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POLYNOMIAL SOLUTION OF THE STANDARD H_2 PROBLEM

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Abstract

A polynomial solution to the standard linear H_2 problem is given, together with a detailed algorithm. The assumptions are very general and the paper includes necessary and sufficient conditions for the existence of a solution.

1. Introduction

In an earlier paper (Kwakernaak, 2000) a polynomial solution is outlined of the standard H_2 problem under quite general conditions. The present paper gives the details of the solution and the algorithm, corrects some mistakes, and states conditions for the existence of a solution not included in the earlier paper. The algorithm applies to a much wider class of problems than earlier solutions by Hunt *et al.* (1994), Kučera (1986, 1996), Kučera and Henrion (2000), and Meinsma (2000).

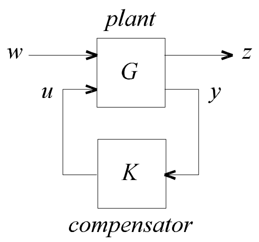

Fig. 1. Standard plant

2. Problem formulation

The open-loop transfer matrix of the standard feedback configuration in Fig. 1 is given by

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{1}$$

with dimensions

$$\begin{array}{c} \\ m_1 \\ m_2 \end{array} \begin{array}{c} \begin{matrix} k_1 & k_2 \end{matrix} \\ \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \end{array} \tag{2}$$

If the compensator transfer matrix is K then the closed-loop transfer matrix from the external signal w to the control output z is $H = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}$. The H_2 optimization problem is to find a feedback controller K that stabilizes the closed-loop system and minimizes the 2-norm $\|H\|_2$ of the closed-loop system. The norm is defined by

$$\|H\|_2^2 = \frac{1}{2\pi} \operatorname{tr} \int_{-\infty}^{\infty} H^T(-j\omega)H(j\omega)\, d\omega \tag{3}$$

3. Assumptions

For the solution of the H_2 problem we introduce the following modest assumptions:

a. G_{21} has full row rank and G_{12} has full column rank.

b. The fixed poles, that is, the uncontrollable and unobservable poles, lie in the closed left-half complex plane, possibly on the imaginary axis.

c. There exists a compensator so that the corresponding closed-loop transfer matrix is strictly stable and strictly proper.

If G_{12} does not have full column rank or G_{21} does not have full row rank then the solution is not unique. In this case the plant input and the measured output may be transformed to meet the assumptions without loss of performance.

Under the assumptions a and b no compensator may exist that makes the closed-loop transfer matrix strictly proper and cancels the fixed poles on the imaginary axis in the closed-loop transfer matrix. Both are needed for the 2-norm of the closed-loop transfer matrix to be finite. In Section 12 we derive necessary and sufficient conditions for the existence of a compensator that makes the closed-loop transfer matrix strictly proper and cancels the fixed poles on the imaginary axis.

The assumptions admit infinite-gain solutions, that is, compensators with a singular denominator.

4. Input data and data preparation

We assume that the generalized plant is specified in the left coprime polynomial matrix fraction form

$$G = D^{-1}N = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{4}$$

Given these data, a unimodular matrix U may be found such that

$$U \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} = \begin{bmatrix} D_{11}^* \\ 0 \end{bmatrix} \tag{5}$$

where D_{11}^* is square row reduced. By multiplication of both D and N on the left by U the left coprime fraction (4) takes the form

$$G = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{6}$$

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It is useful to arrange the fraction so that N_{21} is row reduced. We also need the left and right coprime fractions

$$G_{22} = \breve{D}_{22}^{-1}\breve{N}_{22} = \hat{N}_{22}\hat{D}_{22}^{-1} \tag{7}$$

By inspection of (6) we see that $G_{22} = D_{22}^{-1}N_{22}$. Hence we have $D_{22} = D_o\breve{D}_{22}$, $N_{22} = D_o\breve{N}_{22}$ with D_o a greatest common left divisor of D_{22} and N_{22}.

Obviously, there also exists a right coprime fractional representation of G of the form

$$G = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix} \begin{bmatrix} \bar{D}_{11} & 0 \\ \bar{D}_{21} & \bar{D}_{22} \end{bmatrix}^{-1} \tag{8}$$

with $\bar{N}_{12}$ column reduced. Since $G_{22} = \bar{N}_{22}\bar{D}_{22}^{-1}$ we see that $\bar{D}_{22} = \hat{D}_{22}\bar{D}_o$, $\bar{N}_{22} = \hat{N}_{22}\bar{D}_o$ with $\bar{D}_o$ a greatest common right divisor of $\bar{D}_{22}$ and $\bar{N}_{22}$.

