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J.W. Nieuwenhuis, C. Praagman, H.L. Trentelman
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Open Loop Stabilizability of Unitary Groups

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1 Introduction

In this paper we study the stabilizability problem for systems whose open loop is governed by a unitary group of strongly continuous operators. Unitary groups appear naturally when studying conservative systems, for instance undamped wave and beam equations. We assume here that the input operator is bounded and one-dimensional. It is know from the work of Jacobson & Nett [6] and Nefedov & Scholokhovich [7] that such a system is not stabilizable by a bounded linear feedback operator. Furthermore, from Curtain [1] and Rebarber [8] it is known that such a system is not stabilizable by an unbounded but "admissible" feedback. However, this does not imply that it will never be stabilizable by a feedback which is unbounded and/or nonlinear. In this paper we take a open-loop approach to stabilization: roughly speaking, we ask whether for each initial condition there exists a control so that the resulting state trajectory decays exponentially. If a system cannot be open loop stabilized, it cannot be closed loop stabilized, but there are systems which are not closed loop stabilizable which are open loop stabilizable.

In section 2 we give a detailed description of the problem along with some preliminary results. In particular, we relate the problem of open loop stabilization to an interpolation problem. In section 3 we give necessary conditions for the system to be open loop stabilizable. These conditions are given in terms of the eigenvalues of the infinitesimal generator and the Fourier coefficients of the input operator. In section 4 we give sufficient conditions for a system governed by a unitary group with a with bounded one-dimensional input operator to be open loop stabilizable. In the last section we illustrate these results with a simple example. In this section we also show that some results of Zwart in [11] are incorrect.

2 Problem formulation

In this paper we consider the following abstract differential equation on a separable Hilbert space Z:

$$\dot{z}(t) = Az(t) + bu(t), \qquad t \geq 0, \tag{1}$$

$$z(0) = z_0. \tag{2}$$

We assume that $z_0 \in Z$, $b \in Z$, and that A is the infinitesimal generator of an strongly continuous unitary group $T(t)$, so A is skew-adjoint. We denote this system by $\Sigma(A,b)$.

Since the system (1), (2) is not stabilizable by a bounded or admissible feedback law, we are led to the natural question of whether it is possible to stabilize the system, in the sense of Definition 2.1 below, by more general controls $u(t)$. From the work of Datko [3, Theorem 4] it follows that if we restrict ourselves to controls in $\boldsymbol{L}_2(0,\infty)$, then this will not be possible. Therefore, we have to use a larger class of controls. To insure generality, we choose the class of distributions. We first introduce some notation.

- $\mathcal{D}'$ denotes the class of distributions, see Schwartz [9].
- $\mathbb{C}_\alpha^+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$.
- $\boldsymbol{H}_2(\mathbb{C}_\alpha^+; Z)$ will denote the Hilbert space of all holomorphic functions from $\mathbb{C}_\alpha^+$ to Z such that

$$\sup_{r>\alpha} \int_{-\infty}^{\infty} \|f(r+\jmath\eta)\|^2 d\eta < \infty.$$

- $\boldsymbol{H}_2(\mathbb{C}_\alpha^+)$ denotes the Hardy space $\boldsymbol{H}_2(\mathbb{C}_\alpha^+, \mathbb{C})$.
- $\boldsymbol{H}_2 := \boldsymbol{H}_2(\mathbb{C}_0^+)$.

Since we are dealing with distributions, we have to define the solution of the abstract equation (1). We say that $z(t)$ is the solution of (1), (2) with input u if for every $w \in Z$ the following holds:

$$\langle w, z(t)\rangle = \langle w, T(t)z_0\rangle + \langle w, T(\cdot)b\rangle * u(\cdot),$$

where $\langle \cdot,\cdot\rangle$ is the inner product in Z, $*$ denotes the convolution product, and equality is in the sense of distributions. This formula is of course a generalization of the familiar variation of constants formula.

Definition 2.1 The system $\Sigma(A,b)$ is open-loop stabilizable if there exists $\alpha < 0$ such that for every $z_0 \in Z$ we can find $u \in \mathcal{D}'$ with support in $[0,\infty)$ such that the solution of (1), (2) satisfies

$$\int_0^\infty e^{-2\alpha t}\|z(t)\|^2 dt < \infty.$$

In this definition we allow arbitrary distributions. However, in the next theorem we see that there is no loss of generality if we restrict the controls to derivatives of $\boldsymbol{L}_2(0,\infty)$-functions.

