overline{G}$, we choose m nonadjacent vertices of z from V_1, denoted by $z_1, \ldots, z_m$. By Claim 3, for $1 \leq i \leq m$, z_i and z have at least $4m - 2$ common nonadjacent vertices, and then z_i has at least $3m - 1$ nonadjacent vertices in $X_z - \{z_1, \ldots, z_m\}$. Thus, we may find a matching of m edges in $\overline{G}[N_{\overline{G}}(z)]$ by Lemma 6.2, which together with z forms an F_m in $\overline{G}$, a contradiction. Therefore, for any vertex z of V_1, we have $d_{V_1}(z) \leq m - 1$ in $\overline{G}$. Moreover, we may assume that $m \geq 2$, otherwise $G[V_1]$ is a complete graph which contains F_n, a contradiction. Since $V(H_v) \subseteq V_1$ and $|H_v| \leq 2n - 2m + 3$ by Claim 1, we let $V_1' \subseteq V_1$ such that $V(H_v) \subseteq V_1'$ and $|V_1'| = 2n + 1$.

Now we prove that there exists some $z_0 \in V_1'$ such that $d_{V_1'}(z_0) = 2n$. By Claim 1, $H_v = K_{2n-2m+2}$; or $\overline{H_v} = K_3 \cup (2n - 2m)K_1$; or H_v is a graph of order $2n - m - l + 1$ and at most $3m - 2l - 3$ vertices in $\overline{H_v}$ are of positive degree, where $0 \leq l \leq m - 3$. Since each vertex of $V_1' - V(H_v)$ is of degree at most $m - 1$ in $\overline{G}[V_1']$, then at most $q = \max\{(2m-1)m, (2m-2)m + 3, (m+l)m + (3m - 2l - 3)\}$ vertices are of positive degree in $\overline{G}[V_1']$. Because $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$, $m \geq 2$ and $l \leq m - 3$, it is easy to check that $q \leq 2n$. Thus, there is a vertex $z_0 \in V_1'$ such that $d_{V_1'}(z_0) = 2n$. Since $G[X_v - \{z_0\}]$ is a complete graph of order at least $2n - 3m + 2$, and every vertex of $V_1' - X_v$ has at least $2n - 3m + 2 - (m-1) \geq n$ adjacent vertices in X_v, we can always find a perfect matching in $G[V_1' - \{z_0\}]$, which together with z_0 forms an F_n, a contradiction. This completes the proof. □

6.3 Proof of Theorems 6.5 and 6.8

We first show Theorem 6.5. It is clear that $K_{2n-1} \cup \lceil m/2 \rceil K_1$ contains no nK_2. Suppose its complement contains a W_m. Then, since the cardinality of a maximum independent set of W_m is (at most) $\lfloor m/2 \rfloor$, at most $\lfloor m/2 \rfloor$ vertices of

W_m are in $V(K_{2n-1})$, hence $|V(W_m)| \le m$, a contradiction. This shows that $R(nK_2, W_m) \ge 2n + \lceil m/2 \rceil$. It is easy to check that $K_m \cup (n-1)K_1$ contains no W_m and that its complement contains no nK_2. Thus, $R(nK_2, W_m) \ge n + m$. Let $N = \max\{2n + \lceil m/2 \rceil, n + m\}$. It remains to prove that $R(nK_2, W_m) \le N$.

For this, let G be a graph of order N and suppose, to the contrary, that G contains no nK_2 and $\overline{G}$ contains no W_m. Let $M \subseteq E(G)$ be a maximum matching in G and let $X = V(G) - V(M)$. Then $G[X]$ is an edgeless graph, and $|M| \le n-1$; otherwise G contains nK_2, a contradiction. Moreover, because of the choice of M, for any edge $yz \in M$, we have $\min\{d_X(y), d_X(z)\} \le 1$.

Suppose first that $n \ge \lfloor m/2 \rfloor$. Then $N = 2n + \lceil m/2 \rceil$. Since $|M| \le n-1$ and $2|M| + |X| = 2n + \lceil m/2 \rceil$, we have $|X| \ge \lceil m/2 \rceil + 2$ and $|M| + |X| \ge m+1$. Thus, $\overline{G}[X]$ is a complete graph of order at least $\lceil m/2 \rceil + 2$, and $V(M)$ contains a subset Y of $m + 1 - |X|$ vertices such that each vertex of Y has at least $|X| - 1$ adjacent vertices of X in $\overline{G}$. Thus, there is a vertex $x \in X$ such that x is adjacent to every vertex of $X \cup Y - x$ in $\overline{G}$. By Theorem 1.1, $\overline{G}[X \cup Y - x]$ contains a C_m, which together with x forms a W_m, a contradiction.

Next suppose that $n \le \lfloor m/2 \rfloor - 1$. Then $N = n + m$. Since $|M| \le n-1$ and $2|M| + |X| = n + m$, we have $|X| \ge \lceil m/2 \rceil + 3$ and $|M| + |X| \ge m+1$. We may deduce a contradiction as before. This completes the proof of Theorem 6.5.

Now we establish Theorem 6.8.

Since $3K_{2n}$ contains no F_n and its complement contains no W_m for odd m, $R(F_n, W_m) \ge 6n+1$. It remains to show that $R(F_n, W_m) \le 6n+1$ for m odd and $n \ge (5m+3)/4$.

For this, let G be a graph of order $6n+1$ with $n \ge (5m+3)/4$ and with m odd. It is sufficient to show that either G contains an F_n or $\overline{G}$ contains a W_m. Suppose to the contrary that neither G contains an F_n nor $\overline{G}$ contains a W_m.

Suppose $\Delta(G) \ge 2n + (m+1)/2$. Let v be a vertex with $d(v) = \Delta(G)$. By Theorem 6.5, either $G[N(v)]$ contains nK_2, which together with v forms an F_n, or $\overline{G}[N(v)]$ contains a W_m. Hence, $\Delta(G) \le 2n + (m-1)/2$, implying that $\delta(\overline{G}) \ge 4n - (m-1)/2$. By Theorem 6.7, $\overline{G}$ contains a W_3. Now we choose a vertex u from the W_3 and let H be the subgraph of $\overline{G}$ formed by the set of nonadjacent vertices of u, that is, $H = \overline{G}[N_{\overline{G}}(u)]$. Hence, $|H| \ge 4n - (m-1)/2$, $g(H) = 3$ and H is nonbipartite.

First assume $\delta(H) \geq (|H|+2)/3$. Then by Theorem 1.5, H is weakly pancyclic. By Theorem 1.1, $c(H) \geq \delta(H)+1 \geq m$. Thus, H contains a C_m, which together with u forms a W_m in $\overline{G}$, a contradiction.

Next assume $\delta(H) < (|H|+2)/3$. Then $\overline{H}$ contains a vertex w such that $d_{\overline{H}}(w) = \Delta(\overline{H}) > (2|H|-5)/3$, that is, $d_{\overline{H}}(w) \geq (2|H|-4)/3$. Since $|H| \geq 4n-(m-1)/2$ and $n \geq (5m+3)/4$, we have $d_{\overline{H}}(w) \geq 2n+(m-1)/2$. By Theorem 6.6, either $G[N_{\overline{H}}(w)]$ contains nK_2, which together with w forms an F_n, a contradiction; or $\overline{G}[N_{\overline{H}}(w)]$ contains a C_m, which together with u forms a W_m in $\overline{G}$, also a contradiction. This completes the proof of Theorem 6.8. □

6.4 Proof of Theorem 6.10

Since $4K_{2l}$ contains no F_l and its complement contains no K_5, $R(F_l, K_5) \geq 8l+1$. In the following, we need only to show that $R(F_l, K_5) \leq 8l+1$.

Let G be a graph of order $8l+1$ with $l \geq 5$, we need to show that either G contains an F_l or $\overline{G}$ contains a K_5. Suppose to the contrary that neither G contains a F_l nor $\overline{G}$ contains a K_5.

Let $v \in V(G)$. If $d(v) \leq 2l-1$, then $G-N[v]$ is a graph of order at least $6l+1$. By Theorem 6.7, $\overline{G}-N[v]$ contains a K_4, which implies that $\overline{G}$ contains a K_5, a contradiction. If $d(v) \geq 2l+3$, then a maximum matching M of $G[N(v)]$ contains at least l edges for otherwise $\overline{G}[N(v)-V(M)]$ is a complete graph of order at least 5, which implies that G has an F_l, a contradiction. Therefore, $2l \leq d(v) \leq 2l+2$ for any $v \in V(G)$.

Suppose that G contains a subgraph $H = K_{2l-1}$. Choose $v_0 \in V(G)-V(H)$ such that $d_H(v_0) = \max\{d_H(v) \mid v \in V(G)-V(H)\}$. Obviously, $G-(V(H)\cup\{v_0\})$ is a graph of order $6l+1$. By Theorem 6.7, $G-(V(H)\cup\{v_0\})$ contains an independent set $\{u_1, u_2, u_3, u_4\}$. Since $\overline{G}$ has no K_5, we have $V(H)\cup\{v_0\} \subseteq \cup_{i=1}^{4} N(u_i)$. This implies that $\max\{d_H(u_i) \mid 1 \leq i \leq 4\} \geq \lceil (2l-1)/4 \rceil \geq 3$. By the choice of v_0, we have $d_H(v_0) \geq 3$. If $d_H(v_0) \geq 4$, then there is some u_i having at least two neighbors in $N_H(v_0)\cup\{v_0\}$; If $d_H(v_0) = 3$, then $d_H(u_i) \leq d_H(v_0) = 3$ for $1 \leq i \leq 4$, which implies that there exists some u_i such that $d_H(u_i) \geq 2$ and $N_H(u_i)\cap N_H(v_0) \neq \emptyset$. In both cases, $G[V(H)\cup\{v_0, u_i\}]$ contains an F_l, a contradiction. Hence, G contains no K_{2l-1}.

By Theorem 6.7, G has an independent set $U = \{u_1, u_2, u_3, u_4\}$. For $1 \le i \le 4$, set $X_i = \{v \mid d_U(v) = i,\ v \in V(G)\}$. Obviously,

$$\sum_{i=1}^{4} |X_i| = 8l - 3, \tag{6.1}$$

$$\sum_{i=1}^{4} i|X_i| = \sum_{i=1}^{4} d(u_i). \tag{6.2}$$

Since $\sum_{i=1}^{4} d(u_i) \le 8l + 8$, by (6.1) and (6.2), we have

$$|X_1| \ge 8l - 14 + |X_3| + 2|X_4| \ge 8l - 14. \tag{6.3}$$

Let $X_{1i} = N_{X_1}(u_i)$ for $1 \le i \le 4$. Because $\overline{G}$ has no K_5, $G[X_{1i} \cup \{u_i\}]$ is a complete graph. Since G contains no K_{2l-1}, we have $|X_{1i} \cup \{u_i\}| \le 2l - 2$, which implies that $|X_{1i}| \le 2l - 3$ for $1 \le i \le 4$. Thus, $|X_1| = \sum_{i=1}^{4} |X_{1i}| \le 8l - 12$. By (6.3), we have $|X_3| + 2|X_4| \le 2$. By (6.1),

$$|X_2| \ge 7. \tag{6.4}$$

Assume without loss of generality that $|X_{11}| \ge |X_{12}| \ge |X_{13}| \ge |X_{14}|$. Then $|X_{11}| = |X_{12}| = 2l - 3$, $|X_{13}| + |X_{14}| \ge 4l - 8$ and $|X_{14}| \ge 2l - 5$. Denote by U_i both the vertex set $X_{1i} \cup \{u_i\}$ and the graph $G[X_{1i} \cup \{u_i\}]$ for $1 \le i \le 4$, then U_1, U_2, U_3, U_4 are pairwise vertex-disjoint complete graphs with $|U_1| = |U_2| = 2l - 2$, $|U_3| + |U_4| \ge 4l - 6$ and $|U_4| \ge 2l - 4$.

Let $Y_{ij} = N_{X_2}(u_i) \cap N_{X_2}(u_j)$ for $1 \le i < j \le 4$.

Claim 1. If $|U_i| = 2l - 2$ for some i with $1 \le i \le 4$, then for any $y \in Y_{ij}$, $d_{U_j}(y) \ge 3$ and if $|U_i| = |U_j| = 2l - 2$, then $Y_{ij} = \emptyset$.

Proof. Since G contains no K_{2l-1}, $U_i - N(y) \ne \emptyset$. In this case, $G[U_i \cup U_j - N(y)]$ is a complete graph for otherwise any two nonadjacent vertices in $G[U_i \cup U_j - N(y)]$ together with $U \cup \{y\} - \{u_i, u_j\}$ form a K_5 in $\overline{G}$, a contradiction. Since G has no F_l and both U_i and U_j are complete graphs, we have $d_{U_j}(u) \le 3$ for any $u \in U_i$, which implies that $|U_j - N(y)| \le 3$. Noting that $|U_j| \ge 2l - 4$ and $l \ge 5$, we have $d_{U_j}(y) \ge |U_j| - |U_j - N(y)| \ge (2l - 4) - 3 \ge 3$.

If $|U_i| = |U_j| = 2l - 2$ and $Y_{ij} \ne \emptyset$, then for any $y \in Y_{ij}$, $d_{U_i}(y) + d_{U_j}(y) \le 2l$ since otherwise $G[N[y]]$ contains an F_l with y as center, a contradiction. Thus we have $|U_i \cup U_j - N(y)| \ge |U_i| + |U_j| - 2l \ge 6$ since $l \ge 5$. By the arguments in the first part, we have $|U_i - N(y)| = |U_j - N(y)| = 3$. Thus, $G[U_i \cup (U_j - N(y))]$ contains an F_l with u as center for any $u \in U_i - N(y)$, a contradiction. Hence $Y_{ij} = \emptyset$. □

If $|U_4| = 2l-2$, then by Claim 1, $X_2 = \cup_{1\le i<j\le 4} Y_{ij} = \emptyset$ which contradicts (6.4). Hence we have $2l-4 \le |U_4| \le 2l-3$.

Assume $|U_3| = 2l-2$. By Claim 1, we have $X_2 = \cup_{1\le i<j\le 4} Y_{ij} = Y_{14} \cup Y_{24} \cup Y_{34}$, that is, $X_2 \subseteq N(u_4)$. If $|U_4| = 2l-3$, then since $\sum_{i=1}^4 d(u_i) \le 8l+8$, by (6.1), (6.2) and (6.3), either $|X_2| = 10$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^4 d(u_i) = 8l+7$ or $|X_2| = 9$, $|X_3| = 1$, $|X_4| = 0$ and $\sum_{i=1}^4 d(u_i) = 8l+8$. If $|U_4| = 2l-4$, then for the same reason, we have $|X_2| = 11$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^4 d(u_i) = 8l+8$. Thus we have $|X_2| \ge 9$ in both cases, which implies that $d(u_4) \ge |X_{14}| + |X_2| \ge 2l-5+9 = 2l+4$, a contradiction. Therefore, $|U_3| \le 2l-3$.

Since $|U_3| + |U_4| \ge 4l-6$ and $|U_4| \le 2l-3$, we are now left to consider the case when $|U_3| = |U_4| = 2l-3$. Since $\sum_{i=1}^4 d(u_i) \le 8l+8$, by (6.1), (6.2) and (6.3), we have $|X_2| = 11$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^4 d(u_i) = 8l+8$, which implies $d_{X_2}(u_4) = 6$. Let $N_{X_2}(u_4) = \{y_i \mid 1 \le i \le 6\}$. Since $\overline{G}$ contains no K_5, $G[N_{X_2}(u_4)]$ contains at least one edge, say $y_1y_2 \in E(G)$. Since G has no F_l, $G[\{y_3, y_4, y_5, y_6\}]$ contains no edge. Because $\overline{G}$ has no K_5, we have $|\{y_3, y_4, y_5, y_6\} \cap (N(u_1) \cup N(u_2))| \ge 2$. Assume that $\{y_3, y_4\} \subseteq N(u_1) \cup N(u_2)$. By Claim 1, $d_{U_4}(y_3) \ge 3$ and $d_{U_4}(y_4) \ge 3$, which implies that $d_{X_{14}}(y_3) \ge 2$ and $d_{X_{14}}(y_4) \ge 2$. In this case, there exists $u', u'' \in X_{14}$ such that $u'y_3, u''y_4 \in E(G)$, which implies that $G[U_4 \cup \{y_1, y_2, y_3, y_4\}]$ contains an F_l with u_4 as center, a contradiction.

The proof of Theorem 6.10 is completed. □

Chapter 7

Paths versus kipases

7.1 Introduction

In 1978, Rousseau and Sheehan [116] gave an explicit formula of $R(P_n,K_{1,m})$. We here present how they established the lower bound. Let G denote a graph of the disjoint union of $K_{p_1},K_{p_2},\ldots,K_{p_k}$, that is, $G = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_k}$. Let $p_1 \le p_2 \le \ldots \le p_k \le n-1$, and $p_2 + p_3 + \ldots + p_k \le m-1$. Then we see that G contains no P_n and $\overline{G}$ contains no $K_{1,m}$. Set $N = p_1 + p_2 + \ldots + p_k + 1$. Then $R(P_n,K_{1,m}) \ge N$.

Now we maximise N. If $(k-1)(n-1) \le m-1$, then the maximum value of N is obtained by setting $p_1 = p_2 = \ldots = p_k = n-1$ and k as large as possible. Since $k \le (m-1)/(n-1)+1$, then let $k = \lfloor (m-1)/(n-1) \rfloor + 1$ and the maximum value of N is $(\lfloor (m-1)/(n-1) \rfloor + 1)(n-1)+1$. If $(k-1)(n-1) > m-1$, then the maximum value of N is obtained by setting $p_2 + p_3 + \ldots + p_k = m-1$ and p_1 as large as possible. Since $p_1 \le p_2 \le \lfloor (m-1)/(k-1) \rfloor$, then let $k = \lceil m/(n-1) \rceil + 1$ and the maximum value of N is $m + \lfloor (m-1)/\lceil m/(n-1) \rceil \rfloor$. Thus,

$$R(P_n,K_{1,m}) \ge N = \max\{(\lfloor (m-1)/(n-1) \rfloor + 1)(n-1)+1, m + \lfloor (m-1)/\lceil m/(n-1) \rceil \rfloor\}$$

Very recently, Li and Ning [88] obtained Ramsey numbers $R(P_n,W_m)$ for all m,n. It is easy to check from [88] that $R(P_n,W_m) \le N$ for $m \ge 2n$. Hence, $R(P_n,K_{1,m}) = R(P_n,W_m) = N$ for $m \ge 2n$. Remember that a kipas $\widehat{K}_m = K_1 + P_m$ is a graph of order $m+1$. The term kipas and its notation were adopted from [117]. Kipas is the Malay word for fan; the motivation for the term kipas is that the graph K_1+P_m looks like a hand fan (especially if the path P_m is drawn as part of a circle) but the term fan was already in use for the graphs K_1+nK_2. Since $\widehat{K}_m$ is a subgraph of W_m and contains $K_{1,m}$ as a subgraph, then $R(P_n,\widehat{K}_m)=N$ for $m\geq 2n$.

It is trivial that $R(P_1,\widehat{K}_m)=1$ and $R(P_n,\widehat{K}_1)=n$. Many nontrivial Ramsey numbers of $R(P_n,\widehat{K}_m)$ have been obtained by Salman and Broersma in [117]. Here we completely solve the case by determining all the remaining path-kipas Ramsey numbers. Since $R(P_n,\widehat{K}_m)$ can be easily determined for $m\geq 2n$, in this chapter, we close the gap by proving the following theorem.

Theorem 7.1. $R(P_n,\widehat{K}_m)=\max\{2n-1,\lceil 3m/2\rceil-1,2\lfloor m/2\rfloor+n-2\}$ *for* $m\leq 2n-1$ *and* $m,n\geq 2$.

7.2 Proof of Theorem 7.1

We need Theorems 1.3, 1.15, 1.18 and the following lemmas.

