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H_∞-CONTROL FOR THE WIENER ALGEBRA

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Abstract

In this paper we provide necessary and sufficient conditions for the solvability of the standard H_∞-suboptimal control problem for systems with the transfer function in a subalgebra of the quotient field of the Wiener algebra. These conditions are formulated in terms of the existence of two J-spectral factorizations. Furthermore, a formula for the set of all stabilizing controllers is given.

1 Introduction

The standard H_∞-control problem was introduced in 1984 by J.C. Doyle [8]. Nowadays there are different techniques for solving this problem. Here we use coprime factorizations. The idea of factorizing the transfer function of a (not necessarily stable) system as a ratio of two stable transfer functions was first introduced in 1972 by Vidyasagar [14]. Using this (coprime) factorization Green [9, 10] and Meinsma [13] showed that for rational transfer functions the H_∞-control problem can be solved if and only if two J-spectral factorizations are solvable. This result has been extended to the infinite-dimensional case for systems in state space form $\Sigma(A, B, C, D)$ with B and C bounded by Curtain and Rodriguez [6] and for the Pritchard-Salamon class of state space systems in Weiss [15].

In this paper we provide a self-contained solution to the standard H_∞-suboptimal control problem for systems with the transfer function in a subalgebra of the quotient field of the Wiener algebra. It provides an independent proof of the results in [5] which used the abstract theory of Ball and Helton [1]. However, if one looks for the result quoted from [1] in [5], one realizes that this is not an obvious corollary of the very abstract and general theory in [1].

2 Preliminaries

In this section we quote some general results and introduce our notation. We begin with our class of stable systems. We say that $f \in \mathcal{A}$ if f has the representation

$$f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $f_0 \in \mathbb{C}$, $\int_0^\infty |f_a(t)|dt < \infty$ and δ represents the delta distribution at zero. Let $\hat{f}$ denote the Laplace transform of f. Then $\hat{\mathcal{A}}$ defined as $\hat{\mathcal{A}} := \left\{ \hat{f} \mid f \in \mathcal{A} \right\}$ is our class of stable transfer functions. By the definition of $\mathcal{A}$ it is easy to see that for every $f \in \mathcal{A}$, $\hat{f}$ is well-defined on $\overline{\mathbb{C}}_+ := \{s \in \mathbb{C} \mid Re(s) \geq 0\}$, it is holomorphic and bounded on $\mathbb{C}_+ := \{s \in \mathbb{C} \mid Re(s) > 0\}$, and continuous on $\mathbb{C}_0 := \{s \in \mathbb{C} \mid Re(s) = 0\}$. Furthermore, $\hat{\mathcal{A}}$ is a commutative Banach algebra with identity under pointwise addition and multiplication (see [7], Corollary A.7.48). If for an element of $\hat{\mathcal{A}}$ its inverse exists and it is also stable, it is said that it is *bistable*. For any complex number s we make the following notation $f^\sim(s) = \overline{f(-\overline{s})}$.

Our class of transfer functions is the algebra of fractions $\hat{\mathcal{B}} = \hat{\mathcal{A}}[\hat{\mathcal{A}}_\infty]^{-1}$, where $\hat{\mathcal{A}}_\infty$ is the class of transfer functions in $\hat{\mathcal{A}}$ with the property that they are bounded away from zero at infinity and only finitely many unstable zeros in $\overline{\mathbb{C}}_+$. Any $f \in \hat{\mathcal{B}}$ has a nonzero limit at infinity. For more properties of these classes of transfer functions see [2],[3] and [4].

For theoretical reasons we consider a larger class of transfer functions, known as the *Wiener algebra*

$$\hat{\mathcal{W}} = \left\{ \hat{f} \in L_\infty \mid \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ with } \hat{f}_1, \hat{f}_2^\sim \in \hat{\mathcal{A}} \right\},$$

where

$$L_\infty = \{f : \mathbb{C}_0 \to \mathbb{C} \mid \|f\|_{L_\infty} = \operatorname{ess\,sup}_{s \in \mathbb{C}_0} |f(s)| < \infty\}.$$

$\hat{\mathcal{W}}$ is a Banach algebra under pointwise addition, multiplication, and scalar multiplication. The elements of $\hat{\mathcal{W}}$ are bounded and continuous on the imaginary axis, and their limit at infinity is well-defined.