Not all these polynomial matrices are needed for the actual algorithm, which is summarized in Section 14.

5. The closed-loop system

In signal form the open-loop system is given by

$$\begin{aligned} D_{11}z + D_{12}y &= N_{11}w + N_{12}u \\ D_{22}y &= N_{21}w + N_{22}w \end{aligned} \tag{9}$$

We assume that the compensator is given in the right polynomial matrix fractional form $K = YX^{-1}$. In signal form we have $u = YX^{-1}y$. Defining the latent signal $\xi = X^{-1}y$ the compensator is described by the equations

$$u = Y\xi, \quad y = X\xi \tag{10}$$

Substituting these equations into the open-loop equations and rearranging them we have

$$\begin{aligned} D_{11}z + (D_{12}X - N_{12}Y)\xi &= N_{11}w \\ (D_{22}X - N_{22}Y)\xi &= N_{21}w \end{aligned} \tag{11}$$

Hence, the closed-loop transfer matrix H is given by

$$H = D_{11}^{-1}N_{11} - D_{11}^{-1}(D_{12}X - N_{12}Y)(D_{22}X - N_{22}Y)^{-1}N_{22} \tag{12}$$

Inspection of the equations (11) shows that the generalized plant is stabilizable iff D_{11} is strictly Hurwitz and any greatest common left divisor D_o of D_{22} and N_{22} is strictly Hurwitz. The zeros of D_{11} together with those of D_o are the fixed poles of the system.

The assumptions of Section 3 allow fixed poles on the imaginary axis. The closed-loop transfer matrix H, however, cannot have any poles on the imaginary axis because otherwise its 2-norm is infinite. Hence, the compensator needs to cancel all fixed poles on the imaginary axis in H.

6. Youla-Kucera parametrization

The closed-loop characteristic polynomial matrix is $D_{22}X - N_{22}Y$. If D_{22} and N_{22} have a nontrivial greatest common left divisor D_o then the closed-loop characteristic polynomial is of the form

$$D_oD_{cl} = D_{22}X - N_{22}Y \tag{13}$$

Write $D_{22} = D_o\breve{D}_{22}$, $N_{22} = N_o\breve{N}_{22}$ and let X_o, Y_o be a solution of the Bezout equation $I = \breve{D}_{22}X_o - \breve{N}_{22}Y_o$. Then the general solution of the equation (13) is

$$X = X_oD_{cl} + \hat{N}_{22}P, \quad Y = Y_oD_{cl} + \hat{D}_{22}P \tag{14}$$

where P is any polynomial matrix of the correct dimensions. These expressions define the compensator as

$$K = YX^{-1} = (Y_o + \hat{D}_{22}Q)(X_o + \hat{N}_{22}Q)^{-1} \tag{15}$$

The strictly stable rational matrix $Q = PD_{cl}^{-1}$ is the Youla-Kučera parameter. Substituting X and Y into the closed-loop transfer matrix (12) we have

$$\begin{aligned} H = {} & D_{11}^{-1}N_{11} - D_{11}^{-1}(D_{12}X_o - N_{12}Y_o)D_o^{-1}N_{21} \\ & - D_{11}^{-1}(D_{12}\hat{N}_{22} - N_{12}\hat{D}_{22})QD_o^{-1}N_{21} \end{aligned} \tag{16}$$

This is the desired parametrization of the closed-loop transfer matrix, which may be advantageously rearranged as follows. From the equality

$$\begin{bmatrix} D_{11} & D_{12} & N_{11} & N_{12} \\ 0 & D_{22} & N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \\ -\bar{D}_{11} & 0 \\ -\bar{D}_{21} & -\bar{D}_{22} \end{bmatrix} = 0 \tag{17}$$

it follows that $D_{11}\bar{N}_{12} + D_{12}\bar{N}_{22} - N_{12}\bar{D}_{22} = 0$, or

$$D_{11}^{-1}(N_{12}\hat{D}_{22} - D_{12}\hat{N}_{22}) = \bar{N}_{12}\bar{D}_o^{-1} \tag{18}$$

Thus, we may rewrite the parametrization of the closed-loop transfer matrix as

$$H = H_o + \bar{N}_{12}\bar{D}_o^{-1}QD_o^{-1}N_{21} \tag{19}$$

where H_o is the stable (but not strictly stable) and not necessarily proper transfer matrix

$$H_o = D_{11}^{-1}N_{11} - D_{11}^{-1}(D_{12}X_o - N_{12}Y_o)D_o^{-1}N_{21} \tag{20}$$