Lemma 7.1 (Dirac [38]). *If G is a connected graph, then G contains a path of order at least* $\min\{2\delta(G)+1,|V(G)|\}$

Lemma 7.2 (Parsons [106]). *For* $n\geq m\geq 2$*,* $R(K_{1,m},P_n)=\max\{2m-1,n\}$.

Lemma 7.3 (Salman and Broersma [117]). $R(P_4,\widehat{K}_6)=8$.

The following lemma is the same as Lemma 2.3, whose proof can be found in Chapter 2. We present it here for convenience.

Lemma 7.4 (Zhang et al. [141]). *Let C be a longest cycle of a graph G and* $v_1,v_2\in V(G)-V(C)$. *Then* $|N_{V(C)}(v_1)\cup N_{V(C)}(v_2)|\leq\lfloor|V(C)|/2\rfloor+1$.

Lemma 7.5. *Let G be a graph with* $|V(G)|\geq 6$ *and* $\delta(G)\geq 2$. *Then G contains two vertex-disjoint paths, one with order three and one with order two.*

Proof. If G is connected, by Lemma 7.1, G contains a path of order at least 5. Let $x_1x_2x_3x_4x_5$ be a path in G. Then G contains two vertex-disjoint paths $x_1x_2x_3$ and x_4x_5. If G is disconnected, then each component of G contains a path of order three. This completes the proof of Lemma 7.5. □

We proceed to prove Theorem 7.1. Let $N = \max\{2n-1, \lceil 3m/2\rceil - 1, 2\lfloor m/2\rfloor + n-2\}$, and let $m \le 2n-1$ and $m,n \ge 2$. It suffices to show that $R(P_n, \widehat{K}_m) = N$.

If $n=2$, then $m \le 2n-1$ and $m,n \ge 2$ imply $m=2$ or $m=3$. It is obvious that $R(P_2, \widehat{K}_m) = m+1$, and one easily checks that $m+1 = N$ for these values of m and n. Next we assume that $n \ge 3$. We first show that $R(P_n, \widehat{K}_m) \ge N$. For this purpose, we note that it is straightforward to check that any of the graphs $G \in \{K_{n-1,n-1}, K_{\lfloor m/2\rfloor, \lceil m/2\rceil -1, \lceil m/2\rceil -1}, K_{n-1, \lfloor m/2\rfloor -1, \lfloor m/2\rfloor -1}\}$ contains no $\widehat{K}_m$, whereas $\overline{G}$ contains no P_n. Thus, $R(P_n, \widehat{K}_m) \ge \max\{2n-1, \lceil 3m/2\rceil - 1, 2\lfloor m/2\rfloor + n - 2\} = N$.

It remains to prove $R(P_n, \widehat{K}_m) \le N$. To the contrary, we assume there exists a graph G of order N such that neither G contains a $\widehat{K}_m$, nor $\overline{G}$ contains a P_n.

We first claim that $\Delta(G) \ge N - \lfloor n/2\rfloor$. To prove this claim, assume to the contrary that $\Delta(G) \le N - \lfloor n/2\rfloor - 1$. Then $\delta(\overline{G}) \ge \lfloor n/2\rfloor$. Let H be a largest component of $\overline{G}$. If $|V(H)| \ge n$, since $\delta(H) \ge \lfloor n/2\rfloor$, by Lemma 7.1, H contains a P_n, a contradiction. Thus, $|V(H)| \le n-1$ and $|V(G)| - |V(H)| \ge N - n + 1$. Since $m \le 2n-1$, we have $n \ge \lfloor m/2\rfloor$. From the definition of N we get that $N - n + 1 \ge n$ and $N - n + 1 \ge 2\lfloor m/2\rfloor - 1$, so $N - n + 1 \ge \max\{2\lfloor m/2\rfloor - 1, n\}$. Since $\overline{G} - V(H)$ contains no P_n, by Lemma 7.2, $G - V(H)$ contains a $K_{1,\lfloor m/2\rfloor}$. If $|V(H)| \ge \lceil m/2\rceil$, since every vertex of $V(H)$ is adjacent to every vertex of $V(G) - V(H)$ in G, then G contains a $\widehat{K}_m$, a contradiction. This implies that $|V(H)| \le \lceil m/2\rceil - 1$. Recall that H is a largest component of $\overline{G}$. Thus $\overline{G}$ contains at least four components; otherwise $|V(\overline{G})| \le 3(\lceil m/2\rceil - 1) < \lceil 3m/2\rceil - 1 \le N$, a contradiction. Let H' be a smallest component of $\overline{G}$. Then $|V(H')| \le N/4$ and $|V(G)| - |V(H')| \ge 3N/4 \ge 3/4(\lceil 3m/2\rceil - 1) \ge 9m/8 - 3/4 \ge m - 3/4$. That is, $|V(G)| - |V(H')| \ge m$. Since every component in $\overline{G} - V(H')$ is of order at most $\lceil m/2\rceil - 1$, then every vertex in $\overline{G} - V(H')$ is of degree at most $\lceil m/2\rceil - 2$. Thus, we have $\delta(G - V(H')) > (|V(G)| - |V(H')|)/2$. By Theorem 1.3, $G - V(H')$ contains a P_m, which together with any vertex of $V(H')$ forms a $\widehat{K}_m$ in G, a contradiction. This proves our claim that $\Delta(G) \ge N - \lfloor n/2\rfloor$.

Let u be a vertex of G with $d(u) = d = \Delta(G)$, let $F = G[N(u)]$ and $Z = V(G) - V(F) - \{u\}$. Then $|V(F)| = d \ge N - \lfloor n/2\rfloor = \max\{n + \lceil n/2\rceil - 1, \lceil 3m/2\rceil - \lfloor n/2\rfloor - 1, 2\lfloor m/2\rfloor + \lceil n/2\rceil - 2\}$. We claim that $R(P_m, P_n) > d$; otherwise $R(P_m, P_n) \le d$, and either F contains a P_m, which together with u forms a $\widehat{K}_m$, a contradiction; or $\overline{F}$ contains a P_n, also a contradiction. If $m \le n$, or if $m = n+1$ and m is even, then by Theorem 1.15, $R(P_m, P_n) = \max\{n + \lfloor m/2\rfloor - 1, m + \lfloor n/2\rfloor - 1\} \le n + \lceil n/2\rceil - 1 \le d$, a contradiction. Therefore, it remains to deal with the cases that $m \ge n+2$, and that $m = n+1$ and m is odd. We first deal with the latter case.

Let $m = n+1$ and m odd. Then n is even, hence $n \ge 4$. We claim that $|Z| \ge 1$. If not, $d = N-1 = 2n-2$. By Theorem 1.15, $R(P_m,P_n) = m+n/2-1 \le 2n-2 = d$, a contradiction. This proves our claim that $|Z| \ge 1$. By Theorem 1.18, $R(C_{m-1},P_n) = m-1+n/2-1 = n+n/2-1 \le d$. Since $\overline{F}$ contains no P_n, then F contains a C_{m-1}. Let $C_{m-1} = x_1x_2\dots x_{m-1}x_1$, $Y = V(F)-V(C_{m-1}) = \{y_1,y_2,\dots,y_k\}$. Then $k \ge n/2-1$. If $e(V(C_{m-1}),Y) \ge 1$, say $x_1y_1 \in E(G)$, then $y_1x_1x_2\dots x_{m-1}$ is a path in G, which together with u forms a $\widehat{K}_m$, a contradiction. Thus, $e(V(C_{m-1}),Y) = 0$. If there is an edge in $\overline{G}[V(C_{m-1})]$, say $x_ix_j \in E(\overline{G})$ $(1 \le i < j \le m-1)$, then $x_ix_jy_1x'_1y_2x'_2\dots y_{n/2-1}x'_{n/2-1}$ with $\{x'_k : 1 \le k \le n/2-1\} \subseteq V(C_{m-1})-\{x_i,x_j\}$ is a path of order n in $\overline{G}$, a contradiction. Thus, $G[V(C_{m-1})]$ is a complete graph. Set $z \in Z$. If $e(\{z\},V(C_{m-1})) \ge 1$ in $\overline{G}$, say $zx_1 \in E(\overline{G})$, then $uzx_1y_1\dots x_{n/2-1}y_{n/2-1}$ is a path of order n in $\overline{G}$, a contradiction. Thus, $e(\{z\},V(C_{m-1})) = 0$ in $\overline{G}$, and G contains a path $ux_1zx_2x_3\dots x_{m-2}$, which together with x_{m-1} forms a $\widehat{K}_m$, another contradiction. This completes the case that $m = n+1$ and m is odd. We proceed with the case that $n+2 \le m \le 2n-1$, and first consider the small values of n.

For $n = 3$ and $m = 5$, or $n = 4$ and $m = 7$, or $n = 5$ and $7 \le m \le 9$, we get that $R(P_m,P_n) = m+\lfloor n/2\rfloor-1 \le \lceil 3m/2\rceil-\lfloor n/2\rfloor-1 \le d$, a contradiction. By Lemma 7.3, $R(P_4,\widehat{K}_6) = 8 = N$. Hence it remains to consider the case that $m \ge n+2 \ge 8$.

We first claim that $|Z| \ge 2$. If not, $|Z| \le 1$ and $d = N-1-|Z| \ge N-2$. By Theorem 1.15, $R(P_m,P_n) = m+\lfloor n/2\rfloor-1$. If $m \ge n+3$, then $m+\lfloor n/2\rfloor-1 \le \lceil 3m/2\rceil-3 \le N-2 \le d$, a contradiction; if $n \ge 7$ or $(n,m) = (6,8)$, then $m+\lfloor n/2\rfloor-1 \le 2\lfloor m/2\rfloor+n-4 \le N-2 \le d$, also a contradiction. Thus, for $m \ge n+2 \ge 8$, we have $|Z| \ge 2$.

Since $m \ge n+2 \ge 8$, by Theorem 1.18, $R(C_{2\lfloor m/2\rfloor-2},P_n) = \max\{2\lfloor m/2\rfloor+\lfloor n/2\rfloor-3, n+\lfloor m/2\rfloor-2\} < 2\lfloor m/2\rfloor+\lceil n/2\rceil-2 \le d$. Since $\overline{F}$ contains no P_n, F contains a $C_{2\lfloor m/2\rfloor-2}$. Let C be a longest cycle in F. Then $|V(C)| \ge m-3$. If $|V(C)| \ge m$, then F contains a P_m, which together with u forms a $\widehat{K}_m$ in G, a contradiction. Thus, $m-3 \le |V(C)| \le m-1$. We complete the proof by distinguishing the three cases that $|V(C)| = m-1$, $|V(C)| = m-2$ or $|V(C)| = m-3$. In each case, let $C = x_1x_2\dots x_{|V(C)|}x_1$ and $Y = V(F)-V(C) = \{y_1,y_2,\dots,y_k\}$.

Case 1. $|C| = m-1$.

We have $k = d-(m-1) \geq \lceil n/2 \rceil - 2$. If $e(V(C),Y) \geq 1$, say $x_1y_1 \in E(G)$, then $y_1x_1x_2\ldots x_{m-1}$ is a path in G, which together with u forms a $\widehat{K}_m$, a contradiction. Thus, $e(V(C),Y) = 0$. Let $z_1, z_2 \in Z$. If $e(\{z_1\},V(C)) \geq 1$ in $\overline{G}$, say $z_1x_1 \in E(\overline{G})$, then $z_2uz_1x_1y_1\ldots x_{\lceil n/2\rceil-2}y_{\lceil n/2\rceil-2}x_{\lceil n/2\rceil-1}$ is a path of order at least n in $\overline{G}$, a contradiction. This implies that $e(\{z_1\},V(C)) = 0$ in $\overline{G}$. For the same reason, $e(\{z_2\},V(C)) = 0$ in $\overline{G}$.

We claim that $\delta(\overline{G}[V(C)]) \leq 1$. If not, $\delta(\overline{G}[V(C)]) \geq 2$. Since $m \geq 8$, by Lemma 7.5, there are two vertex-disjoint paths in $\overline{G}[V(C)]$, one with order three and one with order two. Without loss of generality, let $x'_1x'_2x'_3$ and $x'_4x'_5$ be the two paths in $\overline{G}[V(C)]$. Because $m-1 \geq \lceil n/2\rceil+2$, we may assume that $x'_6,\ldots,x'_{\lceil n/2\rceil+2} \in V(C)-\{x'_1,\ldots,x'_5\}$. Then $x'_1x'_2x'_3y_1x'_4x'_5y_2x'_6y_3\ldots x'_{\lceil n/2\rceil+1}y_{\lceil n/2\rceil-2}x'_{\lceil n/2\rceil+2}$ is a path of order at least n in $\overline{G}$, a contradiction. This proves our claim that $\delta(\overline{G}[V(C)]) \leq 1$. That is, there exists a vertex of $V(C)$ which is adjacent to at least $|V(C)|-2$ vertices of $V(C)$. Without loss of generality, let x_1 be a vertex with maximum degree in $G[V(C)]$, and let x_3 be the possible vertex that is nonadjacent to x_1. Then $ux_2z_1x_4z_2x_5x_6\ldots x_{m-1}$ is a path of order m, which together with x_1 forms a $\widehat{K}_m$ in G, our final contradiction in Case 1.

Case 2. $|C| = m-2$.

We have $k = d-(m-2)$. Note that $k \geq \lceil n/2\rceil - 1$ for odd m, and $k \geq \lceil n/2\rceil$ for even m. Let X be the set of all vertices of $V(C)$ that are nonadjacent to Y in G. For $1 \leq i \leq m-2$, either $x_i \in X$, or $x_{i+1} \in X$. Here, $x_{m-1} = x_1$. This is because, if x_i and x_{i+1} have a common neigbor in Y, say y_1, then by replacing x_ix_{i+1} by $x_iy_1x_{i+1}$ in C, we obtain a cycle longer than C, a contradiction; if x_i and x_{i+1} are adjacent to different vertices of Y, say $x_iy_1, x_{i+1}y_2 \in E(G)$, then $y_2x_{i+1}x_{i+2}\ldots x_{m-2}x_1\ldots x_iy_1$ is a path of length m, which together with u forms a $\widehat{K}_m$ in G, also a contradiction. Thus, at least one end of each edge of C is nonadjacent to Y in G. Note that $|X| \geq \lceil n/2\rceil$ and $|Y| \geq \lceil n/2\rceil - 1$ for odd m and $|Y| \geq \lceil n/2\rceil$ for even m. If m is even or n is odd, then we get a path P_n in $\overline{G}[X \cup Y]$. This implies it remains to consider the case that n is even and m is odd, with $m \geq n+3$.

If $|V(C)-X| \geq 2$, say $x_i, x_j \notin X$, then $x_{i+1}, x_{j+1} \in X$. Moreover, $x_{i+1}x_{j+1} \notin E(G)$; otherwise we may obtain either a cycle longer than C in F, or a path of length m in F, which together with u forms a $\widehat{K}_m$ in G, both of which are contradictions. Now let $x'_1, x'_2, \ldots, x'_{|X|-2} \in X-\{x_{i+1}, x_{j+1}\}$. Since $|X|-2 \geq \lceil |V(C)|/2\rceil - 2 \geq$

$n/2-1$, let $P = x_{i+1}x_{j+1}y_1x'_1y_2x'_2\dots y_{n/2-1}x'_{n/2-1}$. Note that P is a path of order n in $\overline{G}$, a contradiction. Thus, $m-3 \le |X| \le m-2$ and there exists a vertex in $V(C)$, say x_1, such that $e(V(C)-\{x_1\}, Y) = 0$.

Since $m \ge n+3 \ge 9$, we have $m-3 \ge \lceil n/2\rceil + 2$. If there is an edge in $\overline{G}[V(C)-\{x_1\}]$, say $x_ix_j \in E(\overline{G})$, then $\overline{G}[X \cup Y]$ contains a path P_n, a contradiction. Thus, $G[V(C)-\{x_1\}]$ is a complete graph of order $m-3$.

Let $z_1, z_2 \in Z$. We claim that $e(\{z_1\}, V(C)-\{x_1\}) = 0$ in $\overline{G}$. If not, say $z_1x_2 \in E(\overline{G})$, then $z_2uz_1x_2y_2x_3y_3\dots x_{n/2-1}y_{n/2-1}x_{n/2}$ is a path of order n in $\overline{G}$, a contradiction implying our claim. For the same reason, $e(\{z_2\}, V(C)-\{x_1\}) = 0$ in $\overline{G}$.

It is easy to check that $x_1ux_3z_1x_4z_2x_5\dots x_{m-2}$ is a path of order m, which together with x_2 forms a $\widehat{K}_m$ in G, our final contradiction in Case 2.

Case 3. $|C| = m-3$.

If $m = n+2 \ge 8$, then m and n have the same parity. In that case, $R(C_{2\lfloor (m-1)/2\rfloor}, P_n) = 2\lfloor (m-1)/2\rfloor + \lfloor n/2\rfloor - 1 \le 2\lfloor m/2\rfloor + \lceil n/2\rceil - 2 \le d$. Since $\overline{F}$ contains no P_n, F contains a $C_{2\lfloor (m-1)/2\rfloor}$. This contradicts that C with $|V(C)| = m-3$ is a longest cycle in F. It remains to consider the case that $m \ge n+3 \ge 9$.

We have $k = d-(m-3) \ge \lceil n/2\rceil$. By Lemma 7.4, any two vertices of Y have at least $\lceil (m-3)/2\rceil - 1 \ge \lceil n/2\rceil - 1$ common nonadjacent vertices of $V(C)$ in G. Since C is a longest cycle in G, any vertex of Y has at least $\lceil (m-3)/2\rceil \ge \lceil n/2\rceil$ nonadjacent vertices of $V(C)$ in G. By these observations, y_1 and y_2 have a common nonadjacent vertex in $V(C)$, say x_1; for $2 \le i \le \lceil n/2\rceil - 1$, y_i and y_{i+1} have a common nonadjacent vertex in $V(C)-\{x_1, x_2, \dots, x_{i-1}\}$, say x_i; $y_{\lceil n/2\rceil}$ have a nonadjacent vertex in $X-\{x_1, x_2, \dots, x_{\lceil n/2\rceil -1}\}$, say $x_{\lceil n/2\rceil}$. Then $y_1x_1y_2x_2\dots y_{\lceil n/2\rceil}x_{\lceil n/2\rceil}$ is a path of order at least n in $\overline{G}$. This final contradiction completes the proof of Case 3 and of Theorem 7.1. □

Chapter 8

Ramsey goodness for the union of some graphs

8.1 Introduction

Throughout this chapter, we let H be a connected graph of order p, $\chi(G)$ the chromatic number of G and $s(G)$ the chromatic surplus of G, that is, the minimum number of vertices in some color class under all proper vertex colorings by $\chi(G)$ colors. Recall that Burr [17] established a general lower bound for $R(H,G)$ that $R(H,G) \geq (|H|-1)(\chi(G)-1)+s(G)$ for $|H| \geq s(G)$. He also defined H to be G-good if the equality holds. A simple but fundamental result was given by Chvátal [32], who proved that for any m, all trees are K_m-good. That is, $R(T_n,K_m) = (n-1)(m-1)+1$.

For a given graph G, sometimes Burr's lower bound can help us obtain new Ramsey numbers $R(F,G)$ in case when F is connected. Suppose that we have known some G-good graphs, a natural question is how to obtain a new graph F from these known G-good graphs such that $R(F,G) = (|F|-1)(\chi(G)-1)+s(G)$? One way to do this is to consider the case when F is a vertex disjoint union of graphs, some of which are G-good, that is, if F is disconnected with some G-good components, when does the equality still hold? Another way to do this is to consider whether a graph F is G-good if F is the union of two G-good graphs with some vertices in common.

Let $c(F)$ be the order of the largest component of a graph F and $k_i(F)$ the number of components of order i in F. Stahl extended Chvátal's result to an arbitrary forest, which is a disjoint union of trees.