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The spaces H_2 and H_∞ denote the standard Hardy spaces on the right-half plane. The space $H_2^\perp$ is the orthogonal complement of H_2 with respect to the inner product in the space of square integrable functions on the imaginary axis. We denote by $H_\infty^{n\times m}$, $\hat{\mathcal{B}}^{n\times m}$, $\hat{\mathcal{A}}^{n\times m}$, $\hat{\mathcal{W}}^{n\times m}$, the classes of $n \times m$ matrices with entries in H_∞, $\hat{\mathcal{B}}$, $\hat{\mathcal{A}}$, $\hat{\mathcal{W}}$, respectively. We omit the size of the matrix when there is no danger of confusion.

For matrix valued functions we define $F^\sim(s) = [F(-\overline{s})]^*$, where $*$ denotes the transpose complex conjugate. This corresponds to the definition for scalar functions.

An element $G \in \hat{\mathcal{B}}$ is said to have a coprime factorization over $\hat{\mathcal{A}}$ if there exist stable D, N, X and Y such that $G = D^{-1}N$ and $DX + NY = I$. Every element of $\hat{\mathcal{B}}$ possesses a coprime factorization (see [7], Chapter 7).

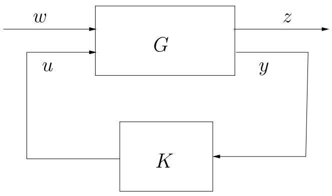

Figure 1: The standard H_∞-control problem

In order to define the standard H_∞-suboptimal control problem we consider the diagram given in Figure 1, where w, u, z and y are vector valued signals as follows: w is the exogenous input, u is the control signal, z is the output to be controlled and y is the measured output.

The transfer matrices G and K are assumed to be in $\hat{\mathcal{B}}$, and G is assumed to be a stabilizable plant.

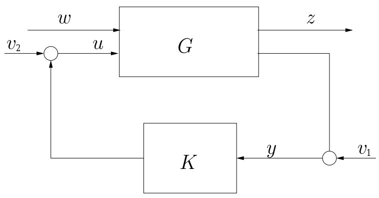

Figure 2: Stability diagram

Definition 2.1 (Standard H_∞-suboptimal control problem) *Given a stabilizable plant $G \in \hat{\mathcal{B}}$ and the positive bound γ, find a compensator $K \in \hat{\mathcal{B}}$ such that the transfer function T_{zw}, from w to z, satisfies $\|T_{zw}\|_{H_\infty} < \gamma$. If it is possible, describe the general form of all stabilizable controllers which satisfies the above inequality.*

By stability of the system we mean the internal stability (see Figure 2) as defined in [7] or [11].

3 Main results

In this section we formulate our main results and provide the outline of the proofs.

Let G be a stabilizable plant and let $G = [D_1\ D_2]^{-1}\,[N_1\ N_2]$ be a left-coprime factorization of G over $\hat{\mathcal{A}}$, where $[D_1\ D_2]$ and $[N_1\ N_2]$ are partitioned with respect to the outputs and inputs, respectively. This factorization exists since $G \in \hat{\mathcal{B}}$.

We consider the matrix

$$J_{\gamma,n,m} = \begin{bmatrix} I_n & 0 \\ 0 & -\gamma I_m \end{bmatrix},$$

where $n, m \in \mathbb{N}$. Sometimes we simply use $J_{n,m}$ when $\gamma = 1$, or J without indices. The following theorem states our main result.

Theorem 3.1 *The standard H_∞-suboptimal control problem has a solution if and only if there exist bistable matrices W and V such that*

$$N_1(j\omega)N_1^\sim(j\omega) - \gamma^2 D_1(j\omega)D_1^\sim(j\omega) = W(j\omega)J_{n_y,n_z}W^\sim(j\omega), \quad (1)$$

for all $\omega \in \mathbb{R}$ with the lower-right $n_z \times n_z$ block M_{22} of $M := W^{-1}\,[\,-N_1 \quad D_1\,]$ bistable, and

$$R^\sim(j\omega)J_{n_w,n_z}R(j\omega) = V^\sim(j\omega)J_{n_y,n_u}V(j\omega) \text{ for } \omega \in \mathbb{R}, \quad (2)$$

where

$$R = \begin{bmatrix} 0 & I_{n_w} \\ I_{n_z} & 0 \end{bmatrix} W^{-1}[-N_2\ D_2] \begin{bmatrix} 0 & I_{n_y} \\ I_{n_u} & 0 \end{bmatrix}. \quad (3)$$

and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable. Moreover the set of all stabilizing controllers is given by

$$\begin{bmatrix} K_n \\ K_d \end{bmatrix} = V^{-1} \begin{bmatrix} U \\ I_{n_y} \end{bmatrix}, \quad (4)$$

with $U \in \hat{\mathcal{A}}$ such that $\|U\|_{H_\infty} < 1$ and $\det K_d \neq 0$.