Theorem 2.2 *(Zwart [10]) The system $\Sigma(A,b)$ is open loop stabilizable if and only if then for every $z_0 \in Z$ there exists a $\xi \in \boldsymbol{H}_2(\mathbb{C}_\alpha^+; Z)$ and a scalar valued function ω such that*

$$z_0 = (sI-A)\xi(s) - b\omega(s), \qquad s \in \mathbb{C}_\alpha^+ \tag{3}$$

and $\omega/(r-\cdot)$ is in $\boldsymbol{H}_2(\mathbb{C}_\alpha^+)$ for every $r < \alpha$.

In this theorem open loop stabilizability is rewritten in the frequency domain, where ω is the Laplace transform of u. The condition that $\omega/(r-\cdot)$ be in $\boldsymbol{H}_2(\mathbb{C}_\alpha^+)$ is equivalent to u being the derivative of a function in $\boldsymbol{L}_2(0,\infty)$ when multiplied by $e^{-\alpha\cdot}$.

Theorem 2.3 *If the system $\Sigma(A,b)$ is open loop stabilizable, then the spectrum of A is pure point spectrum with multiplicity one. Furthermore, the eigenfunctions $\{\phi_n\}$ form an orthonormal basis for Z.*

Proof That the spectrum of A is pure point spectrum follows from Theorem 3.5 in Zwart [10]. Now A is skew-adjoint, this implies that the eigenfunctions form a orthonormal basis. Since open-loop stabilizability implies that the (unstable) eigenvalues are controllable, and since the input space is one-dimensional and A is skew-adjoint, we have that the multiplicity is one. ∎

This theorem shows that if a conservative system is open loop stabilizable, then the resolvent of A must be compact.

3 Necessary conditions for open loop stabilizability

Let $\{\lambda_n\}$ be the eigenvalues of A and $\{\phi_n\}$ be the associated eigenvectors of A. The following theorem shows the relationship between the problem of open loop stabilizability and an interpolation problem.

Theorem 3.1 *Let $r < \alpha$. The system $\Sigma(A,b)$ is open loop stabilizable, if and only if for every z_0 there exists $\tilde{\omega} \in \boldsymbol{H}_2(\mathbb{C}_\alpha^+)$ and $\xi \in \boldsymbol{H}_2(\mathbb{C}_\alpha^+; Z)$ such that*

1. *the (ξ,ω)-representation holds*

$$z_0 = (sI-A)\xi(s) - (r-s)b\tilde{\omega}(s), \quad s \in \mathbb{C}_\alpha^+; \tag{4}$$

2. *there exist constant $M_1, M_2 > 0$ such that*

$$\|\xi\|_{\boldsymbol{H}_2(\mathbb{C}_\alpha^+;Z)} \le M_1\|z_0\|, \tag{5}$$

$$\|\tilde{\omega}\|_{\boldsymbol{H}_2(\mathbb{C}_\alpha^+)} \le M_2\|z_0\|; \textit{ and} \tag{6}$$

3. *the following interpolation holds*

$$\tilde{\omega}(\lambda_n) = -\frac{\langle z_0,\phi_n\rangle}{(r-\lambda_n)\langle b,\phi_n\rangle}, \quad \text{for all } n. \tag{7}$$

Proof *1.* This follows directly from Theorem 2.2 by defining $\tilde{\omega} = \omega/(r-\cdot)$.
2. This follows easily from the Baire Category Theorem.
3. Taking the inner product of (4) with ϕ_n and substituting $s = \lambda_n$ gives

$$\langle z_0,\phi_n\rangle = -(r-\lambda_n)\langle b,\phi_n\rangle\tilde{\omega}(\lambda_n).$$

Since this holds for every z_0 we conclude that $\langle b,\phi_n\rangle \ne 0$, and hence (7) holds. ∎

With this last theorem we can prove one of the main results of this paper.