Theorem 8.1 (Stahl [123]). *If F is an arbitrary forest, then*

$$R(F,K_m) = \max_{1\le j\le c(F)} \{(j-1)(m-2) + \sum_{i=j}^{c(F)} ik_i(F)\}$$

For special cases of forests, Baskoro et al. [5] proved an equivalent form of the formula of Stahl. By substituting a graph G for K_m and replacing the corresponding forests with a disjoint union of G-good graphs, Bielak generalized the two results of Stahl [123] and Baskoro et al. [5] respectively. We present it here.

Theorem 8.2 (Bielak [10]). *Let G be a graph with $\chi(G) = m \ge 2$. If F is a graph with G-good components, then*

$$R(F,G) = \max_{1\le j\le c(F)} \{(j-1)(m-2) + \sum_{i=j}^{c(F)} ik_i(F)\} + s(G) - 1$$

Can we construct a similar equality under the condition that not all the components of F are G-good components? This inspires us to establish the following theorem.

Theorem 8.3. *Let G be a graph with $\chi(G) = m \ge 2$, F a graph with G-good components, $f(j) = (j-1)(m-2) + \sum_{i=j}^{c(F)} ik_i(F)$ and $f(j_0) = \max_{1\le j\le c(F)} f(j)$. Let H be a graph with k components $H_1,\ldots,H_k$ such that for $1 \le t \le k$, $|H_t| \ge j_0$ and $R(H_t,G) \le f(j_0) + s(G) - 1 + \sum_{i=1}^{t} |H_i|$. Then $R(F \cup H,G) = f(j_0) + s(G) - 1 + |H|$.*

We will give the proof of Theorem 8.3 in Section 8.3.1. Since the components of H are not necessarily to be G-good components, Theorem 8.3 may be viewed as an extension of Theorem 8.2. Let us take the Ramsey numbers $R(7C_5 \cup 7K_5,K_5)$ and $R(C_n \cup W_n,C_4)$ ($n \ge 6$) for examples. We don't even know the exact value of $R(K_5,K_5)$, but by [97], $R(K_5,K_5) \le 49$; by [75], C_5 is K_5-good; and by Theorem 8.3, we have $R(7C_5 \cup 7K_5,K_5) = 82$. For $R(C_n \cup W_n,C_4)$ when $n \ge 6$, by [142], $R(W_n,C_4) \le n + \lceil \sqrt{n} \rceil + 1$; by [114], C_n is C_4-good; and by Theorem 8.3, we have $R(C_n \cup W_n,C_4) = 2n+2$ for $n \ge 6$.

Let $C_{p,t}$ be a graph on $p+t$ vertices obtained from C_p by joining exactly one vertex of C_p to all vertices of tK_1. Clearly, $C_{p,0}$ is a graph isomorphic to C_p, and $C_{3,t}$ is $K_{1,t+2}+e$. Bielak give the following results on Ramsey numbers of $C_{5,t}$ versus C_5 or $2C_5$.

Theorem 8.4 (Bielak [10]). *$R(C_{5,t}, C_5) = 2t+9$ for $t \geq 0$.*

Theorem 8.5 (Bielak [10]). *$R(C_{5,t}, 2C_5) = 2t+10$ for $t \geq 2$.*

We prove a generalization of the two results respectively and confirm the following theorems.

Theorem 8.6. *$R(C_{p,t}, C_q) = 2(p+t)-1$ for odd q, $p+t \geq q$ and $(p,q,t) \neq (3,3,0)$.*

Note. By using the same proof method of Theorem 8.6, we can show that $R(C_{p,t}, C_q) = 2(p+t)-1$ for odd q, even p and $3(q+2)/4 \leq p+t < q$.

Theorem 8.7. *$R(C_{p,t}, C_q \cup C_r) = 2(p+t)$ for q,r odd, $p \geq q \geq r \geq 5$ and $t \geq (r-1)/2$.*

Note. If we delete $r \geq 5$ and keep other conditions unchanged, Theorem 8.7 also holds. In fact, we can use the same method to prove the theorem when $r = 3$.

A broom $B_{p,t}$ is a tree on $p+t$ vertices obtained by identifying an end of a path P_p with the hub of a star $S_{t+1} = K_{1,t}$, where the hub is the vertex with maximum degree of the star. The definition of broom was given by Erdős et al. [48]. We see that paths and stars are two special cases of brooms, that is, $B_{1,t}$, $B_{2,t}$, $B_{p,1}$ are S_{t+1}, S_{t+2} and P_{p+1}, respectively. We also see that $B_{p,t}$ is a subgraph of $C_{p,t}$, then Theorems 8.6 and 8.7 also hold if $C_{p,t}$ is replaced by $B_{p,t}$. We need only to check that $2K_{p+t-1}$ and $2K_{p+t-1} \cup K_1$ are two Ramsey graphs for the two theorems respectively. By Theorem 8.6, we may obtain the Ramsey numbers of brooms versus odd cycles, but the broom's order should be no less than the order of the odd cycle. Can we confirm their Ramsey numbers without this condition?

Since there are complete solutions for Ramsey numbers of paths versus cycles, and stars versus odd cycles (the Ramsey numbers of stars versus even cycles have not yet been solved), we now give a complete solution for Ramsey numbers of brooms versus odd cycles.

Theorem 8.8 (Faudree et al. [54]). *$R(P_p, C_q) = \max\{2p-1, q-1+\lfloor p/2 \rfloor\}$ for q odd.*

Theorem 8.9 (Lawrence [87]). *$R(S_p, C_q) = \max\{2p-1, q\}$ for q odd.*

Theorem 8.10. *$R(B_{p,t}, C_q) = \max\{2(p+t)-1, q+\lfloor(p-1)/2\rfloor\}$ for q odd and $p, t \geq 1$.*

For Ramsey numbers of paths or stars versus wheels of even order, four results were obtained as follows. We generalize these results by substituting brooms for paths and stars.

Theorem 8.11 (Chen et al. [30]). *$R(P_p, W_n) = 3p-2$ for n odd and $p \geq n-1$.*

Theorem 8.12 (Zhang [143]). *$R(P_p, W_n) = 3p-2$ for n odd and $n-2 \geq p \geq (n+1)/2$.*

Theorem 8.13 (Chen et al. [28]). *$R(S_p, W_n) = 3p-2$ for n odd and $p \geq n-1$.*

Theorem 8.14 (Hasmawati et al. [74]). *$R(S_p, W_n) = 3p-2$ for n odd and $n-2 \geq p \geq (n+1)/2$.*

Theorem 8.15. *$R(B_{p,t}, W_n) = 3(p+t)-2$ for n odd and $p+t \geq (n+1)/2$.*

8.2 Preliminary Lemmas

In order to prove Theorems 8.3, 8.6, 8.7, 8.10 and 8.15, we need the following lemmas and Theorems 1.1, 1.4, 1.5, 1.17 from Chapter 1, Theorem 2.1 from Chapter 2.

Lemma 8.1 (Bielak [10]). *Let F be a G-good graph and $\chi(G) \geq 2$, then $|F| \geq s(G)+1$.*

Lemma 8.2 (Chvátal [32]). *If G is a graph with $\delta(G) \geq m-1$, then G contains every tree of order m.*

Lemma 8.3 (Chvátal and Harary [36]). $R(C_{p,t}, C_3) = 7$ *for* $(p,t) = (3,1), (4,0)$.

Lemma 8.4 (Clancy [37]). $R(C_{p,t}, C_3) = 9$ *for* $(p,t) = (3,2), (4,1), (5,0)$.

Lemma 8.5 (Dirac [38]). *Let G be a connected graph, then* $p(G) \geq \min\{2\delta(G)+1, |G|\}$.

Lemma 8.6. *Let T_m be a tree on m vertices and n is odd. If $R(T_m, C_n) = 2m-1$, then $R(T_m, W_n) = 3m-2$.*

Proof. If $\delta(G) \geq m-1$, by Lemma 8.2, G contains a T_m and the proof is done. If $\delta(G) \leq m-2$, let $d(v) = \delta(G)$ and $H = G - N[v]$, then $|H| \geq 2m-1$. Hence, either H contains a T_m, or $\overline{H}$ contains a C_n, which together with v forms a W_n in $\overline{G}$. Thus, $R(T_m, W_n) = 3m-2$. □

Note. It was proved by Burr et al. [21] that $R(T_m, C_n) = 2m-1$ for odd n and $m \geq 756n^{10}$. Thus, $R(T_m, W_n) = 3m-2$ for odd n and $m \geq 756n^{10}$.

Lemma 8.7. *Let X and Y be two disjoint vertex subsets of G such that $d_Y(x) \geq m$ for all $x \in X$. Then for any $1 \leq k \leq \min\{|X|-1, 2m-|Y|\}$ and any two vertices x_1, x_2 of X, there is a path of length $2k$ with x_1, x_2 as its ends.*

Proof. Set $X = \{x_1, x_2, \ldots, x_s\}$, then $s \geq k+1$. Since $d_Y(x) \geq m$, any two vertices of X have at least $2m - |Y| \geq k$ common adjacent vertices. Then x_2 and x_3 have a common adjacent vertex in Y, say, y_1; for $3 \leq i \leq k$, x_i and x_{i+1} have a common adjacent vertex in $Y - \{y_1, y_2, \ldots, y_{i-2}\}$, say, y_{i-1}; x_{k+1} and x_1 have a common adjacent vertex in $Y - \{y_1, y_2, \ldots, y_{k-1}\}$, say, y_k. Thus, $x_2 y_1 x_3 y_2 \ldots x_{k+1} y_k x_1$ is a path of length $2k$ with x_1, x_2 as its ends and our proof is done. □

Lemma 8.8. *Let (X,Y) be a partition of $V(G)$ and suppose that $d_Y(x) \leq t-s-1$ for any vertex $x \in X$ in $\overline{G}$, where $|X| = p+s$, $|Y| = p+2t-s$, and $t-s \geq 1$. If there are two disjoint paths $x_1 y_1 y_2 x_2$ and $x_3 y_3 y_4 x_4$ in G such that $x_i \in X$ and $y_i \in Y$ for $1 \leq i \leq 4$, then for $p \geq q \geq r \geq 5$ and q, r odd, G contains a $C_q \cup C_r$.*

Proof. Set $X_1 = X - \{x_3, x_4\}$, $Y_1 = Y - \{y_1, y_2, y_3, y_4\}$, then $|X_1| = p + s - 2$, $|Y_1| = p + 2t - s - 4$, and $d_{Y_1}(x) \geq p + t - 3$ for any vertex $x \in X_1$. Since $1 \leq (r-3)/2 \leq \min\{|X_1| - 1, 2(p+t-3) - |Y_1|\}$, by Lemma 8.7, there is a path of length $r - 3$ in $G[X_1 \cup Y_1]$ with x_1, x_2 as its ends, which together with $x_1y_1y_2x_2$ forms a C_r. Let $X_2 = X - V(C_r)$ and $Y_2 = Y_1 - V(C_r)$. Since there are exactly $(r-1)/2$ vertices of $V(C_r)$ in X and $(r-3)/2$ vertices of $V(C_r)$ in Y_1, we have $|X_2| = p + s - (r-1)/2$, $|Y_2| = p + 2t - s - 4 - (r-3)/2$, and $d_{Y_2}(x) \geq p + t - 3 - (r-3)/2$ for any vertex $x \in X_2$. Since $1 \leq (q-3)/2 \leq \min\{|X_2| - 1, 2(p+t-3) - (r-3) - |Y_2|\}$, by Lemma 8.7, there is a path of length $q - 3$ in $G[X_2 \cup Y_2]$ with x_3, x_4 as its ends, which together with $x_3y_3y_4x_4$ forms a C_q. Thus, G contains a $C_q \cup C_r$. □

Lemma 8.9. *Let (X, Y) be a partition of $V(G)$ and suppose that $d_Y(x) \leq t - s - 1$ for any vertex $x \in X$ in $\overline{G}$, where $|X| = p + s$, $|Y| = p + 2t - s$, and $t - s \geq 2$. If $G[Y]$ contains a $(t-s)K_2$, then for $p \geq q \geq r \geq 5$ and q, r odd, either G contains a $C_q \cup C_r$, or $\overline{G}$ contains a $C_{p,t}$.*

Proof. Let $H = (t-s)K_2$ be a subgraph of $G[Y]$, then $d_H(x) \geq t - s + 1$ for any $x \in X$. For any two vertices $x_1, x_2 \in X$, $d_H(x_1) + d_H(x_2) \geq 2t - 2s + 2$. By the Pigeonhole Principle, there exists one edge y_1y_2 in H such that $e(\{x_1, x_2\}, \{y_1, y_2\}) \geq 3$. Thus, there is a matching covering y_1, y_2, say, $x_1y_1, x_2y_2 \in E(G)$. Let $X' = X - \{x_1, x_2\}$, $H' = H - \{y_1, y_2\}$, then $|X'| = p + s - 2 \geq 3$, $|H'| = 2(t - s - 1) \geq 2$ and $d_{H'}(x) \geq t - s - 1$ for any $x \in X'$. If there exists one edge y_3y_4 in H' such that $e(X', \{y_3, y_4\}) \geq |X'| + 1$, then there is a matching covering y_3, y_4, say, $x_3y_3, x_4y_4 \in E(G)$. Consequently, $x_1y_1y_2x_2$, $x_3y_3y_4x_4$ are two disjoint paths and by Lemma 8.8, our proof is done. Thus, for any edge y_iy_j in H', $e(X', \{y_i, y_j\}) \leq |X'|$. Since $e(X', H') \geq (|X'||H'|)/2$, then for any edge y_iy_j in H', $e(X', \{y_i, y_j\}) = |X'|$. Moreover, if $d_{X'}(y_i) \geq 1$ and $d_{X'}(y_j) \geq 1$, then there is a matching covering y_i, y_j, and our proof is done by Lemma 8.8. Thus, for any edge y_iy_j of H', assume that $d_{X'}(y_i) \geq d_{X'}(y_j)$, then $d_{X'}(y_i) = |X'|$ and $d_{X'}(y_j) = 0$.

Let Z be the set of $t - s - 1$ ends of H which have no adjacent vertex in X'. Since $d_Y(x) \leq t - s - 1$ for any vertex $x \in X$ in $\overline{G}$, then $E(X', Y - Z) = |X'||Y - Z|$. Moreover, both x_1 and x_2 are nonadjacent to Z, otherwise, say, $x_1y_i \in E(G)$. Since there exists a y_j such that $y_iy_j \in E(H)$, then for $x_3, x_4 \in X'$, $x_3y_1y_2x_2$, $x_1y_iy_jx_4$ are two disjoint paths and our proof is done by Lemma 8.8. Therefore, every vertex of X is adjacent to every vertex of $Y - Z$. In this way, $G[Y - Z]$ contains no two independent edges, otherwise our Lemma is solved by Lemma 8.8. Then, the edges in $G[Y - Z]$ form a star or a triangle and then there exists a vertex z such that $\overline{G}[Y - Z - z]$ contains a complete graph with at most one edge deleted. Since $|Y-Z| \geq p+t+1$, then $\overline{G}$ contains a $C_{p,t}$ and our proof is done. □

Lemma 8.10. *Let H be a subgraph of $V(G)$, P a path contained in H, and $F = G - V(H)$. If every vertex of H is adjacent to every vertex of F, $|H| \geq |F| \geq 1$, $|H| - |P| \leq |F| - 1$, and $|P| \geq 2$, then G is pancyclic.*

Proof. Let $P = x_1x_2 \ldots x_l$, $V(H) - V(P) = \{y_1, y_2, \ldots, y_m\}$ and $V(F) = \{z_1, z_2, \ldots, z_n\}$, then $m + l \geq n \geq 1$, $m \leq n-1$ and $l \geq 2$. The following sequence is a cycle:
$C = y_1z_1y_2z_2 \ldots y_{k_1}z_{k_1}x_1z_{m+1}x_2z_{m+2} \ldots x_{k_2}z_{m+k_2}x_{n-m} \ldots x_{n-m+k_3}z_ny_1$
when $0 \leq k_1 \leq m$, $0 \leq k_2 \leq n-m-1$, $0 \leq k_3 \leq m+l-n$ and $k_1+k_2+k_3 \neq 0$. It is easy to check that $|C| = 2k_1 + 2k_2 + k_3 + 2$. If $m+l-n \geq 1$, then $|C|$ can be every number between 3 and $|G|$ by choosing proper k_1, k_2, k_3. Thus, G is pancyclic. If $m+l-n = 0$, then $k_3 = 0$ and then $|C|$ can be every even number between 4 and $|G|$ by choosing proper k_1, k_2. Since $l \geq 2$ and $m+l-n = 0$, we have $n-m \geq 2$. Let $0 \leq k_2 \leq n-m-2$ and keep other conditions unchanged, and insert x_{n-m-1} in the sequence of C between z_{m+k_2} and x_{n-m}. This new sequence also forms a new cycle whose length can be every odd number between 3 and $|G|$ by choosing proper k_1, k_2. Thus, G is pancyclic. □

8.3 Main results

8.3.1 Proof of Theorem 8.3

For $l = \sum_{i=j_0}^{c(F)} ik_i(F) + |H|$, if $(m-2)K_{j_0-1} \cup K_{l-1} \cup K_{s(G)-1}$ contains $F \cup H$, by Lemma 8.1, all the components of F with at least j_0 vertices should be contained in K_{l-1}. Since $|H_t| \geq j_0$ for $1 \leq t \leq k$, H should also be contained in K_{l-1}, but $l - 1 < \sum_{i=j_0}^{c(F)} ik_i(F) + |H|$, a contradiction. Thus, $(m-2)K_{j_0-1} \cup K_{l-1} \cup K_{s(G)-1}$ contains no $F \cup H$. It is not difficult to check that its complement contains no G. Let $N = f(j_0) + s(G) - 1 + |H|$, in the following, we need only to prove that $R(F \cup H, G) \leq N$. Let G' be a graph of order N and suppose to the contrary that neither G' contains $F \cup H$ nor $\overline{G'}$ contains G. Since $R(H_k, G) \leq N$, then G' contains an H_k. For $1 \leq t \leq k-1$, since $R(H_t, G) \leq N - \sum_{i=t+1}^{k} |H_i|$, then $G' - \cup_{i=t+1}^{k} V(H_i)$ contains an H_t. Thus, G' contains H. By Theorem 8.2, $R(F, G) = N - |H|$, then $G' - V(H)$ contains an F. Thus, G' contains $F \cup H$, a contradiction which implies that $R(F \cup H, G) \leq N$. This completes the proof of Theorem 8.3. □

8.3.2 Proof of Theorem 8.6

Because neither $2K_{p+t-1}$ contains a $C_{p,t}$ nor its complement contains a C_q for odd q, we have $R(C_{p,t},C_q) \geq 2(p+t)-1$. In the following, we need only to show that $R(C_{p,t},C_q) \leq 2(p+t)-1$. For $q=3$, if $p+t=3$, then $(p,q,t)=(3,3,0)$, which is excluded here. If $p+t=4$ or 5, our theorem holds by Lemmas 8.3 and 8.4 respectively. If $p+t \geq 6$, since $C_{p,t}$ is a subgraph of W_{p+t-1}, by the monotonicity of Ramsey numbers, $R(C_{p,t},C_q) \leq R(W_{p+t-1},C_q)$. By Theorem 2.1, we have $R(C_{p,t},C_3) \leq 2(p+t)-1$, our theorem holds again. If $t=0$, our theorem is derived by Theorem 1.17. Thus, we need only to consider the theorem for odd $q \geq 5$ and $t \geq 1$. Let G be a graph of order $2(p+t)-1$ with odd $q \geq 5$, $t \geq 1$ and $p+t \geq q$. Suppose to the contrary that G contains no $C_{p,t}$ and its complement contains no C_q. We distinguish two cases.