In order to prove the result stated before we need the following definition.

Definition 3.2 *Consider the Hardy space $H_2^{m+n,\perp}$ with the inner product $\langle\cdot,\cdot\rangle$. A subspace B of $H_2^{m+n,\perp}$ is* strictly positive *with respect to the $J_{\gamma,m,n}$-inner product $[f, g] := \langle f, J_{\gamma,m,n}g\rangle$, $f, g \in H_2^{m+n,\perp}$, if there exists an $\epsilon > 0$ such that for all non-zero $x \in B$*

$$\langle x, J_{\gamma,m,n}x\rangle \geq \epsilon\langle x, x\rangle.$$

The following lemma provides the existence of the bistable matrix W such that (1) is satisfied.

Lemma 3.3 *Assume that* $\det(N_1 N_1^{\sim} - D_1 D_1^{\sim}) \neq 0$ *on* $\mathbb{C}_0 \cup \{\infty\}$. *Then if the space*

$$B_{[-N_1 \quad D_1]} = \left\{ \begin{bmatrix} w \\ z \end{bmatrix} \in H_2^{\perp} \mid [-N_1 \quad D_1] \begin{bmatrix} w \\ z \end{bmatrix} \in H_2 \right\} \tag{5}$$

is strictly positive in the $J_{\frac{1}{\gamma^2}, n_w, n_z}$*-inner product, then there exists a bistable* W *such that the equality (1) is satisfied.*

Proof: We prove, by contradiction, that there is no vector $v \in H_2^{\perp}$ such that

$$\begin{bmatrix} -N_1 & D_1 \end{bmatrix} J \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v \in H_2. \tag{6}$$

Suppose that $v \in H_2^{\perp}$ is a nonzero vector satisfying (6). Since $[-N_1 \ D_1] \in \hat{\mathcal{A}}$ we have that

$$u = J \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v \in B_{\begin{bmatrix} -N_1 & D_1 \end{bmatrix}}.$$

On the other side

$$\begin{aligned} \langle u, Ju \rangle &= \langle J \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v, JJ \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v \rangle \\ &= \langle \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v, J \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v \rangle \\ &= \langle v, \begin{bmatrix} -N_1 & D_1 \end{bmatrix} J \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v \rangle = 0 \end{aligned}$$

since $v \in H_2^{\perp}$ and $\begin{bmatrix} -N_1 & D_1 \end{bmatrix} J \begin{bmatrix} -N_1 & D_1 \end{bmatrix}^{\sim} v \in H_2$. This contradicts the strict positivity of the space $B_{\begin{bmatrix} -N_1 & D_1 \end{bmatrix}}$.

In similar way as in [12] it can be proved that there exists a bistable W such that the equality (1) is satisfied. ■

We state two technical lemma (see [11]).

Lemma 3.4 (Necessary condition) *A necessary condition for the existence of a solution for the standard* H_∞*-suboptimal control problem is that the space given in (5) is strictly positive in the* $J_{\frac{1}{\gamma^2}, n_w, n_z}$*-inner product.*

Lemma 3.5 *Let* $M \in \hat{\mathcal{A}}^{(n_y+n_z)\times(n_w+n_z)}$ *with* $n_y = n_w$*, and suppose that*

$$M(j\omega) J_{\gamma, n_y, n_z} M^{\sim}(j\omega) = J_{n_w, n_z}, \text{ for almost all } \omega \in \mathbb{R}. \tag{7}$$

Consider the equality

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{8}$$

with $H_1 \in \hat{\mathcal{A}}^{n_z \times n_w}$, $H_2 \in \hat{\mathcal{A}}^{n_z \times n_z}$, $U_1 \in \hat{\mathcal{A}}^{n_z \times n_y}$, $U_2 \in \hat{\mathcal{A}}^{n_z \times n_z}$, $M_{11} \in \hat{\mathcal{A}}^{n_y \times n_w}$, $M_{12} \in \hat{\mathcal{A}}^{n_y \times n_z}$, $M_{21} \in \hat{\mathcal{A}}^{n_z \times n_w}$ *and* $M_{22} \in \hat{\mathcal{A}}^{n_z \times n_z}$. *Then the following two conditions are equivalent*

1. H_2 *is bistable and* $\left\| H_2^{-1} H_1 \right\|_{H_\infty} < \gamma$.
2. M_{22} *and* U_2 *are bistable and* $\left\| U_2^{-1} U_1 \right\|_{H_\infty} < 1$.