Theorem 3.2 *If $\Sigma(A,b)$ is open loop stabilizable, then*

1. *there exists a $\delta > 0$ such that*

$$\inf_{n\ne m} |\lambda_n - \lambda_m| = \delta; \tag{8}$$

2. *the following holds for $b_n := \langle b,\phi_n\rangle$:*

$$\inf_n |\lambda_n b_n| > 0, \tag{9}$$

and

$$\sup_{n\ne m} \left|\frac{b_n}{(\lambda_m-\lambda_n)b_m}\right| < \infty. \tag{10}$$

Proof *1.* Taking $z_0 = \phi_m$ and $s = \lambda_n$, $n \ne m$ in (4) and using (7) gives:

$$\phi_m = (\lambda_n I - A)\xi(\lambda_n).$$

Hence $1 = \langle\phi_m,\phi_m\rangle = (\lambda_n-\lambda_m)\langle\xi(\lambda_n),\phi_m\rangle$, and thus

$$\begin{aligned}|\lambda_n-\lambda_m| &= |\langle\xi(\lambda_n),\phi_m\rangle|^{-1}\\ &\ge \sqrt{2\alpha}\,\|\langle\xi(\cdot),\phi_m\rangle\|^{-1}_{\boldsymbol{H}_2(\mathbb{C}_\alpha^+)},\\ &\quad \text{since } \xi \in \boldsymbol{H}_2(\mathbb{C}_\alpha^+;Z)\\ &\ge \frac{\sqrt{2\alpha}}{M_1}, \qquad \text{by Theorem 3.1.b.}\end{aligned}$$

2. Taking $z_0 = \phi_m$ and $s = \lambda_m$ in (4) and using (7) gives

$$\phi_m = (\lambda_m I - A)\xi(\lambda_m) - b\frac{1}{\langle b, \phi_m\rangle}.$$

Hence for $n \neq m$, there holds

$$0 = \langle \phi_m, \phi_n\rangle = (\lambda_m - \lambda_n)\langle \xi(\lambda_m), \phi_n\rangle - \frac{\langle b, \phi_n\rangle}{\langle b, \phi_m\rangle}.$$

Using an argument similar to the proof of part a. we can prove (10). (9) follows from (10) by taking n fixed. ■

Hence, if the system is open loop stabilizable, then the poles have to be separated, and the Fourier coefficients of b go to zero at least as fast as λ_n^{-1}.

In this section we have seen that the property of open loop stabilizability imposes strong conditions on our system. In the next section we give sufficient conditions for the open loop stabilizability of a class of systems.

4 Sufficient conditions for open loop stabilizability

Before we can prove Theorem 4.2 we need the following lemma.

Lemma 4.1 *Let $\alpha \in \mathbf{R}$ and $\{\eta_k\}$ satisfy, for some real γ_1 and γ_2,*

$$\alpha < \gamma_1 < \mathrm{Re}(\eta_k) < \gamma_2,$$

$$\inf_{n\neq m} |\mathrm{Im}(\eta_n - \eta_m)| = \delta > 0.$$

Then there exists $M_3 > 0$ such that for all $\tilde{\omega} \in \boldsymbol{H}_2(\mathbf{C}_\alpha^+)$,

$$\sum_k |\tilde{\omega}(\eta_k)|^2 \leq M_3 \|\tilde{\omega}\|^2_{\boldsymbol{H}_2(\mathbf{C}_\alpha^+)}. \quad (11)$$

Proof We shall use the Carleson measure theorem (see Ho and Russell [5]). Define the following measure on the Borel subsets of $\mathbf{C}_\alpha^+$:

$$\mu(\eta_k) = 1,$$

$$\mu(\{s \mid \mathrm{Re}(s) > \alpha\}) \setminus \{\eta_k\}) = 0.$$

We now follow the proof of Corollary 2.5 in Ho and Russel [5] to see that μ is a Carleson measure and that (11) is therefore true. ■

Now we can give some sufficient conditions for a unitary system to be open loop stabilizable. We shall assume that $\Sigma(A, b)$ is open loop stabilizable, so the conclusions of Theorem 2.3 hold, and $\{\lambda_n\}$ satisfies (8).

Theorem 4.2 *Let $\Sigma(A, b)$ be open loop stabilizable, and let the b_n satisfy $m \leq |\lambda_n b_n| \leq M$ for some positive m and M. Then the system $\Sigma(A, b)$ is open loop stabilizable.*

Proof In order to show that $\Sigma(A, b)$ is open loop stabilizable, we need to show that the interpolation problem (7), (6) is solvable, and that (5) holds for ξ given by (4). For a given $z_0 \in Z$, let $z_n = \langle z_0, \phi_n\rangle$.

Let $\alpha < 0$. It follows from the interpolation results in Garnett [4] that if $m \leq |\lambda_n b_n|$, then the interpolation problem

$$\tilde{\omega}(\lambda_n) = \frac{-z_n}{(r - \lambda_n)b_n}$$

is solvable by an $\tilde{\omega} \in \boldsymbol{H}_2(\mathbf{C}_\alpha^+)$ with $\|\tilde{\omega}\|^2_{\boldsymbol{H}_2(\mathbf{C}_\alpha^+)} \leq M_2^2 \sum |z_k|^2 = M_2^2 \|z_0\|^2$ for some M_2 independent of z_0.