First case, $p \geq t+1$. By Theorem 1.17, $R(C_p,C_q) \leq 2(p+t)-1$, then G contains a C_p. Set $X=V(C_p)$ and $Y=V(G)-X$, then $|Y|=p+2t-1$. If $d_Y(x) \geq t$ for some $x \in X$, then G contains a $C_{p,t}$ which contradicts our assumption. Thus, for any $x \in X$, $d_Y(x) \geq p+t$ in $\overline{G}$. If there is an edge $x_ix_j \in E(\overline{G}[X])$, by Lemma 8.7, $\overline{G}$ contains a path of length $q-1$ joining x_i and x_j, which together with x_ix_j forms a C_q in $\overline{G}$, a contradiction. Thus, $\overline{G}[X]$ is a complete graph. By Theorem 8.1, $R(tK_2,K_{p+1})=p+2t-1$, then either $\overline{G}[Y]$ contains a tK_2, or $G[Y]$ contains a K_{p+1}. Now we consider the following two subcases.

If $\overline{G}[Y]$ contains a tK_2, we set $y_1y_2 \in E(\overline{G}[Y])$ and $Y_1=Y-\{y_1,y_2\}$. Because for any $x \in X$, $d_Y(x) \geq p+t$ in $\overline{G}$, then $d_{Y_1}(x) \geq p+t-2$. By Lemma 8.7, $\overline{G}[X \cup Y_1]$ contains a path of length $q-3$ joining any two vertices of X. If $d_X(y_i) \geq 2$ in $\overline{G}$ for $i=1,2$, then there exist two vertices $x_i,x_j \in X$ such that $x_iy_1y_2x_j$ is a path of length 3. Hence, we may obtain a C_q in $\overline{G}$ which is a contradiction. Thus, for any edge $y_1y_2 \in E(\overline{G}[Y])$, either $d_X(y_1) \leq 1$ or $d_X(y_2) \leq 1$ in $\overline{G}$. Since $\overline{G}[Y]$ contains a tK_2, there exist t vertices, each of which has at most one adjacent vertex from X in $\overline{G}$. Since $|X|=p \geq t+1$, there exists a vertex x in X such that $d_Y(x) \geq t$ in G. Thus, G contains a $C_{p,t}$, a contradiction.

If $G[Y]$ contains a $Y_2=K_{p+1}$, since G contains no $C_{p,t}$ and $\overline{G}[X]$ is a complete graph, it is easy to check that $d_{Y_2}(x) \leq 1$ for any $x \in X$, $d_X(y) \leq 1$ for any $y \in Y_2$, and there are no two independent edges joining X and Y_2. That is, $e(X,Y_2) \leq 1$. For $y \in Y-Y_2$, if it has at least one nonadjacent vertex both in X and Y_2, then $\overline{G}$ contains a C_q, a contradiction. Thus, for any vertex $y \in Y-Y_2$, either it is adjacent to all vertices of X, then let $y \in Z_1$; or it is adjacent to all vertices of Y_2, then let $y \in Z_2$. Since $\max\{|X \cup Z_1|, |Y_2 \cup Z_2|\} \geq p+t$, then G contains a $C_{p,t}$, a contradiction.

Second case, $p \leq t$. We see that $\overline{G}$ is nonbipartite, otherwise $\alpha(\overline{G}) \geq p+t$, which implies that G contains a $C_{p,t}$, a contradiction. If $\delta(\overline{G}) \geq (2p+2t+1)/3$, by Theorem 1.5, $\overline{G}$ is weakly pancyclic with girth 3 or 4. If $\kappa(\overline{G}) \geq 2$, by Theorem 1.1, $c(\overline{G}) \geq 2\delta(\overline{G}) \geq q$. Thus, $\overline{G}$ contains a C_q, a contradiction. If $\kappa(\overline{G}) \leq 1$, then for some $u \in V(G)$, $G-u$ contains a complete bipartite subgraph with two parts X and Y. Assume that $|X| \geq |Y|$, and if equality holds, assume $d_X(u) \leq d_Y(u)$. Then $|X| \geq p+t-1$, $|Y| \geq \delta(\overline{G}) \geq (2p+2t+1)/3$. If $|X| \geq p+t$, there exists one edge x_1x_2 in $G[X]$, since otherwise $\overline{G}[X]$ is complete, which implies that $\overline{G}[X]$ contains a C_q, a contradiction. Choose any $\lceil p/2 \rceil$ vertices from X including x_1, x_2, and any $\lfloor p/2 \rfloor$ vertices from Y, then the graph induced by these p vertices contains a C_p. Since any vertex of Y has degree at least $p+t$, then G contains a $C_{p,t}$, a contradiction. If $|X| = p+t-1$, then $|Y| = p+t-1$, it is easy to check that G contains a $C_{p,t}$ or $\overline{G}$ contains a C_q by considering $d_X(u) \geq |X|-1$ or not. Thus, $\delta(\overline{G}) \leq 2(p+t)/3$ and then $\Delta(G) \geq 4(p+t)/3-2$.

Let $v \in V(G)$ with $d = d(v) = \Delta(G) \geq 4(p+t)/3-2$. Let H be a subgraph of G induced by $N(v)$ and any m vertices of $V(G)-N[v]$, where $m = \min\{\lceil (p-3)/2 \rceil, |G|-d-1\}$. By Theorem 1.17, $R(C_{2\lfloor p/2 \rfloor}, C_q) \leq n$, where $n = \max\{q-1+\lfloor p/2 \rfloor, 4\lfloor p/2 \rfloor - 1\}$. Since $t \geq p \geq 3$ and $p+t \geq q$, it is easy to check that $n \leq |H|$. Since $\overline{H}$ contains no C_q, then H contains a $C_{2\lfloor p/2 \rfloor}$ and set $C = C_{2\lfloor p/2 \rfloor} = v_1v_2 \ldots v_{2\lfloor p/2 \rfloor}v_1$. Since H contains no more than $\lceil (p-3)/2 \rceil$ vertices that are nonadjacent to v, then C contains no more than $\lceil (p-3)/2 \rceil$ vertices that are nonadjacent to v. Thus, there exists some i such that $vv_i, vv_{i+1} \in E(G)$; and some j such that $vv_j, vv_{j+2} \in E(G)$, here subscripts are taken modulo $2\lfloor p/2 \rfloor$. If p is odd, then $v_ivv_{i+1}\overrightarrow{C}v_i$ is a C_p containing v; if p is even, then $v_jvv_{j+2}\overrightarrow{C}v_j$ is a C_p containing v. Since $d(v) \geq 4(p+t)/3-2 \geq p+t$, then G contains a $C_{p,t}$, a final contradiction.

The proof of Theorem 8.6 is complete. □

8.3.3 Proof of Theorem 8.7

Since $K_{p+t-1} \cup K_{p+t-1} \cup K_1$ contains no $C_{p,t}$ and its complement contains no $C_q \cup C_r$ for q, r odd, we have $R(C_{p,t}, C_q \cup C_r) \geq 2(p+t)$. In the following, we need only to prove that $R(C_{p,t}, C_q \cup C_r) \le 2(p+t)$. Let G be a graph of order $2(p+t)$ with q, r odd, $p \ge q \ge r \ge 5$ and $t \ge (r-1)/2$. Suppose by contradiction that G contains no $C_{p,t}$ and its complement contains no $C_q \cup C_r$.

We claim that G contains a K_{p+2}. By Theorem 8.6, $R(C_{p,t}, C_r) = 2(p+t)-1$, then $\overline{G}$ contains a C_r. We use H to denote $G - V(C_r)$, then $|H| = 2(p+t) - r \ge 2p-1$. Since $p \ge q$, by Theorem 1.17, $R(C_p, C_q) = 2p-1$, then H contains a C_p, otherwise $\overline{H}$ contains a C_q which implies that $\overline{G}$ contains a $C_q \cup C_r$, a contradiction. Set $X = V(C_p)$ and $Y = V(G) - X$, then $|Y| = p + 2t$. For any $x \in X$, $d_Y(x) \le t-1$, since otherwise G contains a $C_{p,t}$ which contradicts our assumption. By Lemma 8.9, $\overline{G}[Y]$ contains no tK_2. By Theorem 8.1, $R(K_{p+2}, tK_2) = p + 2t$, then $G[Y]$ contains a K_{p+2}. This proves our claim. Furthermore, if $t = 2$, then G contains a $C_{p,t}$, a contradiction. Thus, $t \ge 3$.

We claim that G contains a K_{p+t-1}. If $t = 3$, the claim is true and then $t \ge 4$. Suppose to the contrary that G contains no K_{p+t-1}. Set H be the largest complete subgraph of G, $Y' = V(G) - V(H)$ and $|H| = p + s$, then $2 \le s \le t-2$, $|Y'| = p + 2t - s$. Moreover, for any $x \in H$, $d_{Y'}(x) \le t - s - 1$, since otherwise G contains a $C_{p,t}$, a contradiction. By Lemma 8.9, $\overline{G}[Y]$ contains no $(t-s)K_2$. By Theorem 8.1, $R(K_{p+s+2}, (t-s)K_2) = p + 2t - s$, then $G[Y]$ contains a K_{p+s+2}, which contradicts the choice of H. This proves our claim that G contains a K_{p+t-1}.

Again, let H be the largest complete subgraph of G and $Y' = V(G) - V(H)$, then $|H| = p + t - 1$ and $|Y'| = p + t + 1$, since otherwise $|H| \ge p + t$ and then G contains a $C_{p,t}$, a contradiction. For the same reason, there are no edges joining H and Y'. If there are two independent edges in $\overline{G}[Y']$, by Lemma 8.8, $\overline{G}$ contains a $C_q \cup C_r$, a contradiction. Then, the edges in $\overline{G}[Y']$ form a star or a triangle and then there exists a vertex y such that $G[Y'-y]$ contains a complete graph with at most one edge deleted. Since $|Y'| = p + t + 1$, then G contains a $C_{p,t}$, a final contradiction.

The proof of Theorem 8.7 is complete. □

8.3.4 Proof of Theorem 8.10

Since $2K_{p+t-1}$ contains no $B_{p,t}$ and its complement contains no C_q for q odd, then $R(B_{p,t}, C_q) \ge 2(p+t)-1$. If $q \le \lfloor (p-1)/2 \rfloor$, then $R(B_{p,t}, C_q) > p - 1 \ge q + \lfloor (p-$

$1)/2\rfloor$. If $q > \lfloor(p-1)/2\rfloor$, then $K_{q-1} \cup K_{\lfloor(p-1)/2\rfloor}$ contains no C_q and its complement contains no $B_{p,t}$. This is because $B_{p,t}$ is a bipartite connected graph with one part $\lfloor(p-1)/2\rfloor+1$ vertices and the other part $\lfloor p/2\rfloor+t$ vertices and $t \geq 1$. Thus, $R(B_{p,t}, C_q) \geq q + \lfloor(p-1)/2\rfloor$. Let $N = \max\{2(p+t)-1, q+\lfloor(p-1)/2\rfloor\}$, then we need only to prove that $R(B_{p,t}, C_q) \leq N$. If $p = 1$ or 2, by Theorem 8.9, our theorem holds; if $t = 1$, by Theorem 1.18, our theorem holds. Hence, we may assume that $p \geq 3$ and $t \geq 2$. Let G be a graph of order N with q odd, $p \geq 3$ and $t \geq 2$. Assume, to the contrary, that G contains no $B_{p,t}$ and its complement contains no C_q.

By Theorem 8.9, $R(S_{\lfloor(N+1)/2\rfloor}, C_q) = \max\{2\lfloor(N+1)/2\rfloor - 1, q\} \leq N$, then G contains $S_{\lfloor(N+1)/2\rfloor}$ as a subgraph. Let $s = 2\lfloor(p+1)/2\rfloor$, then $s \geq p$. By Theorem 1.17, $R(C_s, C_q) = \max\{q-1+s/2, 2s-1\} \leq N$, then G contains C_s as a subgraph. Set u be the hub of $S_{\lfloor(N+1)/2\rfloor}$ and $C_s = v_1v_2\ldots v_sv_1$. If there is a path from u to C_s, without loss of generality, assume that $uP^{(1)}v_1$ is the shortest path from u to C_s, where $uP^{(1)}v_1 = u = v_1$ if u is just on the cycle. Then $P^{(2)} = uP^{(1)}v_1v_2\ldots v_s$ is a path of order at least p. We choose a path $P^{(3)}$ of order p such that $P^{(3)}$ is a segment of $P^{(2)}$ and $P^{(3)}$ has u as one of its ends. Since $d(u) = \lfloor(N+1)/2\rfloor - 1 \geq p+t-1$ and $|P^{(3)} - u| = p-1$, then u has at least t adjacent vertices in $V(G) - V(P^{(3)})$. Thus, G contains a $B_{p,t}$, a contradiction which implies that there is no path connecting u and the cycle. Let H be the connected component of G containing the $S_{\lfloor(N+1)/2\rfloor}$ and $F = G - V(H)$. Then F contains the C_s as a subgraph.

If H contains a cycle of length s, then we may get a contradiction that G contains a $B_{p,t}$ by using the same method as above. Thus, H contains no C_s. Let $h = |H|+1-p$ if $|H|+p$ is odd, and $h = |H|+2-p$ if $|H|+p$ is even. By Theorem 1.17, $R(C_s, C_h) = \max\{h+s/2-1, s+h/2-1\}$. Since $p \geq 3$, $t \geq 2$, $s = 2\lfloor(p+1)/2\rfloor$ and $|H| \geq \lfloor(N+1)/2\rfloor \geq p+t$, it is not difficult to check that $h+s/2-1 \leq |H|$ and $s+h/2-1 \leq |H|$. Thus, $\overline{H}$ contains a cycle of length h, where $h = |H|+1-p$ or $|H|+2-p$. Since $|F| \geq p$, by Lemma 8.10, $\overline{G}$ is pancyclic. Therefore, $\overline{G}$ contains a C_q, a final contradiction.

The proof of Theorem 8.10 is done. □

8.3.5 Proof of Theorem 8.15

We need only to consider the theorem for $p \geq 3$ and $t \geq 2$, since otherwise our theorem holds by Theorems 8.11, 8.12, 8.13, 8.14. By Theorem 8.10, $R(B_{p,t}, C_n) =$

$\max\{2(p+t)-1, n+\lfloor (p-1)/2 \rfloor\}$. If $R(B_{p,t}, C_n) = 2(p+t)-1$, by Lemma 8.6, our theorem holds. Thus, we need only to prove our theorem under the condition that $R(B_{p,t}, C_n) = n + \lfloor (p-1)/2 \rfloor$.

Let G be a graph of order $3(p+t)-2$. Suppose to the contrary that G contains no $B_{p,t}$ and its complement contains no W_n. If $\delta(G) \le p+t-2-\lfloor (p-1)/2 \rfloor$, let $d(v) = \delta(G)$ and $H = G - N[v]$. Since $\Delta(\overline{G}) \ge 2(p+t)-1+\lfloor (p-1)/2 \rfloor \ge n + \lfloor (p-1)/2 \rfloor$ and $R(B_{p,t}, C_n) = n + \lfloor (p-1)/2 \rfloor$, then either H contains a $B_{p,t}$ which is a contradiction, or $\overline{H}$ contains a C_n which together with v forms a W_n, also a contradiction. Thus, $\delta(G) \ge p+t-1-\lfloor (p-1)/2 \rfloor = \lceil (p-1)/2 \rceil + t$.

Since $\overline{G}$ contains no W_n, by Theorems 8.13 and 8.14, G contains a S_{p+t}. Let F be the connected component of G containing S_{p+t}. Since $\delta(F) \ge \lceil (p-1)/2 \rceil + t$, by Lemma 8.5, F contains a path of order p. Choose a path P of order p in F with two ends u, w. If $d_{V(F)-V(P)}(u) \ge t$, then F contains a $B_{p,t}$, a contradiction. Thus, $d_{V(F)-V(P)}(u) \le t-1$. For the same reason, $d_{V(F)-V(P)}(w) \le t-1$. Since $\delta(F) \ge \lceil (p-1)/2 \rceil + t$, then both u and w have at least $\lceil (p+1)/2 \rceil$ adjacent vertices in P. If $uw \in E(F)$, then $G[V(P)]$ is hamiltonian. If $uw \notin E(F)$, by Theorem 1.4, $G[V(P)]$ is hamiltonian. Thus, F contains a C_p.

Set x be the hub of S_{p+t} and $C_p = x_1x_2\ldots x_px_1$. Since F is connected, there is a path from x to C_p, without loss of generality, assume that $xP^{(1)}x_1$ is the shortest path from x to C_p, where $xP^{(1)}x_1 = x = x_1$ if x is just on the cycle. Then $P^{(2)} = xP^{(1)}x_1x_2\ldots x_p$ is a path of order at least p. We choose a path $P^{(3)}$ of order p such that $P^{(3)}$ is a segment of $P^{(2)}$ and $P^{(3)}$ has x as one of its ends. Since $d(x) \ge p+t-1$ and $|P^{(3)} - x| = p-1$, then x has at least t adjacent vertices in $V(G) - V(P^{(3)})$. Thus, G contains a $B_{p,t}$, a final contradiction.

The proof of Theorem 8.15 is done. □

Chapter 9

On planar Ramsey numbers

9.1 Introduction

The computation of classical Ramsey numbers is extraordinarily difficult and even the exact value of $R(K_6, K_6)$ may never be known. The study of generalized Ramsey numbers for sparse graphs seems a little easier, but most of them are still unknown, especially when both F and H have high edge density. For this reason, researchers created a number of variants of Ramsey numbers to take refuge from the formidable difficulties of Ramsey numbers. One variant is to restrict the set of considered graphs. For a graph class $\mathscr{G}$ and two given graphs F, H, the restricted Ramsey number $R_{\mathscr{G}}(F, H)$ is the smallest integer N such that for any graph G in $\mathscr{G}$ of order N, either G contains F or $\overline{G}$ contains H.

If both F and H are complete graphs, then $R_{\mathscr{G}}(K_i, K_j)$ is the same as the definition $R_{\mathscr{G}}(i, j)$, which is due to Belmonte et al. [8]. Belmonte et al. [8] gave an introduction to the progresses of $R_{\mathscr{G}}(i, j)$ when $\mathscr{G}$ is the class of planar graphs, graphs with small maximum degree, and claw-free graphs. Furthermore, they have obtained some new results when $\mathscr{G}$ is the class of perfect graphs and subclasses of claw-free graphs.

If at least one of F, H is noncomplete, many results have been obtained when $\mathscr{G}$ is the class of complete bipartite graphs $K_{s,s}$ (Note that in the literature, sometimes the bipartite Ramsey number refers to s rather than the order $2s$). Another graph class which has been studied is the class of co-connected graphs. A graph G is co-connected if both G and its complement $\overline{G}$ are connected and nontrivial. Some results of connected Ramsey numbers have been obtained in [125], [59] and [71]. This chapter centers on planar Ramsey numbers, which are the Ramsey numbers when $\mathscr{G}$ is the class of planar graphs.

For two given graphs G_1 and G_2, the planar Ramsey number $R(G_1,G_2)$ is the smallest integer N such that for any planar graph G of order N, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G.

In comparison to usual Ramsey number $R(G_1,G_2)$, the planar Ramsey number requires the ground set restricted to planar graphs and so it has no symmetry in general, that is, $PR(G_1,G_2) \neq PR(G_2,G_1)$ in most cases. Since the hardness of some problems in graph theory may become lower when restricted to the plane, one may expect that $PR(G_1,G_2)$ are tractable. This indeed happens when we consider the planar Ramsey numbers for a complete graph versus a complete graph. It is well known that the classical Ramsey numbers are far from being known. However, the planar Ramsey numbers for all pairs of complete graphs were determined by Walker [135] and rediscovered by Steinberg and Tovey [124]. In the case when one of G_1 and G_2 is a sparse graph, Gorgol and Ruciński [63] obtained planar Ramsey numbers for all pairs of cycles. Meanwhile, Dudek and Ruciński [40] calculated and summarized most planar Ramsey numbers $PR(G_1,G_2)$, where G_1 and G_2 belong to the infinite family of graphs $\bigcup_n \{K_n, K_n - e, C_n\}$.