Dual to the parametrization based on the left fractional representation of the generalized plant there exists a parametrization based on the right fractional representation. In this parametrization the compensator is given by the left fraction

$$K = \bar{X}^{-1}\bar{Y} \tag{21}$$

where

$$\bar{X} = \bar{X}_o + \bar{Q}\breve{N}_{22}, \quad \bar{Y} = \bar{Y}_o + \bar{Q}\breve{D}_{22} \tag{22}$$

Here we have $I = \bar{X}_o\hat{D}_{22} - \bar{Y}_o\hat{N}_{22}$. The matrix $\bar{Q} = \bar{D}_{cl}^{-1}\bar{P}$ is the parameter, with $\bar{D}_{cl}$ square strictly Hurwitz and $\bar{P}$ arbitrary. The closed-loop transfer matrix is now parametrized as

$$H = H_o + \bar{N}_{12}\bar{D}_o^{-1}\bar{Q}D_o^{-1}N_{21} \tag{23}$$

where

$$H_o = \bar{N}_{11}\bar{D}_{11}^{-1} - \bar{N}_{12}\bar{D}_o^{-1}(\bar{X}_o\bar{D}_{21} - \bar{Y}_o\bar{N}_{21})\bar{D}_{11}^{-1} \tag{24}$$

7. Reparametrization

The matrices X_o and Y_o that occur in the parametrization (19)–(20) are not unique, and, hence, the parametrization is not unique. We wish the parametrization to have the property that if $Q = 0$ then H has no poles on the imaginary axis and is strictly proper. Assume that for the parametrization (19) the closed-loop matrix H has no poles on the imaginary axis and is strictly proper if the parameter equals Q^o. If such a Q does not exist then the H_2 problem has no solution. We now reparametrize H as

$$H = H_o + \bar{N}_{12}\bar{D}_o^{-1}Q^o D_o^{-1}N_{21} + \bar{N}_{12}\bar{D}_o^{-1}(Q - Q^o)D_o^{-1}N_{21} \quad (25)$$

where $Q - Q^o$ is the new parameter. Since we wish the closed-loop system never to have poles on the imaginary axis we reparametrize H once again in the form

$$H = H_o^o + \bar{N}_{12}\bar{Q}N_{21} \quad (26)$$

where

$$H_o^o = H_o + \bar{N}_{12}\bar{D}_o^{-1}Q^o D_o^{-1}N_{21}, \quad \bar{Q} = \bar{D}_o^{-1}(Q - Q^o)D_o^{-1} \quad (27)$$

so that $\bar{Q}$ is strictly stable.

8. Optimality condition

By replacing the strictly stable parameter $\bar{Q}$ in (26) with $\bar{Q} + \varepsilon\tilde{Q}$, with ε a real parameter and $\tilde{Q}$ a strictly stable perturbation, differentiating $\|H\|_2^2$ with respect to ε and setting ε equal to 0 we obtain the necessary optimality condition

$$\mathrm{tr}\int_{-\infty}^{\infty} H^T(-j\omega)\bar{N}_{12}(j\omega)\tilde{Q}(j\omega)N_{21}(j\omega)\,d\omega = 0 \quad (28)$$

for any strictly stable perturbation $\tilde{Q}$. By the assumptions that G_{12} and G_{21} have full column and full row rank, respectively, the condition is also sufficient. It follows by Cauchy's theorem that

$$N_{21}H^{\sim}\bar{N}_{12} \quad (29)$$

needs to have all its poles in the open left half plane. Here we denote

$$X^{\sim}(s) = X^T(-s) \quad (30)$$

Define the polynomial spectral factorizations

$$N_{21}N_{21}^{\sim} = \Phi\Phi^{\sim}, \quad \bar{N}_{12}^{\sim}\bar{N}_{12} = \Psi^{\sim}\Psi \quad (31)$$

with Φ square Hurwitz and row reduced, and Ψ square Hurwitz and column reduced.

First assume that neither Φ nor Ψ has any roots on the imaginary axis and, hence, both are strictly Hurwitz. Then equivalently to (29) we have the necessary condition that

$$\Phi^{-1}N_{21}H^{\sim}\bar{N}_{12}\Psi^{-1} \quad (32)$$

be stable. Because $N_{12}\Psi^{-1}$ is inner, $\Phi^{-1}N_{21}$ co-inner, and H needs to be strictly proper also (32) needs to be strictly proper. Hence, a necessary and sufficient condition for optimality is that both H and $\Phi^{-1}N_{21}H^{\sim}\bar{N}_{12}\Psi^{-1}$ are strictly stable and strictly proper.