Now we shall use the condition that $|\lambda_n b_n| \leq M$ to show that (5) is true with this $\tilde{\omega}$. Note that $\xi(s)$ in (4) can be written as

$$\xi(s) = (sI - A)^{-1}[z_0 + (r - A)b\tilde{\omega}(s)] - b\tilde{\omega}(s). \quad (12)$$

By construction of $\tilde{\omega}$ we see that

$$\|b\tilde{\omega}(s)\|_{\boldsymbol{H}_2(\mathbf{C}_\alpha^+)} \leq \|b\| M_2 \|z_0\|. \quad (13)$$

Now note that

$$(sI - A)^{-1}[z_0 + (r - A)b\tilde{\omega}(s)] = \sum_k \frac{z_k + (r - \lambda_k)b_k\tilde{\omega}(s)}{s - \lambda_k}\phi_k.$$

Since λ_k is on the imaginary axis, we can write it as $\jmath\omega_k$ with $\omega_k \in \mathbf{R}$. Fix $\varepsilon \in (0, -\alpha)$ such that $\varepsilon < \delta$, where δ is given in the statement of Lemma 4.1. Define the following:

$$I_1 := \sum_k \Big(\int_{-\infty}^{\omega_k - \varepsilon} + \int_{\omega_k + \varepsilon}^{\infty}\Big) \Big|\frac{z_k}{\beta + \jmath(y - \omega_k)}\Big|^2 dy,$$

$$I_2 := \sum_k \Big(\int_{-\infty}^{\omega_k - \varepsilon} + \int_{\omega_k + \varepsilon}^{\infty}\Big) \Big|\frac{(r - \jmath\omega_k)b_k\tilde{\omega}(\beta + \jmath y)}{\beta + \jmath(y - \omega_k)}\Big|^2 dy,$$

$$I_3 := \sum_k \int_{\omega_k - \varepsilon}^{\omega_k + \varepsilon} \Big|\frac{z_k + (r - \jmath\omega_k)b_k\tilde{\omega}(\beta + \jmath y)}{\beta + \jmath(y - \omega_k)}\Big|^2 dy$$

for $\beta > \alpha$ If we can show that

$$I_1 + I_2 + I_3 < C^2 \|z_0\|^2 \quad (14)$$

for some $C > 0$ independent of $\beta > \alpha$, then it follows from the orthonormal basis property of $\{\phi_n\}$ and Fubini's Theorem that

$$\|(sI - A)^{-1}[z_0 + (r - A)b\tilde{\omega}(s)]\|_{\boldsymbol{H}_2(\mathbf{C}_\alpha^+)} \leq C\|z_0\|. \quad (15)$$

To analyze I_1, note that

$$\begin{aligned} I_1 &= \sum_k \Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\Big) \Big|\frac{z_k}{\beta + \jmath y}\Big|^2 dy \\ &\leq C_1 \sum_k |z_k|^2 = C_1 \|z_0\|^2, \end{aligned} \quad (16)$$

where

$$C_1 = (\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})|\frac{1}{y}|^2\, dy.$$

To analyze I_2, note that since $|\lambda_n b_n| \le M$,

$$|(r - \jmath\omega_k)b_k|^2 \le [2r^2\|b\|^2 + 2M^2] := M_4,$$

so

$$\begin{aligned} I_2 &\le M_4(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})\sum_k |\frac{\tilde{\omega}(\beta + \jmath(y+\omega_k))}{\beta + \jmath y}|^2\, dy \\ &\le M_4(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})|\frac{1}{y}|^2 \\ &\quad \sum_k |\tilde{\omega}(\beta + \jmath(y+\omega_k))|^2\, dy. \end{aligned}$$

Using Lemma 4.1 we see that for every $y \in \mathbf{R}$,

$$\sum_k |\tilde{\omega}(\beta + \jmath(y+\omega_k))|^2\, dy \le M_3\|\tilde{\omega}\|^2_{H_2(\mathbf{C}_\alpha^+)}$$

so we can conclude that

$$\begin{aligned} I_2 &\le M_4 M_3 \|\tilde{\omega}\|^2_{H_2(\mathbf{C}_\alpha^+)} \cdot (\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})|\frac{1}{y}|^2\, dy \\ &\le M_4 M_3 M_2^2 C_1 \|z_0\|^2. \end{aligned} \quad (17)$$