The complete graph-wheel planar Ramsey numbers were first studied by Zhou et al., who determined the triangle-wheel planar Ramsey numbers.

Theorem 9.1 (Zhou et al. [144]).

$$PR(K_3,W_n) = \begin{cases} 9 & \textit{for } n = 3, \\ n+5 & \textit{for } 4 \le n \le 6, \\ n+4 & \textit{for } n \ge 7. \end{cases}$$

In this chapter, we first confirm all planar Ramsey numbers for complete graphs versus wheels. Indeed, we need only to consider $PR(K_m,W_n)$ for $m \ge 4$. The proof of the following theorem is postponed to Section 9.2.

Theorem 9.2.

$$PR(K_m,W_n)=\begin{cases}13 & \text{for } m\geq 4 \text{ and } 3\leq n\leq 6,\\ 14 & \text{for } m\geq 4 \text{ and } n=7,\\ n+6 & \text{for } m\geq 4 \text{ and } n\geq 8.\end{cases}$$

Then we determine all planar Ramsey numbers for complete graphs with one edge deleted versus wheels.

Theorem 9.3.

$$PR(K_3^-,W_n)=\begin{cases}7 & \text{for } n=3,\\ 2\lceil n/2\rceil+1 & \text{for } n\geq 4.\end{cases} \tag{9.1}$$

$$PR(K_4^-,W_n)=\begin{cases}10 & \text{for } 3\leq n\leq 5,\\ n+4 & \text{for } n=7,9,10,\\ n+5 & \text{for } n=6,8 \text{ or } n\geq 11.\end{cases} \tag{9.2}$$

$$PR(K_m^-,W_n)=\begin{cases}13 & \text{for } m\geq 5 \text{ and } 3\leq n\leq 6,\\ 14 & \text{for } m\geq 5 \text{ and } n=7,\\ n+6 & \text{for } m\geq 5 \text{ and } n\geq 8.\end{cases} \tag{9.3}$$

We postpone the proof of Theorem 9.3 to Section 9.3. The planar Ramsey graph with respect to the pair (K_m,W_n) ((K_m^-,W_n)) is any planar graph on $PR(K_m,W_n)-1$ ($PR(K_m^-,W_n)-1$) vertices such that it contains no K_m (K_m^-) and its complement contains no W_n. For $n\geq 4$ in the proof of Case (9.1) in Theorem 9.3, $n\geq 7$ in the proof of Theorem 9.1 and Case (9.2) in Theorem 9.3, and $n\geq 9$ in the proof of Theorem 9.2 and Case (9.3) in Theorem 9.3, the planar Ramsey graphs constructed are always the graphs whose complements contains no $K_{1,n}$. Since $PR(F,K_{1,n})\leq PR(F,W_n)$, we have $PR(F,K_{1,n})=PR(F,W_n)$ for $n\geq 9$, where F is K_m or K_m^- and $m\geq 3$. Therefore, we have the following theorem.

Theorem 9.4. *$PR(K_m,H)=PR(K_m,W_n)$ and $PR(K_m^-,H)=PR(K_m^-,W_n)$ for $n\geq 9$, $m\geq 3$, where H is contained in W_n, and H contains $K_{1,n}$.*

By Theorem 9.4, if H is a graph with $K_{1,n}\subseteq H\subseteq W_n$, $n\geq 9$ and $m\geq 3$, then all the planar Ramsey numbers $PR(K_m,H)$ and $PR(K_m^-,H)$ can be determined.

If $|V(H)| \le 9$, then these values can also be calculated by the computer program "Planram" due to Dudek [39]. Hence, we may confirm all the values $PR(K_m,H)$ and $PR(K_m^-,H)$ for any given graph H with $K_{1,n} \subseteq H \subseteq W_n$.

Theorem 9.5. *Let F be a 2-connected graph with $|V(F)| \ge 13$, and H a graph with $K_{1,n} \subseteq H \subseteq W_n$, then $PR(F,H) = n+6$ for $n \ge 45$ or $n = 18,19,29,30,31,40,41,42,43$.*

Since both cycles and wheels are 2-connected graphs, we have the following corollaries immediately.

Corollary 9.1. *$PR(C_m,H) = n+6$ for $m \ge 13$ and $n \ge 45$, where H is a graph with $K_{1,n} \subseteq H \subseteq W_n$.*

Corollary 9.2. *$PR(W_m,H) = n+6$ for $m \ge 12$ and $n \ge 45$, where H is a graph with $K_{1,n} \subseteq H \subseteq W_n$.*

9.2 Proof of Theorem 9.2

To prove Theorem 9.2, we need Theorems 1.4 and 1.6 from Chapter 1, and the following three lemmas.

Lemma 9.1 (Gorgol and Ruciński [63]). *Let G be a planar graph on $n \ge 9$ vertices. Then $\overline{G}$ contains C_{n-2}.*

Let G be a graph and C a longest cycle in G. Suppose G is not hamiltonian and h any vertex of $V(G)-V(C)$. Set $N_C(h) = \{z_1,z_2,\ldots,z_k\}$, where indices follow the orientation of C, $A = \{z_1^+,z_2^+,\ldots,z_k^+\}$ and $B = \{z_1^-,z_2^-,\ldots,z_k^-\}$. We have the following lemmas on Hamilton theory.

Lemma 9.2. *Both $A \cup \{h\}$ and $B \cup \{h\}$ are independent sets.*

Lemma 9.3. *If $\overline{G}$ is a planar graph, then $|N_C(h)| \le 3$. Furthermore, if $|V(C)| \ge 11$, $|V(G)-V(C)| \ge 2$ and $k \ge 2$, then $A \cap B = \emptyset$.*

Proof. If $|N_C(h)| \geq 4$, then $|A \cup \{h\}| \geq 5$. By Lemma 9.2, $\overline{G}$ contains a K_5, a contradiction. Hence, $|N_C(h)| \leq 3$. When $|V(C)| \geq 11$, $|V(G)-V(C)| \geq 2$ and $k \geq 2$, suppose that $A \cap B \neq \emptyset$. Then $z_i^+ = z_{i+1}^-$ for some $1 \leq i \leq k$. Thus, $C' = z_i h z_{i+1} \overrightarrow{C} z_i$ is also a cycle of length $|V(C)|$. Likewise, $|N_{C'}(z_i^+)| \leq 3$. Consequently, z_i^+ is adjacent to at most one vertex of C other than z_i, z_{i+1}. Since $|V(G)-V(C)| \geq 2$, let h' be another vertex in $V(G)-V(C)$, then $|N_C(h')| \leq 3$. Moreover, $|N_C(h) \cup N_C(h') \cup N_C[z_i^+]| \leq 8$. For $|V(C)| \geq 11$, we have $\overline{G}$ contains a $K_{3,3}$, a contradiction. This completes the proof of Lemma 9.3. □

Now we show the theorem. To get the lower bounds, we can see that the icosahedron graph contains no K_4 and its complement contains no W_n for $n = 3,4,6$, which imply that $PR(K_m, W_n) \geq 13$ for $m \geq 4, n = 3,4,6$. Also, we can easily check that the graphs in Figure 9.1 contain no K_4. The complement of the graph in the left subfigure contains no W_5, and the complement of the graph in the right subfigure contains no W_7 and W_8. Thus, $PR(K_m, W_5) \geq 13$ and $PR(K_m, W_n) \geq 14$ for $m \geq 4$ and $n = 7,8$.

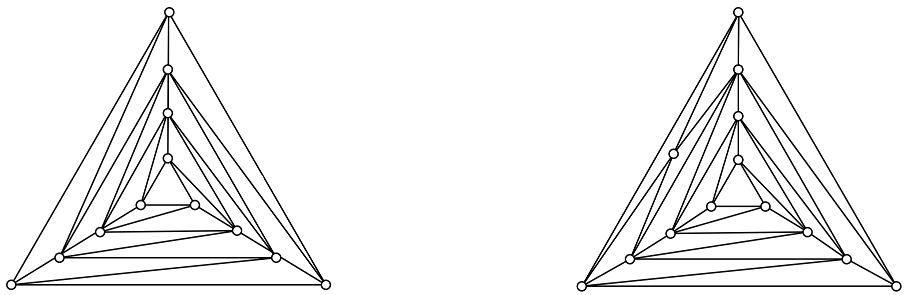

Figure 9.1: The graphs indicate the lower bounds of $PR(K_m, W_n)$ for $n = 5,7,8$

For $n \geq 9$, we give a construction of the Ramsey graph F_n. The graph F'_k consists of two copies of the cycle C_k, denoted by $C_1 = x_1 x_2 \ldots x_k x_1$ and $C_2 = y_1 y_2 \ldots y_k y_1$, where each x_i is connected to y_i and y_{i+1} for $1 \leq i \leq k$ ($y_{k+1} = y_1$). If n is odd, F_n consists of F'_k and two extra vertices z_1, z_2, where $k = (n+3)/2$, z_1 is connected to all vertices of C_1 and z_2 is connected to all vertices of C_2. If n is even, F_n consists of F'_k and three extra vertices z_1, z_2, z_3, where $k = (n+2)/2$, z_1 is connected to all vertices of C_1, z_2 is connected to $z_3, y_1, \ldots, y_4$ and z_3 is connected to $z_2, y_4, \ldots, y_k, y_1$. As in Figure 9.2, it is easy to check that for $n \geq 9$, F_n is a planar graph without K_4, $|V(F_n)| = n+5$ and $\delta(F_n) = 5$. Thus

$\Delta(\overline{F_n}) = n-1$ and $\overline{F_n}$ contains no W_n. In this way, we have $PR(K_m, W_n) \geq n+6$ for $m \geq 4, n \geq 9$.

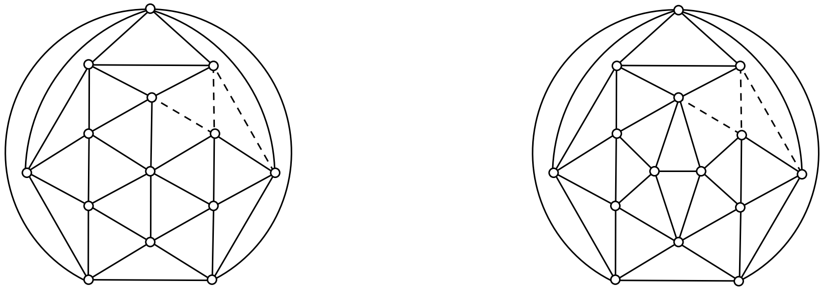

Figure 9.2: The graphs F_n of order $n+5$ for n is odd or even respectively.

Then we prove the upper bounds as follows. By using the program "Planram" due to Dudek [39], we have

$$PR(K_m, W_n) \leq \begin{cases} 13 & \text{for } m \geq 4 \text{ and } 3 \leq n \leq 6, \\ 14 & \text{for } m \geq 4 \text{ and } n = 7, \\ n+6 & \text{for } m \geq 4 \text{ and } 8 \leq n \leq 11. \end{cases}.$$

Let G be a planar graph of order $n+6$ with $n \geq 12$. In the following, we need only to show that $\overline{G}$ contains a W_n. By Euler's formula, $\delta(G) \leq 5$. If there is some vertex v such that $d(v) \leq 3$, then $\overline{G} - N[v]$ is a graph of order at least $n+2$. By Lemma 9.1, $\overline{G} - N[v]$ contains a C_n, which implies that $\overline{G}$ contains a wheel W_n with the hub v. Thus we may assume $4 \leq \delta(G) \leq 5$. For $w_1, w_2 \in N(v)$ with $d(v) = \delta(G)$ we choose a vertex v such that $d(w_1) + d(w_2)$ as large as possible. Without loss of generality, assume $d(w_1) \geq d(w_2)$. For $H = G - N[v]$, let C be a longest cycle in $\overline{H}$. We claim that $|V(C)| \geq n$. By contradiction, suppose to the contrary that $|V(C)| \leq n-1$.

If $\delta(G) = 5$, then $|V(H)| = n$ and $|V(C)| \geq n-2$ by Lemma 9.1. If $|V(C)| = n-2$, set $\{u_1, u_2\} = V(H) - V(C)$. Then by Lemma 9.3, $d_C(u_i) \geq n-5$ for $i = 1,2$. If there is no vertex of degree 5 in $N_C(u_1) \cap N_C(u_2)$, then the degree sum of G is at least $(n-5) \times 2 + 6 \times (n-8) + 5 \times 12 = 8n+2$. Since the degree sum of G is at most $2 \times [3(n+6)-6] = 6n+24$, then $8n+2 \leq 6n+24$ which implies $n \leq 11$, a contradiction. Thus, there is at least one vertex of degree 5 in $N_C(u_1) \cap N_C(u_2)$.

By the choice of v, $d(w_1)+d(w_2) \geq d_C(u_1)+d_C(u_2) \geq 2n-10$. And then the degree sum of G is at least $(n-5)\times 4+5\times(n+2) = 9n-10$. Since $9n-10 \leq 6n+24$, we have $n \leq 11$, a contradiction. If $|V(C)| = n-1$, set $\{u\} = V(H)-V(C)$. Then by Lemma 9.3, $d_C(u) \geq n-4$. By Euler's formula, there is a vertex of degree 5 in $N_C(u)$. By the choice of v, $d(w_1)+d(w_2) \geq (n-4)+5 = n+1$. And then the degree sum of G is at least $(n-4)+(n+1)+5\times(n+3) = 7n+12$. Since $7n+12 \leq 6n+24$, we have $n \leq 12$ which implies $n = 12$. Furthermore, $d(u) = 8$ and every vertex in $V(C)$ is of degree 5, otherwise there would also be a contradiction with the degree sum. Thus, the closure of $\overline{H}$ is a complete graph. By Theorem 1.4, $\overline{H}$ contains a Hamilton cycle, which contradicts the maximality of C.

If $\delta(G) = 4$, then $|V(H)| = n+1$ and $|V(C)| \geq n-1$ by Lemma 9.1. Accordingly, $|V(C)| = n-1$. We may assume that $\{u_1,u_2\} = V(H)-V(C)$ and $d_C(u_1) \geq d_C(u_2)$. Then by Lemma 9.3, $d_C(u_2) \geq n-4$. We distinguish two subcases: $d_C(u_2) \geq n-3$ and $d_C(u_2) = n-4$.

If $d_C(u_2) \geq n-3$, then $|N_C(u_1)\cap N_C(u_2)| \geq n-5$, which implies that G contains $K_{2,n-5}$. For any planar embedding of G, the $K_{2,n-5}$ separates the plane into $n-5$ regions. Suppose that every vertex in $N(u_1)\cap N(u_2)$ is of degree at least 5, then at least $\lceil (n-5)/2\rceil$ regions have vertices inside them. If a region has exactly one vertex in it, let x denote the only vertex. For $d(x) \geq 4$, we have $d(x) = 4$ and $x \in N(u_1)\cap N(u_2)$, which contradicts the hypothesis. Thus, if a region is nonempty, it has at least two vertices. Besides, there exists one region which contains at least 5 vertices. This is because $|N[v]| = 5$. Hence, $|V(G)| \geq 2\times[\lceil (n-5)/2\rceil -1]+5+(n-3)$. Since $|V(G)| = n+6$, we have $n \leq 11$, a contradiction. This shows that there is a vertex of degree 4 in $N(u_1)\cap N(u_2)$ if $d_C(u_2) \geq n-3$. By the choice of v, $d(w_1)+d(w_2) \geq d(u_1)+d(u_2)$. If $d_C(u_2) \geq n-2$, then $d_C(u_1)+d_C(u_2) \geq 2n-4$. If w_1 is adjacent to u_1 and u_2, $d(u_1)+d(u_2) \geq 2n-2$. By the choice of v, $d(w_1) \geq n-1$. If not, $d(w_1) \geq n-2$. In both cases, $|N_C(u_1)\cap N_C(u_2)\cap N_C(w_1)| \geq n-9$. For $n \geq 12$, G contains a $K_{3,3}$, a contradiction. If $d_C(u_2) = n-3$, assume that $z_1, z_2 \in V(C)-N(u_2)$ and $U = \{u_2\}\cup V(C)$. If $d_U(z_1^+)+d_U(z_2^+) \geq |U|$ in $\overline{G}$, then the closure of $\overline{G}[U]$ contains the edge $z_1^+z_2^+$. Since $z_1u_2z_2\overleftarrow{C}z_1^+z_2^+\overrightarrow{C}z_1$ is a Hamilton cycle in the closure of $\overline{G}[U]$, by Theorem 1.4, $\overline{G}[U]$ contains a cycle of length $|U| = n$, a contradiction. Thus, $d_U(z_1^+)+d_U(z_2^+) \geq n-1$ in G. By symmetry, $d_U(z_1^-)+d_U(z_2^-) \geq n-1$. By Lemma 9.3, $z_1^+, z_2^+, z_1^-, z_2^-$ are four distinct vertices. Thus, the degree sum of G is at least $\sum_{i=1}^{2}[d(w_i)+d(u_i)+d(z_i^+)+d(z_i^-)]+4\times[(n+6)-8] \geq 10n-22$. Since

$10n-22 \le 6n+24$, we have $n \le 11$, a contradiction.

If $d_C(u_2) = n-4$, assume that $z_1, z_2, z_3 \in V(C)-N(u_2)$. By Lemma 9.2, $\{u_2, z_1^+, z_2^+, z_3^+\}$ and $\{u_2, z_1^-, z_2^-, z_3^-\}$ form a K_4 respectively. And by Lemma 9.3, $z_1^+, z_2^+, z_3^+, z_1^-, z_2^-, z_3^-$ are six distinct vertices. If $u_1u_2 \notin E(G)$, then $u_1z_1^+$, $u_1z_2^+$, $u_1z_3^+ \in E(G)$, otherwise there would be a cycle longer than $|V(C)|$ in $\overline{H}$. Thus, $G[\{u_1, u_2, z_1^+, z_2^+, z_3^+\}]$ forms a $K_5 - e$, where $e = u_1u_2$. Since $|N_C(u_1) \cap N_C(u_2)| \ge n-7$, there is a vertex $x \notin \{z_1^+, z_2^+, z_3^+\}$ such that u_1xu_2 is a path in G, which implies that G contains a subdivision of K_5, a contradiction. Therefore, $u_1u_2 \in E(G)$. For $u_1z_1^+, u_1z_2^+, u_1z_3^+$, at least two of them are in $E(G)$, otherwise there would be a cycle longer than $|V(C)|$ in $\overline{H}$; and at most two of them are in $E(G)$, otherwise there would be a K_5 in G. Set $U_1 = \{u_1, u_2, z_1^+, z_2^+, z_3^+\}$, $U_2 = \{u_1, u_2, z_1^-, z_2^-, z_3^-\}$. Then $G[U_1]$ is a $K_5 - e_1$, where $e_1 = u_1z_i^+$ for some $i \in \{1,2,3\}$; by symmetry, $G[U_2]$ is a $K_5 - e_2$, where $e_2 = u_1z_j^-$ for some $j \in \{1,2,3\}$. A path is called a $U_1 \cup U_2$-free path if its internal vertices are not in $U_1 \cup U_2$. We claim that there is a $U_1 \cup U_2$-free path joining z_i^+ and $N(v)$. If z_i^+ is nonadjacent to $N(v)$, since $d(z_i^+) \ge 4$, it has to be adjacent to some vertex of $V(C)-U_1$, say, y. Then $y \ne z_j^-$ and $yu_1 \notin E(G)$, otherwise there is a path $z_i^+z_j^-z_k^-u_1$ ($k \in \{1,2,3\}-\{j\}$) or $z_i^+yu_1$ joining z_i^+ and u_1, which together with $G[U_1]$ forms a subdivision of K_5. Moreover, $y \notin U_2$. Since $d_C(u_1) \ge n-4$, $V(C)-N(u_1) = \{z_i^+, z_j^-, y\}$. To avoid a subdivision of K_5, y is nonadjacent to any vertex of $V(C)-U_1$, and is adjacent to at most three vertices of $U_1 - \{u_1\}$. Since $d(y) \ge 4$, y is adjacent to $N(v)$. This proves our claim that there is a $U_1 \cup U_2$-free path joining z_i^+ and $N(v)$. By symmetry, there is also such a path joining z_j^- and $N(v)$. Thus, there is a $U_1 \cup U_2$-free path P joining z_i^+ and z_j^-. Then the path $z_i^+Pz_j^-z_k^-u_1$ and $G[U_1]$ form a subdivision of K_5, a final contradiction.