We formulate a special two-block problem.

Definition 3.6 (A two-block problem) *Let* $L \in \hat{\mathcal{A}}^{(n_z+n_w)\times(n_u+n_y)}$, $(n_y = n_w)$ *and consider the equality*

$$\begin{bmatrix} H_2 \\ H_1 \end{bmatrix} = L \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} \tag{9}$$

with $H_1 \in \hat{\mathcal{A}}^{n_z \times n_w}$, $H_2 \in \hat{\mathcal{A}}^{n_w \times n_w}$, $\tilde{K}_d \in \hat{\mathcal{A}}^{n_y \times n_y}$, $\tilde{K}_n \in \hat{\mathcal{A}}^{n_u \times n_y}$. *The two-block problem is to find a controller* $K \in \hat{\mathcal{B}}$ *with the right-coprime factorization* $K = \tilde{K}_n \tilde{K}_d^{-1}$ *over* $\hat{\mathcal{A}}$ *such that* H_2 *is bistable and* $\left\| H_1 H_2^{-1} \right\|_{H_\infty} < 1$.

We can reformulate the standard H_∞-suboptimal control problem into an equivalent problem involving the coprime factorization of G and K (see [11]).

Definition 3.7 *Given a transfer matrix* $[-N_1 \quad D_1 \quad -N_2 \quad D_2] \in \hat{\mathcal{A}}^{(n_z+n_y)\times(n_w+n_z+n_u+n_y)}$ *find a compensator* $K \in \hat{\mathcal{B}}^{n_u \times n_y}$ *with a left-coprime factorization* $K_d^{-1} K_n$*, such that the transfer matrix* T_{zw} *(see Figure 1) from* w *to* z *induced by the frequency domain equation*

$$A \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} N_1 \\ 0 \end{bmatrix} w \tag{10}$$

satisfies $\|T_{zw}\|_{H_\infty} < \gamma$*, where* A *is given by*

$$A = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix}. \tag{11}$$

and it is bistable.

The following theorem provides sufficient conditions to rewrite the H_∞-suboptimal control problem as a two-block problem. A proof can be found in [11].

Theorem 3.8 (Reduction to a two-block problem) *Consider the standard* H_∞*-suboptimal control problem of Definition 3.7. If there exists a bistable matrix* W *such that (1) holds, with the lower right* $n_z \times n_z$ *block of the matrix* $W^{-1}[-N_1 \ D_1]$ *bistable, then there exists a stabilizing controler* $K \in \hat{\mathcal{B}}$ *for the given plant such that the standard* H_∞*-suboptimal control problem is solved if and only if the two-block problem of Definition 3.6, with*

$$L = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \tag{12}$$

has a solution.

Proof: The proof will be given in four steps.

In step 1 we obtain the equivalent condition for the stability of the closed-loop system. In step 2 a similar result is obtained but now for a system related to the two-block problem. In the last two steps the necessary and sufficient condition of the theorem is proved.

Step 1: Let $K \in \hat{\mathcal{B}}$ be any controller for the given plant, and let $K = K_d^{-1} K_n$ be a left-coprime factorization over $\hat{\mathcal{A}}$. The closed-loop system is given by

$$\begin{bmatrix} -N_1 & D_1 & D_2 & -N_2 \\ 0 & 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} w \\ z \\ y \\ u \end{bmatrix} = 0. \qquad (13)$$

(see [11]). Denote

$$\Omega = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix}. \qquad (14)$$

We know that $K \in \hat{\mathcal{B}}$ has a right coprime factorization, i.e. $K = \tilde{K}_n \tilde{K}_d^{-1}$. Furthermore, by [7] Lemma A.7.44, page 661, there exists a bistable U of the form

$$U = \begin{bmatrix} \tilde{K}_d & * \\ \tilde{K}_n & * \end{bmatrix}. \qquad (15)$$

such that

$$\begin{bmatrix} -K_n & K_d \end{bmatrix} U = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix}.$$