To analyze I_3, note that since $\tilde{\omega}$ satisfies the interpolation problem (7),

$$\frac{z_k + (r - \jmath\omega_k)b_k\tilde{\omega}(\beta + \jmath y)}{\beta + \jmath(y - \omega_k)}$$

$$= (r - \jmath\omega_k)b_k \frac{-\tilde{\omega}(\jmath\omega_k) + \tilde{\omega}(\beta + \jmath y)}{\beta + \jmath(y - \omega_k)}$$

Therefore, by the hypothesis on $\{b_k\}$

$$\begin{aligned} I_3 &\le M_4 \sum_k \int_{\omega_k - \varepsilon}^{\omega_k + \varepsilon} |\frac{\tilde{\omega}(\jmath\omega_k) - \tilde{\omega}(\beta + \jmath y)}{\beta + \jmath(y - \omega_k)}|^2\, dy \\ &\le 2\varepsilon M_4 \sum_k |\frac{\tilde{\omega}(\jmath\omega_k) - \tilde{\omega}(\beta + \jmath y_k)}{\beta + \jmath(y_k - \omega_k)}|^2 \end{aligned} \quad (18)$$

for some $y_k \in [\omega_k - \varepsilon, \omega_k + \varepsilon]$. We now need to consider two cases.

Case 1: $\beta^2 + \varepsilon^2 < \alpha^2$.
In this case let

$$\delta_0 \in (\sqrt{\beta^2 + \varepsilon^2}, -\alpha), \quad (19)$$

and let

$$\Gamma = \{\delta_0 e^{i\theta} \mid \theta \in [0, 2\pi)\},$$

$$\Gamma_k = \{\jmath\omega_k + \delta_0 e^{i\theta} \mid \theta \in [0, 2\pi)\}.$$

Since $|\beta + \jmath(y_k - \omega_k)| < \sqrt{\beta^2 + \varepsilon^2} < \delta_0$, the term under the summation sign in (18) becomes

$$\begin{aligned} &|\frac{1}{(\beta + \jmath(y_k - \omega_k))} \times \\ &\frac{1}{2\pi\jmath}\int_{\Gamma_k} \tilde{\omega}(\eta)[\frac{1}{\eta - \jmath\omega_k} - \frac{1}{\eta - \beta - \jmath y_k}]\, d\eta|^2 \\ &= |\frac{1}{2\pi}\int_{\Gamma_k} \frac{\tilde{\omega}(\eta)}{(\eta - \jmath\omega_k)(\eta - \beta - \jmath y_k)}\, d\eta|^2 \\ &= |\frac{1}{2\pi}\int_{\Gamma} \frac{\tilde{\omega}(\eta + \jmath\omega_k)}{\eta(\eta + \jmath\omega_k - \beta - \jmath y_k)}\, d\eta|^2. \end{aligned} \quad (20)$$

Now we note that for $\eta \in \Gamma$, $|\eta| = \delta_0$ and (using (19)) $|\eta + \jmath\omega_k - \beta - \jmath y_k| > \delta_0 - \sqrt{\beta^2 + \varepsilon^2}$. Therefore, (20) is

$$\begin{aligned} &\le (\frac{1}{2\pi\delta_0(\delta_0 - \sqrt{\beta^2 + \varepsilon^2})})^2 \int_\Gamma |\tilde{\omega}(\eta + \jmath\omega_k)|^2\, |d\eta| \\ &= (\frac{1}{\delta_0 - \sqrt{\beta^2 + \varepsilon^2}})^2 |\tilde{\omega}(\delta_0 e^{\jmath\theta_k} + \jmath\omega_k)|^2 \end{aligned}$$

for some $\theta_k \in [0, 2\pi]$. Therefore, from (18) we see that there exists $M_5(\beta)$ such that

$$I_3 \le M_5(\beta)\sum_k |\tilde{\omega}(\delta_0 e^{\jmath\theta_k} + \jmath\omega_k)|^2.$$

Using Lemma 4.1, we see that there exists $M_6(\beta)$ such that

$$I_3 \le M_6(\beta)\|\tilde{\omega}\|^2_{H_2(\mathbf{C}_\alpha^+)} \le M_6(\beta) M_2^2 \|z_0\|^2. \quad (21)$$