Therefore, $\overline{H}$ contains a cycle C of length at least $n \ge 12$. Since H is a planar graph, $|E(H)| \le 3|V(H)|-6$. Then $|E(\overline{H})| \ge \binom{|V(H)|}{2} - (3|V(H)|-6) > (|V(H)|-1)^2/4+1$ and $\overline{H}$ is nonbipartite. By Theorem 1.6, $\overline{H}$ contains a C_n, which together with v forms a W_n in $\overline{G}$.

This proves the theorem. □

9.3 Proof of Theorem 9.3

In order to prove Theorem 9.3, we need Theorem 1.6 and the following lemmas.

Lemma 9.4 (Kuratowski [85]). *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

Lemma 9.5 (Dudek and Ruciński [40]). *Let G be a planar graph on $n \geq 9$ vertices. Then $\overline{G}$ contains cycles of every length between 3 and $n-2$.*

Lemma 9.6. *Let G be a simple planar graph with at least four vertices, and G contains no K_4^-. Then $|E(G)| \leq 12(|V(G)|-2)/5$ and $\delta(G) \leq 4$.*

Proof. The number of vertices and edges of G are denoted by v, e, respectively. For a planar embedding of G, let f_1 denote the number of triangles, that is, the faces whose boundary contains exactly three edges, and f_2 denote the number of faces whose boundary contains at least four edges. Letting $\{p_i\}$ be the list of face lengths, this yields

$$2e = \sum p_i \geq 3f_1 + 4f_2 \tag{9.4}$$

If there exist two faces which are triangles and have more than one common edges, then they have exactly three common edges. Thus, G is a triangle, which contradicts the order of G. If there exist two faces which are triangles and have exactly one common edge, then G contains a K_4^-, also a contradiction. Thus, G is a planar graph without two triangles sharing an edge. We have

$$e \geq 3f_1 \tag{9.5}$$

By (9.4) + (9.5)/3, we have $7e/3 \geq 4(f_1 + f_2)$, that is, $f_1 + f_2 \leq 7e/12$. Substituting into Euler's formula $v - e + f_1 + f_2 \geq 2$ yields $e \leq 12(v-2)/5$. Moreover, $\delta(G) \leq 2e/v < 5$, i.e., $\delta(G) \leq 4$. □

Lemma 9.7. *Let G be a planar graph which contains no K_4^-, C a longest cycle in $\overline{G}$ with a given orientation, and $\{u_1, u_2\} = V(G) - V(C)$. Then $d_C(u_i) \geq |V(C)| - 2$ for $i = 1, 2$. Furthermore, if $u_1u_2 \notin E(G)$, then $|N_C(u_1) \cap N_C(u_2)| \geq |V(C)| - 2$.*

Proof. If $d_C(u_1) \leq |V(C)| - 3$, then u_1 is nonadjacent to at least three vertices of $V(C)$, denoted by $s_1, \ldots, s_l$, $l \geq 3$. Hence, u_1 must be adjacent to s_1^+, s_2^+, s_3^+. If not, say, u_1 is nonadjacent to s_1^+, then $s_1u_1s_1^+\overrightarrow{C}s_1$ is a cycle longer than C in $\overline{G}$, a contradiction. Moreover, $s_1^+s_2^+, s_2^+s_3^+, s_1^+s_3^+ \in E(G)$. If not, say, $s_1^+s_2^+ \notin E(G)$, then $s_1u_1s_2\overleftarrow{C}s_1^+s_2^+\overrightarrow{C}s_1$ is a cycle longer than C in $\overline{G}$, also a contradiction. Thus,

$G[\{u_1, s_1^+, s_2^+, s_3^+\}]$ forms a K_4, which contradicts the fact that G contains no K_4^-. Therefore, $d_C(u_1) \geq |V(C)| - 2$. For the same reason, $d_C(u_2) \geq |V(C)| - 2$.

If $u_1u_2 \notin E(G)$, suppose that there are three vertices of $V(C)$ which are nonadjacent to u_1 or u_2, say, s_1, s_2, s_3. Then u_1 must be adjacent to s_1^+, s_2^+, s_3^+. If not, say, u_1 is nonadjacent to s_1^+, then $s_1u_1s_1^+\overrightarrow{C}s_1$ or $s_1u_2u_1s_1^+\overrightarrow{C}s_1$ is a cycle longer than C in $\overline{G}$, a contradiction. Moreover, $s_1^+s_2^+, s_2^+s_3^+, s_1^+s_3^+ \in E(G)$ for the same reason and by the same analysis. Thus, $G[\{u_1, s_1^+, s_2^+, s_3^+\}]$ forms a K_4, a final contradiction. Therefore, $|N_C(u_1) \cap N_C(u_2)| \geq |V(C)| - 2$ if $u_1u_2 \notin E(G)$. □

Now we begin to prove Theorem 9.3. Note that for any planar graph G of order N, if G contains no K_3^-, then $\overline{G}$ contains $K_N - \lfloor N/2 \rfloor K_2$ as a subgraph. Since W_3 is a subgraph of $K_7 - 3K_2$, W_n is a subgraph of $K_{n+1} - n/2K_2$ for even $n \geq 4$ and a subgraph of $K_{n+2} - (n+1)/2K_2$ for odd $n \geq 4$, we have $PR(K_3^-, W_3) \leq 7$ and $PR(K_3^-, W_n) \leq 2\lceil n/2 \rceil + 1$ for $n \geq 4$. The corresponding lower bounds are provided by the planar Ramsey graphs $3K_2$ and $\lceil n/2 \rceil K_2$, respectively. Turning to case (3), by the monotonicity of planar Ramsey numbers, we have $PR(K_{m-1}, W_n) \leq PR(K_m^-, W_n) \leq PR(K_m, W_n)$. Thus, the case for $m \geq 5$ follows directly from Theorem 9.2.

Now we consider the most challenging case $PR(K_4^-, W_n)$. A planar Ramsey graph with respect to both (K_4^-, W_3) and (K_4^-, W_4) consists of a 3-cycle and a 6-cycle, denoted by $C_1 = x_1x_2x_3x_1$ and $C_2 = y_1y_2 \ldots y_6y_1$, where each x_i is connected to y_{2i-1} and y_{2i} for $1 \leq i \leq 3$. It is easy to check that this graph is a planar graph without K_4^- and its complement contains no W_n, where $n = 3, 4$. Thus, $PR(K_4^-, W_n) \geq 10$ for $n = 3, 4$. Since $PR(K_4^-, W_n) \geq PR(K_3, W_n)$, by Theorem 9.1, we may establish the lower bounds that $PR(K_4^-, W_n) \geq n + 5$ for $n = 5, 6$ and $PR(K_4^-, W_n) \geq n + 4$ for $n = 7, 9, 10$.

Let every graph G in Figure 9.3 be of order $n + 4$. And we can easily check that each graph G is planar graph which does not contain K_4^-, and $\delta(G) = 4$. Thus, $\Delta(\overline{G}) = n - 1$ and $\overline{G}$ contains no W_n. In this way, we have $PR(K_4^-, W_n) \geq n + 5$ for $n = 8, 11, 13, 14$. In the following, we give a construction of (K_4^-, W_n) planar Ramsey graphs for $n = 12$ or $n \geq 15$. For every graph G in Figure 9.3, subdivide the edge v_iv_{i+1} by inserting a new vertex x_i between v_i and v_{i+1} for $1 \leq i \leq 4$, where $v_5 = v_1$ in Figure 9.3 (a)(b). And we join x_i and x_{i+1} for $1 \leq i \leq 4$, where $x_5 = x_1$. It is easy to check that each new graph G is planar graph which does not contain K_4^-, and $\delta(G) = 4$. Hence, $PR(K_4^-, W_n) \geq n + 5$ for

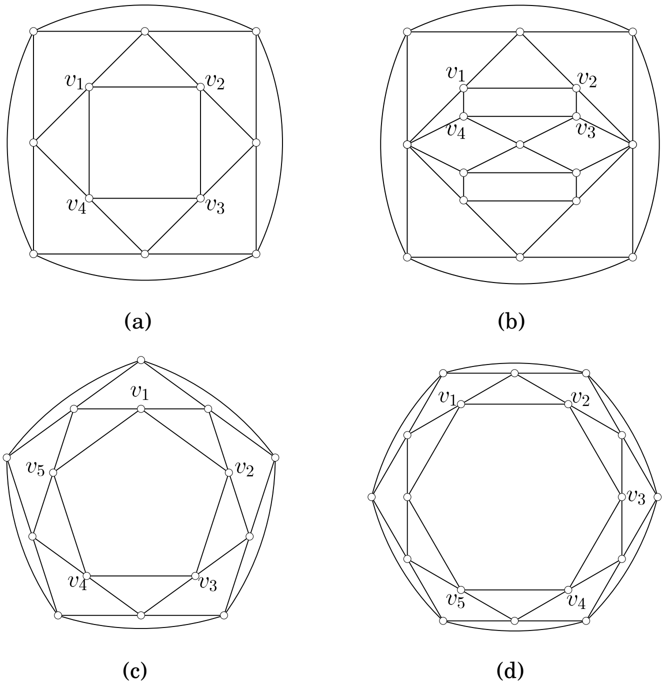

Figure 9.3: Planar Ramsey graphs with respect to (K_4^-, W_n) for $n = 8, 13, 11, 14$ respectively.

$n = 12, 15, 17, 18$. Repeat this procedure again by subdividing the edge $x_i x_{i+1}$ and inserting a new vertex y_i between x_i and x_{i+1} for $1 \le i \le 4$, where $x_5 = x_1$. And then join y_i and y_{i+1} for $1 \le i \le 4$, where $y_5 = y_1$. By the same analysis, $PR(K_4^-, W_n) \ge n + 5$ for $n = 16, 19, 21, 22$. By this iteration technique, we have $PR(K_4^-, W_n) \ge n + 5$ for $n = 8$ or $n \ge 11$.

Then we prove the upper bounds as follows. By using the program "Planram" due to Dudek [39], we have

$$PR(K_4^-, W_n) \le \begin{cases} 10 & \text{for } n = 3, 4, 5, \\ n + 4 & \text{for } n = 7, 9, 10, \\ n + 5 & \text{for } n = 6, 8. \end{cases}$$

Let G be a planar graph of order $n + 5$ with $n \ge 11$, and let G contain no K_4^-. In the following, we need only to show that $\overline{G}$ contains a W_n. By Lemma 9.6,

$\delta(G) \le 4$. If there is some vertex v such that $d(v) \le 2$, then $\overline{G} - N[v]$ is a graph of order at least $n+2$. By Lemma 9.5, $\overline{G} - N[v]$ contains a C_n, which implies that $\overline{G}$ contains a wheel W_n with the hub v. Thus we may assume $3 \le \delta(G) \le 4$. We choose one edge $vw \in E(G)$ such that $d_G(v) = \delta(G)$ and $d_G(w)$ as large as possible. Set $H = G - N[v]$, then $n \le |V(H)| \le n+1$. Let C be a longest cycle in $\overline{H}$, then by Lemma 9.5, $|V(C)| \ge n-2$ if $|V(H)| = n$, and $|V(C)| \ge n-1$ if $|V(H)| = n+1$.

We claim that $|V(C)| \ge n$. By contradiction, suppose to the contrary that $|V(C)| \le n-1$. Then there are three cases.

(a) $\delta(G) = 4$, $|V(H)| = n$ and $|V(C)| = n-1$;

(b) $\delta(G) = 4$, $|V(H)| = n$ and $|V(C)| = n-2$;

(c) $\delta(G) = 3$, $|V(H)| = n+1$ and $|V(C)| = n-1$.

For the first case, set $\{u\} = V(H) - V(C)$. Then by Lemma 9.7, $d_C(u) \ge n-3$. If there is no vertex of degree 4 in $N_C(u)$, that is, every vertex in $N_C(u)$ is of degree at least 5, then the degree sum of G is at least $5 \times (n-3) + (n-3) + 4 \times 7 = 6n+10$. By Lemma 9.6, the degree sum of G is at most $2 \times 12[(n+5)-2]/5 = 24(n+3)/5$, then $6n+10 \le 24(n+3)/5$ which implies $n \le 3$, a contradiction. Thus, there is at least one vertex of degree 4 in $N_C(u)$. By the choice of v, $d(w) \ge d_C(u) \ge n-3$. And then the degree sum of G is at least $(n-3) \times 2 + 4 \times (n+3) = 6n+6$. Since $6n+6 \le 24(n+3)/5$, we have $n \le 7$, also a contradiction.

For the last two cases, set $\{u_1, u_2\} = V(H) - V(C)$. Then by Lemma 9.7, $d_C(u_i) \ge n-4$ for $i = 1,2$. It follows that $|N_C(u_1) \cap N_C(u_2)| \ge n-7$. For $n \ge 11$, u_1 and u_2 are nonadjacent in G, otherwise the graph $G[\{u_1, u_2, s_1, s_2\}]$ contains a K_4^- in G, where s_1, s_2 are any two common adjacent vertices of u_1, u_2. Let $\{t_1, t_2, \ldots, t_l\} = V(C) - (N_C(u_1) \cap N_C(u_2))$, by Lemma 9.7, $l \le 2$. There is no edge in $G[N_C(u_1) \cap N_C(u_2)]$, since otherwise the two ends of the edge together with u_1, u_2 would form a K_4^- in G. Moreover, $|N_C(u_1) \cap N_C(u_2) \cap N_C(N[v])| \le 2$, otherwise the graph $G[N[v] \cup \{u_1, u_2, s_1, s_2, s_3\}]$ contains a $K_{3,3}$-subdivision in G which contradicts Lemma 9.4, where s_1, s_2, s_3 are any three vertices in $N_C(u_1) \cap N_C(u_2) \cap N_C(N[v])$. For the same reason, $|N_C(u_1) \cap N_C(u_2) \cap N_C(t_i)| \le 2$ for $1 \le i \le l$. Since $l \le 2$, $|N_C(u_1) \cap N_C(u_2)| \ge n-4$ and $n \ge 11$, there is at least one vertex in $N_C(u_1) \cap N_C(u_2)$ who has no adjacent vertex other than u_1, u_2. In other words, there is a vertex of degree two, which contradicts $\delta(G) \ge 3$.

Therefore, $\overline{H}$ contains a cycle C of length at least $n \geq 11$. Since H is a planar graph which contains no K_4^-, by Lemma 9.6, $|E(H)| \leq 12(|V(H)|-2)/5$. Then $|E(\overline{H})| \geq \binom{|V(H)|}{2} - 12(|V(H)|-2)/5 > (|V(H)|-1)^2/4+1$ and $\overline{H}$ is nonbipartite. By Theorem 1.6, $\overline{H}$ contains a C_n, which together with v forms a W_n in $\overline{G}$.

This proves the theorem. □

9.4 Proof of Theorem 9.5

By Theorem 9.2, $PR(K_5, W_n) = n+6$ for $n \geq 8$. That is to say, for any planar graph G on $n+6 \geq 14$ vertices, $\overline{G}$ contains W_n and hence $\overline{G}$ contains H. Thus, the upper bound in Theorem 9.5 can be reached without difficulty.

To prove the lower bound, we need only to construct a planar graph G on $n+5$ vertices, of which every block has at most 12 vertices and $\delta(G) = 5$. Then $\Delta(\overline{G}) = n-1$, $\overline{G}$ contains no $K_{1,n}$ and hence $\overline{G}$ contains no H. Notice that the icosahedron graph is a planar graph on 12 vertices and the minimum degree is 5. Then we can form planar Ramsey graphs based on the icosahedron. In Figure 9.4, we substitute a triangle with grayish filling in it for a planar drawing of the icosahedron. And the three vertices of the grayish triangle correspond to the three vertices on the outer face of the icosahedron. As in Figure 9.4, every graph G is a planar graph which contains no F as a subgraph, where F is any 2-connected graph on at least 13 vertices. And since $\delta(G) = 5$, $\overline{G}$ contains no $K_{1,n}$, where $n = |V(G)|-5$. In this way, $PR(F, K_{1,n}) \geq n+6$ for $45 \leq n \leq 50$.

For $51 \leq n \leq 55$, a planar Ramsey graph can be obtained from five copies of the icosahedron graph by identifying precisely one vertex of i copies of the icosahedron, where $i = 56-n$. Through checking without difficulty, we have $PR(F, K_{1,n}) \geq n+6$ for $51 \leq n \leq 55$. Now we have 11 planar Ramsey graphs with respect to $(F, K_{1,n})$ for $45 \leq n \leq 55$, respectively. For $n \geq 56$, there exist integers x, y with $1 \leq y \leq 11$, such that $n = 44+11x+y$. One planar Ramsey graph with respect to $(F, K_{1,n})$ can be formed by identifying exactly one vertex of each graph in $\mathscr{G}$, where $\mathscr{G}$ consists of $x+1$ graphs including x copies of the icosahedron, and the (F, W_{44+y}) planar Ramsey graph which was constructed above. For example, if $n = 100$, then $x = 5$ and $y = 1$. We first select one vertex arbitrarily from each of the following six graphs: the (F, W_{45}) planar Ramsey graph and five copies of the icosahedron. After identifying the six vertices, we

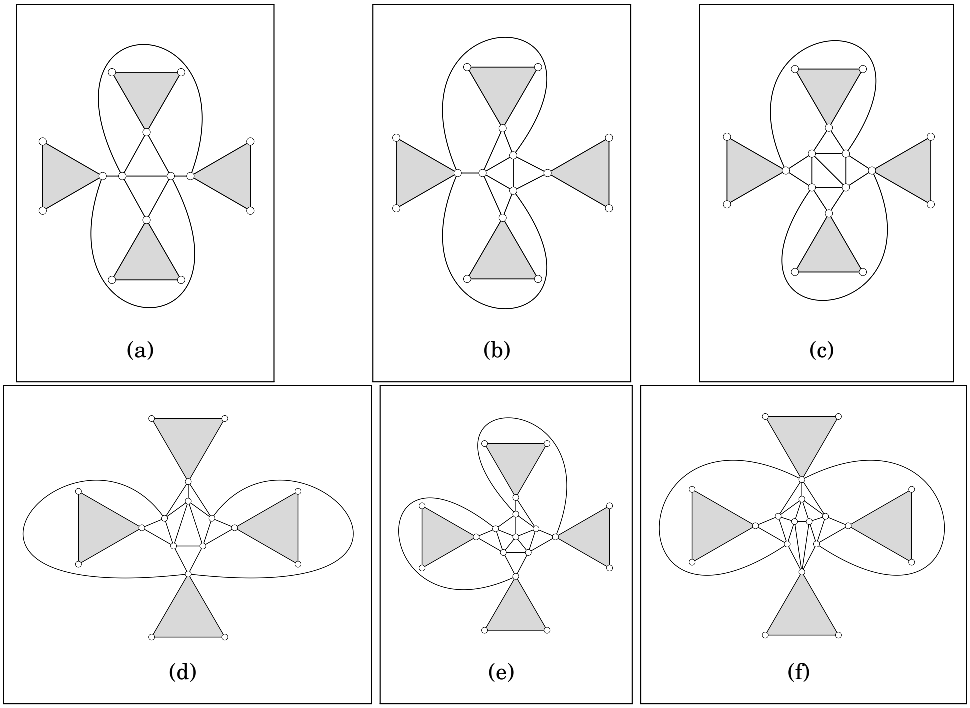

Figure 9.4: Planar Ramsey graphs with respect to $(F, K_{1,n})$ for $45 \le n \le 50$ respectively.

obtain the (F, W_{100}) planar Ramsey graph. Therefore, if F is a 2-connected graph with $|V(F)| \ge 13$, then $PR(F, K_{1,n}) \ge n+6$ for $n \ge 45$.