Defining the signals l_1 and l_2 using this U, via

$$\begin{bmatrix} y \\ u \end{bmatrix} = U \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},$$

we obtain the following equivalent representation for the system

$$\begin{bmatrix} -N_1 & D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n & * \\ 0 & 0 & 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} w \\ z \\ l_1 \\ l_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = U \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}.$$

However $I_{n_u} l_2 = 0$ is the same as $l_2 = 0$, and this representation becomes

$$\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} \begin{bmatrix} z \\ l_1 \end{bmatrix} = N_1 w$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} l_1$$

$$l_2 = 0.$$

Since

$$\Omega \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n & * \\ 0 & 0 & I_{n_u} \end{bmatrix}$$

and U bistable, we have the following equivalence

Ω is bistable if and only if $\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix}$ is bistable .

Step 2: Using the bistable matrix W which satisfy relation (1) we define

$$\begin{bmatrix} -\tilde{N}_1 & \tilde{D}_1 & \tilde{D}_2 & -\tilde{N}_2 \end{bmatrix} = W^{-1} \begin{bmatrix} -N_1 & D_1 & D_2 & -N_2 \end{bmatrix}.$$

Furthermore, for the plant

$$P = \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} & \tilde{N}_2 \end{bmatrix},$$

we define the closed-loop system (see [11])

$$\begin{bmatrix} \begin{bmatrix} I_{n_w} \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{n_z} \\ 0 \end{bmatrix} & \begin{matrix} \tilde{D}_2 \\ K_n \end{matrix} & \begin{matrix} -\tilde{N}_2 \\ K_d \end{matrix} \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \tilde{z} \\ y \\ u \end{bmatrix} = 0, \qquad (16)$$

where K is a controller of the form $K = K_d^{-1} K_n$, $\tilde{w}$ is the new exogenous input, and the new to be controlled output is $\tilde{z}$. We define the matrices $\tilde{H}_1$ and $\tilde{H}_2$ via

$$\begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} = \begin{bmatrix} \tilde{D}_2 & -\tilde{N}_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix}, \qquad (17)$$

where $\tilde{K}_n$ and $\tilde{K}_d$ are given in (15).
Denote

$$\tilde{\Omega} = \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \\ 0 \end{bmatrix} & \begin{matrix} \tilde{D}_2 \\ K_n \end{matrix} & \begin{matrix} -\tilde{N}_2 \\ K_d \end{matrix} \end{bmatrix}. \qquad (18)$$

Then

$$\tilde{\Omega} \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \\ 0 \end{bmatrix} & \begin{matrix} \tilde{D}_2 \\ K_n \end{matrix} & \begin{matrix} -\tilde{N}_2 \\ K_d \end{matrix} \end{bmatrix} \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} \qquad (19)$$

$$= \begin{bmatrix} 0 & \tilde{H}_2 & * \\ I_{n_z} & \tilde{H}_1 & * \\ 0 & 0 & I_{n_u} \end{bmatrix}. \qquad (20)$$

Since U is bistable, the following equivalence holds

$$\tilde{\Omega} \text{ is bistable if and only if } \tilde{H}_2 \text{ is bistable .} \qquad (21)$$

Using

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = U^{-1} \begin{bmatrix} y \\ u \end{bmatrix} \qquad (22)$$

and (19), (20), we obtain the equivalent representation for the new system (16)

$$\begin{bmatrix} 0 & \tilde{H}_2 & * \\ I_{n_z} & \tilde{H}_1 & * \\ 0 & 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} \tilde{z} \\ l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -I_{n_w} \\ 0 \\ 0 \end{bmatrix} \tilde{w} \qquad (23)$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = U \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}. \qquad (24)$$

Suppose that the controller K stabilizes the closed-loop system (16). It can be proved that (see [11]) $\tilde{\Omega}$ is bistable, and thus $\tilde{H}_2$ is bistable, see (21). Consequently, we can write

$$l_2 = 0,\ l_1 = -\tilde{H}_2^{-1}\tilde{w},\ \tilde{z} = -\tilde{H}_1 l_1 \tag{25}$$

which give us the transfer function from $\tilde{w}$ to $\tilde{z}$, namely

$$T_{\tilde{z}\tilde{w}} = \tilde{H}_1\tilde{H}_2^{-1}. \tag{26}$$

Step 3: In this step we show that if the standard H_∞-suboptimal control problem is solvable, then the two-block problem is solvable (see [11]).