Next consider the case that, say, Φ has roots on the imaginary axis. Then N_{21} has the same roots and, in fact, Φ and N_{21} have a common left divisor which contains all these roots. This common left divisor cannot cancel within $N_{21}H^{\sim}\bar{N}_{12}$ because H cannot have any poles on the imaginary axis, but it cancels within the co-inner function $\Phi^{-1}N_{21}$. Hence, $N_{21}H^{\sim}\bar{N}_{12}$ is strictly stable iff $\Phi^{-1}N_{21}N_{21}H^{\sim}\bar{N}_{12}$ is strictly stable. A similar argument applies if Ψ has roots on the imaginary axis. The conclusion is that it is necessary and sufficient for optimality that both H and $\Phi^{-1}N_{21}N_{21}H^{\sim}\bar{N}_{12}$ are strictly stable and strictly proper, regardless of whether or not Φ and Ψ have roots on the imaginary axis.

9. Computation of the optimal parameter

Substituting $H = H_o + \bar{N}_{12}\bar{D}_o^{-1}QD_o^{-1}N_{21}$ into (32) we find that necessary and sufficient for optimality is that

$$(\Psi^{\sim})^{-1}\bar{N}_{12}^{\sim}H_o N_{21}^{\sim}(\Phi^{\sim})^{-1} + \Psi\bar{D}_o^{-1}QD_o^{-1}\Phi \quad (33)$$

be strictly anti-stable and strictly proper. Note that we have gone back to the original parametrization (19). Given $R = (\Psi^{\sim})^{-1}\bar{N}_{12}^{\sim}H_o N_{21}^{\sim}(\Phi^{\sim})^{-1}$ we may uniquely decompose

$$R = (\Psi^{\sim})^{-1}\bar{N}_{12}^{\sim}H_o N_{21}^{\sim}(\Phi^{\sim})^{-1} = R_+ + R_- \quad (34)$$

where R_+ is strictly proper and strictly anti-stable and R_- is stable (but not necessarily strictly stable) and typically non-proper. Then the optimality condition is satisfied iff we choose the parameter Q so that

$$R_- + \Psi\bar{D}_o^{-1}QD_o^{-1}\Phi = 0 \quad ,(35)$$

that is,

$$Q = -\bar{D}_o\Psi^{-1}R_-\Phi^{-1}D_o \quad (36)$$

10. Decomposition of R

The decomposition of R into its strictly anti-stable and stable parts requires the solution of two two-sided linear polynomial equations. We write

$$\begin{aligned} H_o &= D_{11}^{-1}N_{11} - D_{11}^{-1}(D_{12}X_o - N_{12}Y_o)D_o^{-1}N_{21} \\ &= \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{R_1}\underbrace{\begin{bmatrix} D_{11} & D_{12}X_o - N_{12}Y_o \\ 0 & D_o \end{bmatrix}^{-1}}_{T}\underbrace{\begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix}}_{R_2} \end{aligned} \quad (37)$$

so that

$$R = (\Psi^{\sim})^{-1}\bar{N}_{12}^{\sim}R_1T^{-1}R_2N_{21}^{\sim}(\Phi^{\sim})^{-1} \quad (38)$$

Note that the denominators $\Phi^{\sim}$ and $\Psi^{\sim}$ are strictly anti-Hurwitz while T is Hurwitz. By successively solving the two-sided linear polynomial matrix equations

$$A_1T + \Psi^{\sim}A_2 = \bar{N}_{12}^{\sim}R_1 \quad (39)$$

$$B_1\Phi^{\sim} + TB_2 = R_2N_{21}^{\sim} \quad (40)$$

for A_1, A_2 and B_1, B_2 we may write

$$R = (\Psi^{\sim})^{-1}(A_1 R_2 N_{21}^{\sim} + \bar{N}_{12}^{\sim} R_1 B_2 - A_1 T B_2)(\Phi^{\sim})^{-1} + A_2 T^{-1} B_1 \quad (41)$$

We study the term

$$(\Psi^{\sim})^{-1}(A_1 R_2 N_{21}^{\sim} + \bar{N}_{12}^{\sim} R_1 B_2 - A_1 T B_2)(\Phi^{\sim})^{-1} \quad (42)$$