In addition, for the other nine small values of n, planar Ramsey graphs are obtained from l copies of the icosahedron graph by identifying precisely one vertex of its i copies, where $2 \le l \le 4$ and $1 \le i \le l$. Again by the same analysis, $PR(F, K_{1,n}) \ge n+6$ for $n = 18, 19, 29, 30, 31, 40, 41, 42, 43$.

This completes the proof of Theorem 9.5. □

Chapter 10

On star-critical and upper size Ramsey numbers

10.1 Introduction

To study various graph properties in graph Ramsey theory, we adopt the following definition and notation. This arrowing notation will be used in many other definitions in the sequel.

Definition 6. Given two graphs G_1 and G_2, we say that a graph G arrows the pair (G_1, G_2), denoted by $G \to (G_1, G_2)$, if in any red-blue coloring of the edges of G, there is either a red copy of G_1 or a blue copy of G_2.

For two given graphs G_1 and G_2, the most extensively investigated notion within Ramsey theory is the graph Ramsey number $R(G_1, G_2)$. This is the smallest integer r such that, for any graph G of order r, either G contains G_1 as a subgraph or $\overline{G}$ contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. For simplicity, we now restate this definition of $R(G_1, G_2)$ in the language of arrowing.

Definition 7.

$$r = R(G_1, G_2) = \min\{|V(G)| \mid G \to (G_1, G_2)\}.$$

Let r denote the Ramsey number $R(G_1, G_2)$ throughout this chapter. A dynamic survey on Ramsey numbers can be found in [110]. We see that the Ramsey number $R(G_1, G_2)$ is the smallest number of vertices in a graph G such that

$G \to (G_1, G_2)$. Analogously, the size Ramsey number $\hat{r}(G_1, G_2)$ is the smallest number of edges in a graph G such that $G \to (G_1, G_2)$, a definition that was introduced by Erdős et al. [47].

Definition 8 (Erdős et al. [47]).

$$\hat{r}(G_1, G_2) = \min\{|E(G)| \mid G \to (G_1, G_2)\}.$$

The size Ramsey number was also widely studied. For a survey of results on size Ramsey numbers we refer to [60]. In the context of $G \to (G_1, G_2)$, some other graph parameters of G were introduced by Burr et al. [24] in 1976, including the minimum degree of G, a parameter that attracted considerable attention again since 2006.

Definition 9 (Burr et al. [24]).

$$s(G_1, G_2) = \min\{\delta(G) \mid G \to (G_1, G_2)\}.$$

The above definitions inspired researchers to introduce several other definitions. In 1991, Erdős and Faudree [46] considered two related definitions: the upper size Ramsey number $u(G_1, G_2)$ and the lower size Ramsey number $\ell(G_1, G_2)$. The latter was considered earlier by Faudree and Sheehan [61] under the term restricted size Ramsey number.

Definition 10 (Erdős and Faudree [46]).

$$u(G_1, G_2) = \min\{q \mid \text{if } G \subseteq K_r \text{ and } E(G) \geq q, \text{ then } G \to (G_1, G_2)\}.$$

Definition 11 (Erdős and Faudree [46]).

$$\ell(G_1, G_2) = \min\{|E(G)| \mid G \subseteq K_r \text{ and } G \to (G_1, G_2)\}.$$

Clearly, $\hat{r}(G_1, G_2) \leq \ell(G_1, G_2) \leq u(G_1, G_2) \leq \binom{r}{2}$. Moreover, if $\ell(G_1, G_2) \leq q < u(G_1, G_2)$, then there exist two graphs $H_1, H_2 \subseteq K_r$ with $|E(H_1)| = |E(H_2)| = q$ such that $H_1 \to (G_1, G_2)$, but $H_2 \nrightarrow (G_1, G_2)$. Erdős and Faudree performed a detailed investigation on the two functions $u(G_1, G_2)$ and $\ell(G_1, G_2)$, and determined some exact values, and some reasonable upper and lower bounds. With respect to general lower bounds, they established the following theorem. There are example graphs showing that these bounds cannot be improved in general.

Theorem 10.1 (Erdős and Faudree [46]). *For any pair of graphs G_1 and G_2 without isolated vertices,*

$$u(G_1,G_2) \geq \binom{r-1}{2} + \delta(G_1) + \delta(G_2) - 1,$$

$$\ell(G_1,G_2) \geq |E(G_1)| + |E(G_2)| - 1.$$

Since $K_r \to (G_1,G_2)$, but $K_{r-1} \nrightarrow (G_1,G_2)$, a natural problem is to consider G such that $K_{r-1} \subseteq G \subseteq K_r$ and $G \to (G_1,G_2)$. To study this, Hook and Isaak [77] proposed the definition of the star-critical Ramsey number $r_*(G_1,G_2)$, which is the smallest k such that $K_{r-1} \sqcup K_{1,k} \to (G_1,G_2)$. Recall that $K_{r-1} \sqcup K_{1,k}$ is the graph obtained from K_{r-1} and an additional vertex v by joining v to k vertices of K_{r-1}.

Definition 12 (Hook and Isaak [77]).

$$r_*(G_1,G_2) = \min\{\delta(G) \mid G \subseteq K_r \text{ and } G \to (G_1,G_2)\}.$$

We show that Definition 12 is in fact the same as Hook and Isaak's original definition. If $K_{r-1} \sqcup K_{1,k} \to (G_1,G_2)$, by Definition 12, $r_*(G_1,G_2) \leq k$. On the other hand, if $K_{r-1} \sqcup K_{1,k} \nrightarrow (G_1,G_2)$, then any graph $G \subseteq K_r$ with $\delta(G) \leq k$ is a subgraph of $K_{r-1} \sqcup K_{1,k}$, and hence $G \nrightarrow (G_1,G_2)$. Thus, $r_*(G_1,G_2) \geq k+1$. Moreover, we can also define $r_*(G_1,G_2)$ as follows.

$$r_*(G_1,G_2) = \min\{\delta(G) \mid K_{r-1} \subseteq G \subseteq K_r \text{ and } G \to (G_1,G_2)\}.$$

The reason is that, for any graph $G \subseteq K_r$ and $G \to (G_1,G_2)$, there exists a graph G' such that $K_{r-1} \subseteq G' \subseteq K_r$, $G' \to (G_1,G_2)$ and $\delta(G) = \delta(G')$.

We see that both $\hat{r}(G_1,G_2)$ and $\ell(G_1,G_2)$ are the smallest $|E(G)|$ with $G \to (G_1,G_2)$, but the latter is restricted to considering subgraphs of K_r. Correspondingly, both $s(G_1,G_2)$ and $r_*(G_1,G_2)$ are the smallest $\delta(G)$ with $G \to (G_1,G_2)$, but the latter is restricted to considering subgraphs of K_r. Let $\mu(G_1,G_2) = \binom{r-1}{2} + r_*(G_1,G_2)$, then it is easy to check that $\ell(G_1,G_2) \leq \mu(G_1,G_2) \leq u(G_1,G_2)$.

Moreover, Hook and Isaak [77] found the following results.

$$r_*(T_n,K_m) = (n-1)(m-2)+1, \qquad \text{for any tree on } n \text{ vertices.} \tag{10.1}$$

$$r_*(nK_2,mK_2) = m, \qquad \text{for } n \geq m \geq 1. \tag{10.2}$$

$$r_*(nK_3,mK_3) = 3n+2m-1, \qquad \text{for } n \geq m \geq 1 \text{ and } n \geq 2. \tag{10.3}$$

$$r_*(P_n,C_4) = 3, \qquad \text{for } n \geq 3. \tag{10.4}$$

Wu et al. [137] obtained the star-critical Ramsey numbers for cycles versus a quadrilateral.

$$r_*(C_n,C_4) = 5, \qquad \text{for } n \geq 4. \tag{10.5}$$

Li and Li [90] determined several other star-critical Ramsey numbers. The main results of their paper read as follows.

$$r_*(K_n,mK_2) = n+2m-3, \qquad \text{for } m \geq 1 \text{ and } n \geq 3. \tag{10.6}$$

$$r_*(F_n,K_3) = 2n+2, \qquad \text{for } n \geq 2. \tag{10.7}$$

$$r_*(nK_4,mK_3) = 3n+2m+\max\{m,n\}, \qquad \text{for } m \geq 1 \text{ and } n \geq 2. \tag{10.8}$$

We now first classify six of the above eight results into two classes. The first class is based on the observation that there are many pairs of graphs (G_1,G_2) with $u(G_1,G_2) = \binom{r}{2}$, and hence we introduce the following definition.

Definition 13. A pair of graphs (G_1,G_2) is called Ramsey-full if $K_r \rightarrow (G_1,G_2)$, but $K_r - e \nrightarrow (G_1,G_2)$, where $K_r - e$ is K_r with one edge deleted.

If (G_1,G_2) is Ramsey-full, then $\ell(G_1,G_2) = \mu(G_1,G_2) = u(G_1,G_2) = \binom{r}{2}$, and $r_*(G_1,G_2) = r-1$. Conversely, if one of $\hat{r}(G_1,G_2)$, $\ell(G_1,G_2)$, $\mu(G_1,G_2)$, $u(G_1,G_2)$ equals $\binom{r}{2}$, or $r_*(G_1,G_2) = r-1$, then (G_1,G_2) is Ramsey-full.

Erdős and Faudree [46] stated that it is an interesting problem to determine all the pairs of Ramsey-full graphs, or at least some infinite families of such pairs. It was first observed by Chvátal, and later proved by Erdős et al. [47] that (K_n,K_m) is Ramsey-full. Erdős and Faudree [45] showed that (K_n,mK_2) is Ramsey-full for $n \geq 3$. Since $R(K_n,mK_2) = n+2m-2$ for $n \geq 2$ by [93], Equation (10.6) is in fact an immediate consequence. In [46], Erdős and Faudree proved that $(K_{1,n}, K_{1,m})$ is Ramsey-full if both m and n are even. For $n \geq m \geq 1$ and $n \geq 2$, since $R(nK_3, mK_3) = 3n+2m$ by [25], Equation (10.3) implies that (nK_3, mK_3) is Ramsey-full. For $m \geq 1$ and $n \geq 2$, since $R(nK_4, mK_3) = 3n+2m+\max\{m,n\}+1$ by [94], Equation (10.8) implies that (nK_4, mK_3) is Ramsey-full. We will prove the following more general result: (nK_k, mK_l) is Ramsey-full for $k,l \geq 3$ and large m,n.

Theorem 10.2. *Given two integers $k,l \geq 3$, if m and n are large, then*

$$r_*(nK_k, mK_l) = (k-1)n + (l-1)m + \max\{m,n\} + R(K_{k-1}, K_{l-1}) - 3.$$

We postpone the proof of Theorem 10.2 to the next section. Besides (K_n, K_m), (nK_k, mK_l) is a second example of graph pairs for which exact values of Ramsey numbers are unknown (with small exceptions) but that are Ramsey-full. We next give another example.

Burr et al. [23] showed that $R(K_n \sqcup K_{1,k}, K_m \sqcup K_{1,l}) = R(K_n, K_m)$ for $m,n \geq 3$, $m+n \geq 8$, $k \leq \lceil n/(m-1) \rceil$ and $l \leq \lceil m/(n-1) \rceil$. Let $r = R(K_n, K_m)$. From $K_r - e \nrightarrow (K_n, K_m)$, it follows that $K_r - e \nrightarrow (K_n \sqcup K_{1,k}, K_m \sqcup K_{1,l})$, and hence $(K_n \sqcup K_{1,k}, K_m \sqcup K_{1,l})$ is Ramsey-full for $m,n \geq 3$, $m+n \geq 8$, $k \leq \lceil n/(m-1) \rceil$ and $l \leq \lceil m/(n-1) \rceil$.

The second class is based on the notion of goodness in Ramsey theory. Let $\chi(G_1)$ be the chromatic number of G_1, and let G_2 be a connected graph. Chvátal and Harary [36] proved that $R(G_1, G_2) \geq (\chi(G_1)-1)(|V(G_2)|-1)+1$. Burr [17] generalized this lower bound by adding another parameter $\sigma(G_1)$. Let $V_1, V_2, \ldots, V_{\chi(G_1)}$ be the color classes of G_1 under a proper vertex coloring by $\chi(G_1)$ colors and assume that $|V_1| \leq |V_2| \leq \ldots \leq |V_{\chi(G_1)}|$. Choose a vertex coloring such that $|V_1|$ is as small as possible, and let $\sigma(G_1)$ denote this $|V_1|$; this $\sigma(G_1)$ is called the chromatic surplus of G_1. For a connected graph G_2 with $|V(G_2)| \geq \sigma(G_1)$, Burr [17] showed that $R(G_1, G_2) \geq (\chi(G_1)-1)(|V(G_2)|-1)+\sigma(G_1)$. Moreover, Burr defined G_2 to be G_1-good if equality holds in the latter inequality.

Under all proper vertex colorings with $|V_1| = \sigma(G_1)$, let $\tau(G_1) = \min\{|E(v, V_i)| \mid v \in V_1, 2 \leq i \leq \chi(G_1)\}$, which is the minimum degree of some vertex of V_1 in V_i for some $2 \leq i \leq \chi(G_1)$. Our second result provides the following general lower bound for $r_*(G_1, G_2)$.

Theorem 10.3. *Let G_2 be a nontrivial connected graph which is G_1-good. If $\sigma(G_1)=1$, or if $\sigma(G_1)\ge 2$ and $\delta(G_2)=1$, or if $\sigma(G_1)\ge 2$, $\delta(G_2)\ge 2$ and $\kappa(G_2)\ge 2$, then*

$$r_*(G_1,G_2)\ge(\chi(G_1)-2)(|V(G_2)|-1)+\sigma(G_1)+\delta(G_2)+\tau(G_1)-2.$$

We postpone the proof of Theorem 10.3 to Section 10.3. Since $u(G_1,G_2)\ge\mu(G_1,G_2)=\binom{r-1}{2}+r_*(G_1,G_2)$, using Theorem 10.3 we may obtain a better lower bound for $u(G_1,G_2)$ under certain conditions than the lower bound stated in Theorem 10.1. Motivated by Theorem 10.3, we now introduce the notion of size goodness, corresponding to the notion of goodness.

Definition 14. A nontrivial connected graph G_2 is called G_1-size good, if G_2 is G_1-good, and the equality holds in (10.3) under one of the following condition-s:

(1) $\sigma(G_1)=1$;
(2) $\sigma(G_1)\ge 2$ and $\delta(G_2)=1$;
(2) $\sigma(G_1)\ge 2$, $\delta(G_2)\ge 2$ and $\kappa(G_2)\ge 2$.

By the definition of size goodness, we see that Equation (10.1) implies that T_n is K_m-size good; Equation (10.4) implies that P_n is C_4-size good for $n\ge 3$; Equation (10.7) implies that F_n is K_3-size good for $n\ge 2$. Since there are many other pairs of graphs (G_1,G_2) such that G_2 is G_1-good, a natural question is whether in such cases G_2 is G_1-size good as well. It is obvious that there exist graph pairs (G_1,G_2) such that G_2 is G_1-good, but G_2 is not G_1-size good. For instance, C_n is C_4-good for $n\ge 6$, but Equation (10.5) implies that C_n is not C_4-size good. In fact, for even $m\ge 4$ and odd $n\ge 3m/2$, or for even m,n, $n\ge m\ge 4$ and $(m,n)\ne(4,4)$, it has been proved by Rosta [114] (and also by Faudree and Schelp [58] independently) that C_n is C_m-good. We claim that for these values of m and n, C_n is not C_m-size good. For this purpose, let $r=n+m/2-1$, and let U_1,U_2 be a partition of $V(K_{r-1})$ with $|U_1|=n-1$ and $|U_2|=m/2-1$. Assign colors to the edges of the K_{r-1} as follows: color the edges of U_1 blue, the edges from a vertex $u_1\in U_1$ to each vertex of U_2 blue, and all the other edges red. Let u_0 be an additional vertex, which is adjacent to U_2 with $m/2-1$ red edges, adjacent to u_1 with a red edge, and adjacent to $U_1-\{u_1\}$ with one red edge and one blue edge. It is easy to check that there is neither a blue C_n nor a red C_m, and hence $r_*(C_n,C_m)\ge m/2+3$. By Definition 14, C_n is not C_m-size good for even m.

Our next result deals with cases for which C_n is C_m-size good. For large cycles versus small odd cycles, we prove that C_n is C_m-size good for m odd, $n \geq m \geq 3$ and $(m,n) \neq (3,3)$. Actually, we have the following stronger theorem, which will be proved in Section 10.4.

Theorem 10.4. *For m odd, $n > m \geq 3$, $u(C_n, C_m) = 2(n-1)^2 + 2$. For m odd, $n \geq m \geq 3$ and $(m,n) \neq (3,3)$, $r_*(C_n, C_m) = n+1$.*

It is interesting to further investigate the existence of graph pairs (G_1, G_2) for which G_2 is G_1-good, and $\tau(G_1) = 1$ implies that G_2 is also G_1-size good. For a disjoint union of some graphs, like forests versus complete graphs, even though they do not fit the definition of goodness, one may also obtain their star-critical Ramsey numbers. In many of the situations, the same techniques that are used for obtaining their Ramsey numbers may give corresponding results for their star-critical Ramsey numbers. Usually, Turán-type results and some additional tricks are needed.

10.2 Proof of Theorem 10.2

To prove Theorem 10.2, our key ingredient is the following result of Burr [19].

Lemma 10.1 (Burr [19]). *Given two integers $k, l \geq 3$, if m and n are large, then*

$$R(nK_k, mK_l) = (k-1)n + (l-1)m + \max\{m,n\} + R(K_{k-1}, K_{l-1}) - 2.$$

We only consider the case $m \leq n$, because the other case is symmetric. Let $r = (k-1)n + (l-1)m + \max\{m,n\} + R(K_{k-1}, K_{l-1}) - 2$. We first give a coloring of the edges of K_{r-1}. Partition $V(K_{r-1})$ into three sets V_1, V_2, V_3 with $|V_1| = kn-1$, $|V_2| = (l-1)m - 1$ and $|V_3| = R(K_{k-1}, K_{l-1}) - 1$. Color all edges of $G[V_1]$ blue, all edges of $G[V_2]$ red, all edges joining V_1 and $V_2 \cup V_3$ red, and all edges joining V_2 and V_3 blue. Since $|V_3| = R(K_{k-1}, K_{l-1}) - 1$, we can give the edges of $G[V_3]$ a coloring such that $G[V_3]$ contains neither a blue K_{k-1} nor a red K_{l-1}. Under such a coloring, we see that K_{r-1} contains no blue nK_k. Since every red K_l needs at least $l-1$ vertices of V_3, it is clear that K_{r-1} does not contain m vertex-disjoint red K_ls. Let v be a vertex of V_3, and u an additional vertex such that u has the same red and blue adjacencies as v, and u is nonadjacent to v.