Step 4: In this step we show that if the two-block problem is solvable, then the H_∞-suboptimal control problem is solvable. Using the notation of (17) we know that $\tilde{H}_2$ is bistable and $\|\tilde{H}_1\tilde{H}_2^{-1}\|_{H_\infty} < 1$. Furthermore, let $K = K_d^{-1}K_n$ be a left-coprime factorization of the controller $K = \tilde{K}_n\tilde{K}_d^{-1}$ which solves the two-block problem. Using step 2 and Lemma 4.4 from [11] we see that $K = K_d^{-1}K_n$ is a stabilizing controller for the system (16) and that $\|T_{\tilde{z}\tilde{w}}\|_{H_\infty} < 1$. We have to prove that $\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix}$ is bistable and $\|T_{zw}\|_{H_\infty} \le \gamma$. Define

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I \end{bmatrix} W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}. \tag{27}$$

Using the properties the matrix $W^{-1}\begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ and the assumption that $\|T_{\tilde{z}\tilde{w}}\| < 1$ we can apply Lemma 3.5 and obtain that

$$\|H_2^{-1}H_1\|_{H_\infty} < \gamma \text{ and } H_2 \text{ is bistable}.$$

From (27) we see that $\|T_{zw}\|_{H_\infty} = \|H_2^{-1}H_1\|_{H_\infty}$. Also from (27) we have that

$$\begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I \end{bmatrix} W^{-1} D_1 = H_2 \tag{28}$$

and using (17) it follows that

$$\begin{bmatrix} I & 0 \end{bmatrix} W^{-1}(D_2\tilde{K}_d - N_2\tilde{K}_n) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} = \tilde{H}_2. \tag{29}$$

Combining (28) and (29) the following equality holds

$$\begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I_{n_z} \\ I_{n_w} & 0 \end{bmatrix} W^{-1} \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} = \begin{bmatrix} H_2 & 0 \\ * & \tilde{H}_2 \end{bmatrix}. \tag{30}$$

Since H_2, $\tilde{H}_2$ and W are bistable, we have that $\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix}$ is bistable. Using the step 1 we conclude that the standard H_∞-suboptimal control problem is solved. ■

Sufficient conditions for solving the two-block problem are given in the following lemma (see [11]).

Lemma 3.9 *Let L be*

$$L = W^{-1}[-N_2\ D_2], \tag{31}$$

and

$$R = \begin{bmatrix} 0 & I_{n_w} \\ I_{n_z} & 0 \end{bmatrix} L \begin{bmatrix} 0 & I_{n_y} \\ I_{n_u} & 0 \end{bmatrix}. \tag{32}$$

If there exists a bistable matrix V such that

$$R^{\sim}(j\omega)J_{n_w,n_z}R(j\omega) = V^{\sim}(j\omega)J_{n_y,n_u}V(j\omega) \text{ for } \omega \in \mathbb{R}, \tag{33}$$

and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable, then the set of all controllers which solves the two-block problem (see Definition 3.6) is given by $K = K_nK_d^{-1}$ where K_n and K_d satisfy (4) with $U \in \hat{\mathcal{A}}$ such that $\|U\|_{H_\infty} < 1$ and $\det K_d \neq 0$.

Using Lemma 3.3 we provide the outline of the proof for the main result.

Proof of Theorem 3.1: The sufficiency follows from Theorem 3.8 and Theorem 3.9. For the necessity we see that strict positivity of the space $B_{\begin{bmatrix} -N_1 & D_1 \end{bmatrix}}$ is a necessary condition (from Lemma 3.4). Applying now Lemma 3.3 we have that there exists a bistable W such that the equality (1) is satisfied. Using Lemma 3.5 and the equality (27) we obtain that the lower-right $n_z \times n_z$ block M_{22} of $M := W^{-1}[-N_1 \quad D_1]$ is bistable.

In similar way we can prove that the equality (33) holds and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable.

The fact that all controllers are of the form (4) follows from Lemma 3.9. ■

4 Conclusion

Necessary and sufficient conditions for the solvability of the standard H_∞-suboptimal control problem for systems with the transfer function in a subalgebra of the quotient field of the Wiener algebra are provided. These conditions are formulated in terms of the existence of two J-spectral factorizations. Furthermore, a formula for the set of all stabilizing controllers is given. An independent self-contained proof of the results in [5], which used the abstract theory of Ball and Helton [1], is given.

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