Case 2: $\beta^2 + \varepsilon^2 \ge \alpha^2$.
In this case

$$|\beta + \jmath(y_k - \omega_k)| \ge \beta^2 \ge \alpha^2 - \varepsilon^2,$$

so we see that (see (18)

$$I_3 \le \frac{4\varepsilon^2 M_4}{\alpha^2 - \varepsilon^2}\sum_k |\tilde{\omega}(\jmath\omega_k) - \tilde{\omega}(\beta + \jmath y_k)|^2.$$

Using Lemma 4.1, we see that in this case

$$I_3 \le M_7\|\tilde{\omega}\|^2_{H_2(\mathbf{C}_\alpha^+)} \le M_7 M_2^2 \|z_0\|^2. \quad (22)$$

is true. Let $M_8 = \max(\{M_6(\beta) \mid |\beta| < \alpha\} \cup \{M_7\})$ Then (22) is true with M_7 replaced by M_8.

Putting together (16), (17) and (22), we see that (14) is true, proving (15). Combining (15) and (13) we get (5), finishing the proof. ∎

If b is not in the class defined by $m \le |\lambda_n b_n| \le M$, it is possible that the system might not be open-loop stabilizable. This is illustrated by Example 5.1 in the next section.

5 Open loop stabilizability for the wave equation

We consider the following model for the undamped wave equation:

$$\begin{aligned} &\frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) + \tilde{b}(x)u(t), \\ &0 \le x \le 1,\ t \ge 0, \\ &w(0,t) = w(1,t) = 0, \end{aligned} \tag{23}$$

where we assume that $\tilde{b} \in \boldsymbol{L}_2(0,1)$.

This equation can be rewritten in the abstract form (1) with state space $Z = \boldsymbol{D}(A_0^{\frac{1}{2}}) \oplus \boldsymbol{L}_2(0,1)$, where $A_0 h = -\frac{d^2h}{dx^2}$ for h in the domain of A_0 given by

$$\boldsymbol{D}(A_0) = \{h \in \boldsymbol{H}^2(0,1) \mid h(0) = 0 = h(1)\}.$$

The system operator A is given by

$$A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{24}$$

with $\boldsymbol{D}(A) = \boldsymbol{D}(A_0) \oplus \boldsymbol{D}(A_0^{\frac{1}{2}})$. The input operator b is given by

$$bu = \begin{pmatrix} 0 \\ \tilde{b}u \end{pmatrix}.$$

This system operator is the infinitesimal generator of a strongly continuous unitary group $T(t)$ on $Z = \boldsymbol{D}(A_0^{\frac{1}{2}}) \oplus \boldsymbol{L}_2(0,1)$ (see, for instance, [2]). A simple calculation shows that A is skew adjoint, and has the eigenvalues $\{\lambda_n = \jmath n\pi, n = \pm 1, \pm 2, \ldots\}$ with associated eigenvectors

$$\{\phi_n(x) = \frac{1}{\lambda_n}\begin{pmatrix} \sin(n\pi x) \\ \lambda_n \sin(n\pi x) \end{pmatrix}, n = \pm 1, \pm 2, \ldots\}.$$

It is easy to see that $b_n := \langle b, \phi_n \rangle_Z = \langle \tilde{b}(\cdot), \sin(n\pi \cdot) \rangle_{L_2}$.

In Zwart [11] it was claimed that the system is open loop stabilizable if and only if equation (9) was satisfied. In the next example we show that this is not true.

Example 5.1 Assume that the Fourier coefficients of b are given by

$$b_n = \begin{cases} |k|^{-1} & \text{for } n = 2k+1, \\ |k|^{-\frac{3}{4}} & \text{for } n = 2k. \end{cases}$$

It is easy to see that $b \in Z$, and that (9) is satisfied. For this b we see that for $n = 2k$

$$\sup_n \left| \frac{b_n}{(\lambda_{n+1} - \lambda_n) b_{n+1}} \right| = \sup_k |k|^{1/4} = \infty. \tag{25}$$

This contradicts (10), which is a necessary condition for open loop stabilizability, so this system is not open loop stabilizable.

We can now apply Theorem 4.2 to this system to show that the wave equation can be open loop stabilizable with for certain choices of $\tilde{b} \in \boldsymbol{L}_2[0,1]$.

Theorem 5.2 *Suppose* $b_n = \langle \tilde{b}(\cdot), \sin(n\pi \cdot) \rangle$ *satisfies*

$$m \le n|b_n| \le M$$

for some positive m *and* M. *Then the undamped wave equation (23) is open loop stabilizable.*

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