There is no blue K_k which contains u, and no red K_l which contains u. Thus, $d(u) = r-2$, and neither a blue nK_k nor a red mK_l is contained in the graph. Hence we conclude that (nK_k, mK_l) is Ramsey-full and $r_*(nK_k, mK_l) = r-1$. This completes the proof of Theorem 10.2. □

10.3 Proof of Theorem 10.3

Let $k = \chi(G_1)$, $t = |V(G_2)|$, $s = \sigma(G_1)$. Since G_2 is nontrivial, we have $t \geq 2$. If $k = 1$, then the inequality holds trivially. Thus, assume $k \geq 2$. If $t \leq s$, then $(k-1)(t-1)+s \leq (k-1)(s-1)+s < |V(G_1)|$, but $R(G_1,G_2) \geq |V(G_1)|$. Thus, G_2 is not G_1-good, which contradicts the condition that G_2 is G_1-good. For this reason, $t \geq s+1$. We see that $\tau(G_1) \geq 1$; otherwise there exists a vertex $v_0 \in V_1$ such that v_0 has no adjacent vertex in V_i for some i with $2 \leq i \leq k$. Then we can recolor v_0 with the same color of V_i and hence either $\chi(G_1) \leq \chi(G_1)-1$, a contradiction; or $s \leq |V_1|-1$, contradicting the choice of V_1.

Let $r = R(G_1,G_2) = (k-1)(t-1)+s$ and color a K_{r-1} as follows. Color the edges of a $(k-1)K_{t-1} \cup K_{s-1}$ blue, and all other edges red. Let $U_1, U_2, \ldots, U_{k-1}$ denote the vertex sets of $k-1$ copies of the blue K_{t-1}, and let U_k be the vertex set of the blue K_{s-1}. Now let u be an additional vertex, which is adjacent to each vertex of $U_2 \cup U_3 \cup \ldots \cup U_{k-1}$ with a red edge, adjacent to each vertex of U_k with a blue edge, and adjacent to U_1 with $\min\{\tau(G_1)-1, t-\delta(G_2)\}$ red edges and $\delta(G_2)-1$ blue edges. If $\tau(G_1)-1 \geq t-\delta(G_2)$, we have colored a graph $G = K_r$. If $\tau(G_1)-1 < t-\delta(G_2)$, we have colored a graph $G = K_{r-1} \sqcup K_{1,m}$, where $m = (\chi(G_1)-2)(|V(G_2)|-1)+\sigma(G_1)+\delta(G_2)+\tau(G_1)-3$.

We claim that the graph G with the above coloring contains no blue G_2 as a subgraph. If $\delta(G_2) = 1$, since $t \geq s+1$, every blue component has at most $|V(G_2)|-1$ vertices. Accordingly, there is no blue G_2 in that case. If $\delta(G_2) \geq 2$, the only blue component which has at least $|V(G_2)|$ vertices is formed by the blue edges of $U_1 \cup U_k \cup \{u\}$. If $s = 1$, then $U_k = \emptyset$, and this blue component has exactly $|V(G_2)|$ vertices. Then it has a vertex of degree $\delta(G_2)-1$, hence no blue G_2 is contained in that case. If $s \geq 2$, by the condition of Theorem 10.3, $\kappa(G_2) \geq 2$. Since both $G[U_1 \cup \{u\}]$ and $G[U_k \cup \{u\}]$ contain no blue G_2, if a blue G_2 is contained in $G[U_1 \cup U_k \cup \{u\}]$, then G_2 has at least one vertex in U_1 and at least one vertex in U_k. Thus, $\kappa(G_2) \leq 1$, contradicting the condition $\kappa(G_2) \geq 2$.

This proves our claim.

On the other hand, denote the subgraph of G formed by all the red edges by G^R. Suppose that G^R contains G_1 as a subgraph. If $s = 1$ and $\tau(G_1) = 1$, then $U_k = \emptyset$, and $\chi(G^R) = k-1$. Thus, G^R does not contain G_1 as a subgraph. If $s = 1$ and $\tau(G_1) \geq 2$, or if $s \geq 2$, then $\chi(G^R) = k$. Since G_1 is a subgraph of G^R, then any proper vertex coloring of G^R restricted to G_1 is also a proper vertex coloring of G_1. For $1 \leq i \leq k$, color the vertices of each U_i with color i, and color the vertex u with color k. We see that $\chi(G^R) = k$, $\sigma(G^R) = s$, and u is adjacent to at most $\tau(G_1)-1$ vertices in the color class U_1. When restricting this vertex coloring to the subgraph G_1, by the definition of $\tau(G_1)$, we have $\tau(G_1) \leq \tau(G_1)-1$, a contradiction which implies that G^R does not contain G_1 as a subgraph. Thus, $G \nrightarrow (G_1,G_2)$.

If $\tau(G_1)-1 \geq t-\delta(G_2)$, then $G = K_r$ and hence $K_r \nrightarrow (G_1,G_2)$, a contradiction. Thus, we have $\tau(G_1)-1 < t-\delta(G_2)$ and $G = K_{r-1} \sqcup K_{1,m}$, where $m = (\chi(G_1)-2)(|V(G_2)|-1)+\sigma(G_1)+\delta(G_2)+\tau(G_1)-3$. Since $G \nrightarrow (G_1,G_2)$, our proof is complete. □

10.4 Proof of Theorem 10.4

To prove Theorem 10.4, we need Theorems 1.2, 1.5 and 1.6 from Chapter 1, and we will also use the the following lemmas.

Lemma 10.2 (Faudree et al. [55])**.** *Let G be a graph of order $n \geq 6$. Then* $\max\{c(G), c(\overline{G})\} \geq \lceil 2n/3 \rceil$.

Lemma 10.3 (Ore [105])**.** *Every graph G of order n with $|E(G)| \geq (n-1)(n-2)/2+2$ contains a Hamilton cycle.*

Lemma 10.4. *Every graph G of order n with $|E(G)| \geq (n-1)(n-2)/2+2$ is pancyclic.*

Proof. If G is bipartite, then $|E(G)| \leq n^2/4 < (n-1)(n-2)/2+2$. Hence, G is nonbipartite. By Theorem 1.6, G is weakly pancyclic with $g(G) = 3$. By Lemma 10.3, G contains a Hamilton cycle. Thus, G is pancyclic. □

We first consider the case that m is odd and $n \geq m \geq 3$, and we prove that $r_*(C_n, C_m) \geq n+1$ and $u(C_n, C_m) \geq 2(n-1)^2+2$. In K_{2n-2}, color the edges of a complete bipartite graph $K_{n-1,n-1}$ blue and all the remaining edges red. Let u be an additional vertex which is adjacent to one bipartition class of $K_{n-1,n-1}$ with $n-1$ blue edges, and adjacent to the other class of $K_{n-1,n-1}$ with exactly one red edge. Now we have the graph $K_{2n-2} \sqcup K_{1,n}$. It contains neither a red C_n nor a blue C_m. Therefore, $r_*(C_n, C_m) \geq n+1$, and $u(C_n, C_m) \geq \binom{2n-2}{2} + r_*(C_n, C_m) \geq 2(n-1)^2+2$.

Next, let G be a subgraph of K_{2n-1} with $|E(G)| = 2(n-1)^2+2$. We show that $G \to (C_n, C_m)$ for m odd and $n > m \geq 3$. Since $|E(K_{2n-2})| < |E(G)|$, G has exactly $2n-1$ vertices. For any red-blue edge coloring of G, let G^R (G^B) be the graph whose vertex set is $V(G)$ and edge set consists of all red (blue) edges of G, respectively. Suppose to the contrary that neither G^R contains a C_n nor G^B contains a C_m. We see that G^R cannot be a bipartite graph; otherwise one of its bipartition classes, say U, has at least n vertices. Since $|E(G)| = |E(K_{2n-1})| - (n-3)$, then $G[U]$ is a nearly complete graph with at most $n-3$ edges deleted and all the other edges blue. By Lemma 10.4, $G[U]$ contains a blue C_m, a contradiction. Therefore, G^R is a nonbipartite graph. The same conclusion can be drawn for G^B. If $|E(G^R)| \geq (n-1)^2+2$, then by Theorems 1.2 and 1.6, G^R contains a C_n, a contradiction. If $|E(G^B)| \geq (n-1)^2+2$, then by Theorems 1.2 and 1.6, G^B contains a C_m, also a contradiction. Thus, we have $|E(G^R)| = |E(G^B)| = (n-1)^2+1$.

First assume $m = 3$, and let u be a maximum degree vertex in G^B. Then $d_{G^B}(u) \geq 2|E(G^B)|/(2n-1)$ and hence $d_{G^B}(u) \geq n-1$. Since G contains no blue triangle, $G[N_{G^B}(u)]$ contains no blue edges. If $d_{G^B}(u) \geq n$, since $|E(G)| = |E(K_{2n-1})| - (n-3)$, then $G[N_{G^B}(u)]$ is a nearly complete graph with at most $n-3$ edges deleted and all the other edges red. By Lemma 10.4, $G[N_{G^B}(u)]$ contains a red C_n, a contradiction. Thus, $d_{G^B}(u) = n-1$. If there are at most $n+1$ vertices whose blue degree is $n-1$, then all the other vertices are of degree at most $n-2$ and hence $(n+1)(n-1)+(n-2)(n-2) \geq 2(n-1)^2+2$, that is, $-1 \geq 0$, a contradiction. Thus, there are at least $n+2$ vertices whose blue degree is $n-1$. Let w be a vertex in $N_{G^B}(u)$ whose blue degree is $n-1$. Then $N_{G^B}(w)$ and $N_{G^B}(u)$ are vertex disjoint sets of order $n-1$. Let z be the remaining vertex which is neither in $N_{G^B}(w)$ nor in $N_{G^B}(u)$. Suppose that z is adjacent to $N_{G^B}(w)$ with k_1 blue edges, and adjacent to $N_{G^B}(u)$ with k_2 blue edges, and $k_1 \leq k_2$. Since there is no blue edge in $N_{G^B}(z)$, G^B contains at most $(n-1)^2 - k_1k_2 + k_1 + k_2$ blue edges. Since $|E(G^B)| = (n-1)^2 + 1$, we have $k_1 \le 1$. We see that neither $N_{G^B}(u)$ nor $N_{G^B}(w)$ contains blue edges. If $k_1 = 0$ or $G[N_{G^B}(w) \cup \{z\}]$ is a nearly complete graph with at most $n-4$ edges deleted, by Lemma 10.4, $G[N_{G^B}(w) \cup \{z\}]$ contains a red C_n, a contradiction. Thus, $k_1 = 1$ and $G[N_{G^B}(w) \cup \{z\}]$ is a nearly complete graph with exactly $n-3$ edges deleted. Then $G[N_{G^B}(u)]$ is a complete red graph K_{n-1}. Let $y \in N_{G^B}(w)$ such that yz is a blue edge. It follows that either $G[N_{G^B}(u) \cup \{y\}]$ contains a red C_n, or $G[N_{G^B}(u) \cup \{z\}]$ contains a red C_n, a contradiction. Therefore, we may assume that $m \ge 5$.

If $\delta(G^B) \ge (2n+1)/3$, by Theorem 1.5, G^B is weakly pancyclic with $g(G^B) \le 4$. Since $|E(G^B)| = (n-1)^2 + 1$, by Theorem 1.2, $c(G^B) \ge n$. Thus, G^B contains C_m, a contradiction. This implies that there is a vertex v in G^B with $d_{G^B}(v) \le \lfloor 2n/3 \rfloor$. We claim that $G^B - v$ is not a bipartite graph; otherwise let X, Y be the two bipartition classes of $G^B - v$ with $|X| \ge |Y|$. If $|X| \ge n$, since $|E(G)| = |E(K_{2n-1})| - (n-3)$, then $G[X]$ is a nearly complete graph with at most $n-3$ edges deleted and all the other edges red. By Lemma 10.4, $G[X]$ contains a red C_n, a contradiction. Thus we have $|X| = |Y| = n-1$. Since $|E(G^B)| = (n-1)^2 + 1$ and v is incident with at most $\lfloor 2n/3 \rfloor$ blue edges, then $|E(G^B - v)| \ge (n-1)^2 + 1 - \lfloor 2n/3 \rfloor$. There are at most $\lfloor 2n/3 \rfloor - 1$ red edges joining X and Y. This is because at most $|X||Y| = (n-1)^2$ edges join X and Y. Let H_1, H_2 be the subgraphs of G^R induced by $X \cup \{v\}$ and $Y \cup \{v\}$, respectively. Without loss of generality, assume that $|E(H_1)| \ge |E(H_2)|$. It follows that $|E(H_1)| + |E(H_2)| \ge |E(G^R)| - (\lfloor 2n/3 \rfloor - 1) = (n-1)^2 - \lfloor 2n/3 \rfloor + 2$. Since $n \ge 4$, we can check that $|E(H_1)| \ge \binom{n-1}{2} + 2$. By Lemma 10.3, H_1 contains a red C_n, a contradiction. This proves our claim that $G^B - v$ is a nonbipartite graph. Since $|E(G^B - v)| \ge (n-1)^2 + 1 - \lfloor 2n/3 \rfloor$, by Theorems 1.2 and 1.6, $G^B - v$ contains C_m, a final contradiction. Therefore, $u(C_n, C_m) = 2(n-1)^2 + 2$.

Since $u(C_n, C_m) \ge \binom{2n-2}{2} + r_*(C_n, C_m)$, we have $r_*(C_n, C_m) \le n+1$ for m odd and $n > m \ge 3$. It remains to consider the case $n = m$, which is $r_*(C_n, C_n) \le n+1$ for odd $n \ge 5$. For the graph $G = K_{2n-2} \sqcup K_{1,n+1}$, let x be the vertex with degree $n+1$. For any red-blue edge coloring of G, let G^R (G^B) be the graph whose vertex set is $V(G)$, and edge set consists of all red (blue) edges of G, respectively. We show that $G^R - x$ ($G^B - x$) is a nonbipartite graph. If $G^R - x$ is a bipartite graph, let X', Y' be the two bipartition classes of $G^R - x$. Since both X' and Y' are complete graphs whose edges are all blue, then $|X'| = |Y'| = n-1$. There is at most one blue edge joining X' and Y'; otherwise G contains a blue C_n. If x is adjacent to X' with one red edge, and adjacent to Y' with one red edge, then G contains a red C_n. Since x is incident with $n+1$ edges, then x is adjacent to one of X', Y' with at least two blue edges and hence G contains a blue C_n. Thus, $G^R - x$ is a nonbipartite graph and so is $G^B - x$. If $|E(G^R - x)| \geq ((2n-3)(n-1)+1)/2$, by Theorems 1.2 and 1.6, $G^R - x$ contains a red C_n. Thus, $|E(G^R - x)| \leq (2n-3)(n-1)/2$. For the same reason, $|E(G^B - x)| \leq (2n-3)(n-1)/2$. Since $G^R - x$ is the complement graph of $G^B - x$, we have $|E(G^R - x)| \geq (2n-3)(n-1)/2$ and so $|E(G^R - x)| = |E(G^B - x)| = (2n-3)(n-1)/2$. By Theorem 1.6, both $G^R - x$ and $G^B - x$ are weakly pancyclic with girth 3. By Lemma 10.2, G contains a monochromatic C_n, and our proof is complete. □

Summary

This thesis contains new contributions to Ramsey theory, in particular results that establish exact values of graph Ramsey numbers that were unknown to date. Such numbers show that there is structure in every sufficiently large graph, in the following sense. For every pair of graphs G_1 and G_2, there exists an integer N such that every graph on at least N vertices contains G_1, or its complement contains G_2; the smallest such integer is called the Ramsey number (for G_1 and G_2) and denoted by $R(G_1,G_2)$.

Obtaining exact values for $R(G_1,G_2)$ is usually difficult, and a popular area of research within the field of graph theory. This is explained in more detail in Chapter 1 of the thesis, where the main results of this thesis are presented together with the relevant terminology and (references to) related results from literature.

Burr showed back in 1984 that the problem of determining the exact value of $R(G_1,G_2)$ for arbitrary graphs G_1 and G_2 is NP-hard. This is another motivation for obtaining such values analytically by applying graph theoretical approaches. To be more specific, using techniques and results from graph theory, since the early 1970s researchers have been trying to confirm the exact value of Ramsey numbers for many well-studied families of graphs, including cycles, wheels, stars, paths, trees, fans and kipases. This is also the main motivation and approach behind most of the results in this thesis.

To give an example, there are many known degree conditions and edge conditions guaranteeing that a graph contains a cycle of (at least) a given length.

These existence results can be applied to establish exact values of Ramsey numbers involving cycles, but also involving wheels. For proving the existence of a wheel of a given order, it is sufficient to prove the existence of the associated cycle in the neighborhood of a vertex with a sufficiently high degree. This is the basic idea behind the newly obtained results on $R(C_m,W_n)$ scattered around Chapters 2, 3 and 4. These results can be summarized in the following way.

$$R(C_m,W_n)=\begin{cases} 2n+1 & \text{for } m \text{ odd, } n\geq 3(m-1)/2 \text{ and } (m,n)\neq(3,3),(3,4),\\ 3m-2 & \text{for } n \text{ odd, and } m<n\leq 3m/2-1,\\ 2m-1 & \text{for } n \text{ even, and } m\geq n+502,\\ 17 & \text{for } (m,n)=(7,8),\\ 16 & \text{for } (m,n)=(6,9). \end{cases}$$

Moreover, it is proved that $R(C_4,W_n)=R(C_4,K_{1,n})$ for $n\geq 6$. This shows that the study of $R(C_4,W_n)$ and $R(C_4,K_{1,n})$ coincides for $n\geq 6$, which is a nice meta-result. For the case of larger cycles versus generalized wheels, other researchers have established the Ramsey numbers of large cycles versus generalized even wheels. Analogous results for large cycles versus generalized odd wheels are obtained in the thesis. Other obtained results in this part of the thesis fill a considerable part of the gap in the existing knowledge on exact values of $R(C_m,K_{1,n})$.

The main result of Chapter 5 is that $R(T_n,F_m)=2n-1$ for all integers $n\geq 3m^2-2m-1$. Using similar proof techniques, it is shown that $R(F_n,F_m)=4n+1$ for $n\geq\max\{m^2-m/2,11m/2-4\}$ in Chapter 6. In the same chapter, exact values of Ramsey numbers are obtained for large fans versus wheels of even order, and for fans versus a complete graph of order five.

In Chapter 7, the gap in the existing knowledge on exact values of the Ramsey numbers $R(P_n,\widehat{K}_m)$ is completely closed by determining the exact values for the remaining open cases, that is $R(P_n,\widehat{K}_m)=\max\{2n-1,\lceil 3m/2\rceil-1,2\lfloor m/2\rfloor+n-2\}$ for $m\leq 2n-1$ and $m,n\geq 2$. The proof makes full use of the Ramsey numbers for cycles versus paths, and paths versus paths, together with some other

known results.

Exact values of Ramsey numbers involving arbitrary trees are hard to obtain, hence there are only a few known results for these Ramsey numbers. Therefore, the research is usually focused on restricted classes of trees, including paths and stars, and the ES-trees of Chapter 5 and brooms of Chapter 8. As one of the main results of Chapter 8, the exact values of all Ramsey numbers for brooms versus odd cycles have been established.

Using the notation of Chapter 10, $R(G_1,G_2)$ is the smallest number of vertices of a graph G such that $G \to (G_1,G_2)$. In case $R(G_1,G_2)$ is difficult to obtain for general graphs G, partial results may be obtained by restricting G to some special graph classes. In Chapter 9, G is restricted to be a planar graph. In this way all exact values are obtained for planar Ramsey numbers of complete graphs versus wheels, and for complete graphs with one edge deleted versus wheels. If G_1 is not a planar graph, then the planar Ramsey number $PR(G_1,G_2)$ is in fact the smallest $|V(G)|$ such that G contains G_2 and $\overline{G}$ is planar. Using this, more planar Ramsey numbers have been established.

In the final chapter, two new definitions in graph Ramsey theory are introduced: Ramsey-full graphs and size Ramsey good graphs. A comprehensive study on the two definitions is performed. Since these newly introduced notions have close ties with graph Ramsey numbers, more results may be obtained using these notions in the near future.

Throughout the thesis, the reader can find several open problems and conjectures, including the conjecture in Chapter 2 on the Ramsey numbers for cycles versus wheels, and the conjecture in Chapter 5 on the Ramsey numbers for trees versus fans. Hopefully, these challenges will spur the future research in this fascinating area.

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Yanbo Zhang
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