Inspection shows that if both Φ and Ψ have no roots on the imaginary axis then this expression is strictly antistable. If, say, Φ has roots on the imaginary axis then as argued previously Φ and N_{21} have a common left divisor containing all these roots which cancels in the first term of the expression. Moreover, inspection of (40) shows that also $B_2^{\sim}$ has this left divisor, which cancels in the second and third terms of (42). Therefore, any roots of Φ on the imaginary axis cancel within (42), and, hence, are not poles. It similarly may be argued that any roots of Ψ on the imaginary axis also cancel and therefore there are not poles of (42). Thus, (42) is strictly antistable, regardless of whether or not Φ and Ψ have roots on the imaginary axis.

Denoting the polynomial part of (42) as P we see that the polynomial and stable part of *R* is given by

$$R_{-} = P + A_2 T^{-1} B_1 \quad (43)$$

The solution of the bilateral equations (39) and (40) may be simplified. Consider the bilateral equation (39). We detail it as

$$\begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12}X_o - N_{12}Y_o \\ 0 & D_o \end{bmatrix} + \Psi^{\sim} \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} = \bar{N}_{12}^{\sim} \begin{bmatrix} I & 0 \end{bmatrix} \quad (44)$$

or

$$\begin{aligned} A_{11}D_{11} + \Psi^{\sim} A_{21} &= \bar{N}_{12}^{\sim} \\ A_{12}D_o + \Psi^{\sim} A_{22} &= -A_{11}(D_{12}X_o - N_{12}Y_o) \end{aligned} \quad (45)$$

After the first equation has been solved for A_{11} and A_{21} the second may be solved for A_{12} and A_{22}.

The bilateral equation (40) may be rendered as

$$\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} \Phi^{\sim} + \begin{bmatrix} D_{11} & D_{12}X_o - N_{12}Y_o \\ 0 & D_o \end{bmatrix} \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} N_{21}^{\sim} \quad (46)$$

This, in turn, we rewrite as

$$\begin{aligned} B_{11}\Phi^{\sim} + D_{11}B_{21} + (D_{12}X_o - N_{12}Y_o)B_{22} &= N_{11}N_{21}^{\sim} \\ B_{12}\Phi^{\sim} + D_o B_{22} &= \Phi\Phi^{\sim} \end{aligned} \quad (47)$$

Inspection shows that the second equation has the solution $B_{12} = \Phi$, $B_{22} = 0$, so that the first equation reduces to

$$B_{11}\Phi^{\sim} + D_{11}B_{21} = N_{11}N_{21}^{\sim} \quad (48)$$

11. Computation of the compensator

The parameter

$$Q = -\bar{D}_o \Psi^{-1}(P + A_2 T^{-1} B_1)\Phi^{-1} D_o \quad (49)$$

may now be computed as follows. Convert

$$\Phi^{-1} D_o = d_o \phi^{-1}, \quad \bar{D}_o \Psi^{-1} = \psi^{-1} \bar{d}_o \quad (50)$$

It follows that

$$Q = -\psi^{-1}(\bar{d}_o P d_o + \bar{d}_o A_2 T^{-1} B_1 d_o)\phi^{-1} \quad (51)$$

We have

$$\bar{d}_o A_2 T^{-1} B_1 d_o = \underbrace{\begin{bmatrix} \bar{d}_o A_{21} D_{11}^{-1} & \bar{d}_o A_{22} \end{bmatrix}}_{a_2} \begin{bmatrix} I & D_{12}X_o - N_{12}Y_o \\ 0 & I \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} B_{11} d_o \\ \phi \end{bmatrix}}_{b_1} \quad (52)$$

This shows that the factor D_o in the denominator of this expression cancels. By using the dual parametrization it follows that also D_{11} cancels. Hence, a_2 is polynomial. We thus have

$$\bar{d}_o A_2 T^{-1} B_1 d_o = a_2 \begin{bmatrix} I & -D_{12}X_o + N_{12}Y_o \\ 0 & I \end{bmatrix} b_1 = q \quad (53)$$

Hence,

$$Q = -\psi^{-1} \underbrace{(\bar{d}_o P d_o + q)}_{p} \phi^{-1} = -\bar{p}\bar{\psi}^{-1}\phi^{-1} \quad (54)$$

where $\psi^{-1} p = \bar{p}\bar{\psi}^{-1}$. It follows that the optimal compensator is given by $K = \tilde{Y}\tilde{X}^{-1}$, where

$$\tilde{Y} = Y_o \phi \bar{\psi} - \hat{D}_{22}\bar{p}, \quad \tilde{X} = X_o \phi \bar{\psi} - \hat{N}_{22}\bar{p} \quad (55)$$

12. Existence tests

To derive necessary and sufficient conditions for the existence of a compensator that makes the closed-loop transfer matrix *H* strictly stable and strictly proper we consider the expression $H = H_o + \bar{N}_{12}\bar{D}_o^{-1} Q D_o^{-1} N_{21}$ for the closed-loop transfer matrix. Given the wide row reduced polynomial matrix N_{21} of full row rank we define another wide row reduced polynomial matrix M_{21} of maximal full row rank such that $N_{21}M_{21}^{\sim} = 0$. Besides the spectral co-factorization $N_{21}N_{21}^{\sim} = \Phi\Phi^{\sim}$ we introduce the co-factorization $M_{21}M_{21}^{\sim} = \bar{\Phi}\bar{\Phi}^{\sim}$ and define the square rational matrix

$$V = \begin{bmatrix} \Phi^{-1} N_{21} \\ \bar{\Phi}^{-1} M_{21} \end{bmatrix} \quad (56)$$

It may be proved that V is nonsingular, inner, co-inner, strictly stable, and has no zeros on the imaginary axis. Because V is inner nonsingular we have $V^{-1} = V^{\sim}$. V is biproper and neither V nor V^{-1} has poles on the imaginary axis.

Similarly, given the tall full rank matrix $\bar{N}_{12}$ we define the tall column reduced matrix $\bar{M}_{21}$ with maximal and full column rank so that $\bar{M}_{12}^{\sim}\bar{N}_{12} = 0$. Besides the spectral factorization $\bar{N}_{12}^{\sim}\bar{N}_{12} = \Psi^{\sim}\Psi$ we introduce the factorization $\bar{M}_{12}^{\sim}\bar{M}_{12} = \bar{\Psi}^{\sim}\bar{\Psi}$, and define the square rational matrix

$$\bar{V} = \begin{bmatrix} \bar{N}_{12}\Psi^{-1} & \bar{M}_{12}\bar{\Psi}^{-1} \end{bmatrix} \quad (57)$$

$\bar{V}$ is nonsingular, inner, co-inner, strictly stable, and has no zeros on the imaginary axis. Because it is inner nonsingular we have $\bar{V}^{-1} = \bar{V}^{\sim}$. $\bar{V}$ is biproper and neither $\bar{V}$ nor $\bar{V}^{-1}$ has poles on the imaginary axis.

Obviously, H is strictly proper and has no poles on the imaginary axis if and only if $\bar{V}^{\sim} H V^{\sim}$ is strictly proper and has no poles on the imaginary axis. Consider

$$\bar{V}^{\sim} H V^{\sim} = \bar{V}^{\sim} H_o V^{\sim} + \begin{bmatrix} \Psi \\ 0 \end{bmatrix} \bar{D}_o^{-1} Q D_o^{-1} [\Phi \quad 0] \tag{58}$$

The optimal parameter Q is so constructed that the (1,1) block of this expression is strictly anti-stable and strictly proper, and, hence, has no poles on the imaginary axis and is strictly proper. Therefore, a necessary and sufficient condition for the corresponding candidate optimal compensator to make H strictly stable and strictly proper is that the remaining three blocks have no poles on the imaginary axis and are strictly proper. Obviously, the second block row $(\bar{\Psi}^{\sim})^{-1} \bar{M}_{12}^{\sim} H_o V^{\sim}$ has no poles on the imaginary axis and is strictly proper if and only if

$$(\bar{\Psi}^{\sim})^{-1} \bar{M}_{12}^{\sim} H_o \tag{59}$$

has no poles on the imaginary axis and is strictly proper. Similarly, the second block column $\bar{V}^{\sim} H_o M_{21}^{\sim} (\bar{\Phi}^{\sim})^{-1}$ has no poles on the imaginary axis and is strictly proper if and only if

$$H_o M_{21}^{\sim} (\bar{\Phi}^{\sim})^{-1} \tag{60}$$

has no poles on the imaginary axis and is strictly proper.

First consider

$$H_o M_{21}^{\sim} (\bar{\Phi}^{\sim})^{-1} = D_{11}^{-1} N_{11} M_{21}^{\sim} (\bar{\Phi}^{\sim})^{-1} \tag{61}$$

It is not difficult to check whether this expression is strictly proper and has no poles on the imaginary axis.

We next consider a convenient and symmetric way of checking (59). Based on the dual parametrization (23)–(24) the expression (59) may be rendered as

$$(\bar{\Psi}^{\sim})^{-1} \bar{M}_{12}^{\sim} \bar{N}_{11} \bar{D}_{11}^{-1} \tag{62}$$

It is also easy to check whether this expression is strictly proper and has no poles on the imaginary axis.

The strict properness and absence of poles on the imaginary axis of (61) and (62) guarantee the existence of a compensator that makes the closed-loop transfer matrix H both strictly stable and strictly proper. The conditions may be tested before executing the actual algorithm for the computation of the optimal compensator.

13. The case when $\bar{N}_{12}$ or N_{21} has zeros on the imaginary axis

Any zeros of G_{12} on the imaginary axis are also zeros of $\bar{N}_{12}$. Even if G_{12} has no zeros on the imaginary axis then $\bar{N}_{12}$ still may have zeros on the imaginary axis. Any zeros of $\bar{N}_{12}$ are also zeros of the spectral factor Ψ. Because all or some of the zeros of Ψ are closed-loop poles the closed-loop system may no longer be strictly stable if $\bar{N}_{12}$ has zeros on the imaginary axis. In this case no strictly optimal solution exists but there exists a stabilizing compensator that approaches the minimal H_2 norm arbitrarily closely.

Similarly, if N_{21} has zeros on the imaginary axis then these may reappear as closed-loop poles so that no strictly optimal solution exists.

14. Summary of the algorithm

1. We assume that the generalized plant is specified in the left coprime polynomial matrix fraction form

$$G = D^{-1} N = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{63}$$

Determine a unimodular matrix U so that

$$U \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} = \begin{bmatrix} D_{11}^{*} \\ 0 \end{bmatrix} \tag{64}$$

where D_{11}^{*} is square row reduced. By multiplication of both D and N on the left by U the left coprime fraction (63) takes the form

$$G = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{65}$$

Determine another unimodular matrix V such that VN_{21} is row reduced. Multiply the second blocks of rows of the denominator and numerator matrices by V so as to make N_{21} row reduced. If D_{11} has zeros in the open right-half plane then exit.

Reduce $D_{22}^{-1} N_{22} = \hat{N}_{22} \hat{D}_{22}^{-1} = \breve{D}_{22}^{-1} \breve{N}_{22}$ with $\hat{N}_{22}$ and $\hat{D}_{22}$ right coprime and $\breve{N}_{22}$ and $\breve{D}_{22}$ left coprime. Compute D_o (by right division) so that $D_{22} = D_o \breve{D}_{22}$, $N_{22} = D_o \breve{N}_{22}$. Convert

$$D_{11}^{-1} (N_{12} \hat{D}_{22} - D_{12} \hat{N}_{22}) = \bar{N}_{12} \bar{D}_o^{-1} \tag{66}$$

to find $\bar{D}_o$ and $\bar{N}_{12}$ such that $\bar{N}_{12}$ is column reduced. If D_o has roots in the open right-half plane then exit.

3. If it is desired to check the existence of an optimal compensator then determine a right coprime factorization of the plant of the form

$$G = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix} \begin{bmatrix} \bar{D}_{11} & 0 \\ \bar{D}_{21} & \bar{D}_{22} \end{bmatrix}^{-1} \tag{67}$$

with $\bar{N}_{12}$ column reduced.

Compute a tall column reduced matrix $\bar{M}_{21}$ with maximal and full column rank so that $\bar{M}_{12}^{\sim} \bar{N}_{12} = 0$, and check whether $(\bar{\Psi}^{\sim})^{-1} \bar{M}_{12}^{\sim} \bar{N}_{11} \bar{D}_{11}^{-1}$ is strictly proper and has no poles on the imaginary axis. $\bar{\Psi}$ is the spectral-factor defined by $\bar{M}_{12}^{\sim} \bar{M}_{12} = \bar{\Psi}^{\sim} \bar{\Psi}$. It is not necessary to compute this spectral factor actually because (a) it is guaranteed to have no zeros on the imaginary axis, and (b) to test for strict properness only the column degrees of $\bar{\Psi}$ are needed, which are equal to the column degrees of $\bar{M}_{12}$.

Compute a wide row reduced polynomial matrix M_{21} of maximal full row rank such that $N_{21}\tilde{M}_{21} = 0$, and check whether $D_{11}^{-1} N_{11} \tilde{M}_{21} (\bar{\Phi}^{\sim})^{-1}$ is strictly proper and has no roots on the imaginary axis. $\bar{\Phi}$ is the spectral co-factor defined by $M_{21}\tilde{M}_{21} = \bar{\Phi}\bar{\Phi}^{\sim}$. It is not necessary to compute this spectral co-factor because (a) it is guaranteed to have no zeros on the imaginary axis, and (b) to test for strict properness only the row degrees of $\bar{\Phi}$ are needed, which are equal to the row degrees of M_{21}.

If either test fails then exit because the H_2 problem does not have a solution.

4. Perform the spectral factorizations $N_{21}\tilde{N}_{21} = \Phi\Phi^{\sim}$, $\tilde{\bar{N}}_{12}\bar{N}_{12} = \Psi^{\sim}\Psi$, with Φ square strictly Hurwitz and row reduced, and Ψ square strictly Hurwitz and column reduced. Solve the Diophantine equation $I = \breve{D}_{22} X_o - \breve{N}_{22} Y_o$ for X_o and Y_o.

5. Let

$$R_1 = \begin{bmatrix} I_{m_1 \times m_2} & 0_{m_1 \times m_2} \end{bmatrix}$$
$$T = \begin{bmatrix} D_{11} & D_{12}X_o - N_{12}Y_o \\ 0_{m_2 \times m_1} & D_o \end{bmatrix}, \quad R_2 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} \tag{68}$$

Solve the two-sided polynomial matrix equations

$$\begin{aligned} A_{11}D_{11} + \Psi^{\sim} A_{21} &= \tilde{\bar{N}}_{12} \\ A_{12}D_o + \Psi^{\sim} A_{22} &= -A_{11}(D_{12}X_o - N_{12}Y_o) \end{aligned} \tag{69}$$

After the first equation has been solved for A_{11} and A_{21} the second may be solved for A_{12} and A_{22}. Let

$$A_1 = [A_{11} \quad A_{12}], \quad A_2 = [A_{21} \quad A_{22}] \tag{70}$$

Solve the bilateral equation

$$B_{11}\Phi^{\sim} + D_{11}B_{21} = N_{11}\tilde{N}_{21} \tag{71}$$

and let

$$B_1 = \begin{bmatrix} B_{11} \\ \Phi \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ 0_{m_2 \times m_2} \end{bmatrix} \tag{72}$$

Compute the polynomial part *P* of

$$(\Psi^{\sim})^{-1}(A_1 R_2 \tilde{N}_{21} + \tilde{\bar{N}}_{12} R_1 B_2 - A_1 T B_2)(\Phi^{\sim})^{-1} \tag{73}$$

6. Convert $\Phi^{-1} D_o = d_o \phi^{-1}$, $\bar{D}_o \Psi^{-1} = \psi^{-1} \bar{d}_o$, and let

$$a_2 = \begin{bmatrix} \bar{d}_o A_{21} D_{11}^{-1} & \bar{d}_o A_{22} \end{bmatrix}, \quad b_1 = \begin{bmatrix} B_{11} d_o \\ \phi \end{bmatrix} \tag{74}$$

The factor D_{11} in the denominator of a_2 is guaranteed to cancel. Let

$$p = \bar{d}_o P d_o + a_2 \begin{bmatrix} I & -D_{12}X_o + N_{12}Y_o \\ 0 & I \end{bmatrix} b_1 \tag{75}$$

and convert $\psi^{-1} p = \bar{p} \bar{\psi}^{-1}$.

Compute the numerator *Y* and denominator *X* of the optimal compensator $K = YX^{-1}$ according to

$$Y = Y_o \phi \bar{\psi} - \hat{D}_{22} \bar{p}, \quad X = X_o \phi \bar{\psi} - \hat{N}_{22} \bar{p} \tag{76}$$

15. Conclusions

The very mild assumptions on the H_2 problem allow the use of nonproper weighting functions and shaping filters with poles on the imaginary axis. This makes the algorithm of this paper an excellent tool for the application of H_2 optimization to the design of linear feedback systems with good robustness and performance (Kwakernaak, 2000).

The algorithm has been implemented with the help of the Polynomial Toolbox for MATLAB. A macro for use with version 2 of the Toolbox is available at the websites `www.polyx.com` or `www.polyx.cz`. From the same locations a document may be downloaded that describes a number of examples and applications of the polynomial solution of the standard H_2 problem.

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