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# P.O.D. FOR LINEAR SYSTEMS

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## Abstract

In this paper we show that for linear systems there is a strong relation between P.O.D. approximation and balanced truncation. Using this relation we obtain an error estimate for the P.O.D. approximation in the $\mathcal{H}_\infty$-norm. A small $\mathcal{H}_\infty$-norm is needed in order to guarantee that a controller design for the reduced system will perform well on the original system

## 1 Introduction

For systems there are several approximation techniques e.g. linearization, modal approximation, balanced approximation, etc. Recently, one sees that the technique of proper orthogonal decomposition (P.O.D.) is applied to systems, see e.g. Volkwein [5]. However, P.O.D. is a technique for finding a simpler representation of signals. Although systems and signals are closely related, it is not easy to indicate on which signals one should apply the P.O.D. method. For non-linear models Lall, Marsden and Glavaški [4] described how one can obtain model reduction using P.O.D.'s. For linear systems the situation is a little bit easier, because almost any linear, time-invariant system can be written in the convolution form

$$y = h * u, \tag{1}$$

where $*$ denotes the convolution product, and the $u$ and $y$ are the input and output, respectively. Hence it seems to be very logical to take as signal which one must approximate the impulse response $h$. This is what we do in this paper. We show that this has several advantages. For instance, one can obtain error bounds for the reduced order approximation. These error bounds are in terms of the associated transfer functions, and thus are essential for robust controller design. The idea that an approximation must be useful for controller design is the principal motivation in the paper by Ruth Curtain [1].

Since in most partial differential applications one is interested in the behavior of the full system, and not merely in the behavior of an output, we consider the abstract differential equation

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0 \qquad t \geq 0. \tag{2}$$

We assume that $A$ is the infinitesimal generator of the $C_0$-semigroup $T(t)$ on a Hilbert space $Z$, and $b$ is an element of $Z$. Note that many partial differential equations can be written in this format. For information on this and on the equation (2) we refer to [2].

The impulse response of (2) is given by

$$h(t) = T(t)b, \qquad t \geq 0. \tag{3}$$

where $T(t)$ is the $C_0$-semigroup generated by $A$. In the rest of this paper we need the following assumptions:

- $T(t)$ is strongly stable, i.e. $T(t)x_0 \to 0$ when $t \to \infty$ for all $x_0 \in Z$.
- $h(t)$ is square integrable, i.e. $h \in L^2(0,\infty); Z)$.

Althought for the square integrability of the impulse response it is not neccesary that $b \in Z$, we have assumed it anyway in order to simplify some arguments.

In the next section we show that applying the P.O.D. approximation on the impluse response is related to a Lyapunov equation. In Section 3, we introduce a different approximation technique which has as an advantage that it gives an $L^2$-error bound between the orginal and approximated impulse response. Using the theory of balanced realizations we obtain in Section 4 an $\mathcal{H}_\infty$-error bound between the Laplace transform of the orginal impulse response and P.O.D. approximation.

## 2 P.O.D. approximations

Given a signal $X(\cdot)$ in $L^2(\mathbb{T}, H)$, where $\mathbb{T}$ is the time axis, and $H$ is a Hilbert space. The first P.O.D. approximation is the function of the form $\alpha(\cdot)\phi$, with $\alpha \in L^2(\mathbb{T})$, $\phi \in H$ with $\|\phi\| = 1$ such that

$$\|X(\cdot) - \alpha(\cdot)\phi\|_{L^2(\mathbb{T},H)} \tag{4}$$

is minimized. In order to describe the solution we introduce the operator $L$ on $H$ defined as

$$\langle \psi_1, L\psi_2\rangle_H = \langle\langle \psi_1, X(\cdot)\rangle_H, \langle \psi_2, X(\cdot)\rangle_H\rangle_{L^2(\mathbb{T})}. \tag{5}$$

It is easy to see that $L$ is a self-adjoint, non-negative operator on $H$. If the norm of this operator equals the largest eigenvalue, then the minimization problem has a solution which is given by

$$\alpha_{\text{opt}}(\cdot) = \langle X(.), \phi\rangle_H, \tag{6}$$

where $\phi$ is the (normalized) eigenvector corresponding to the largest eigenvalue of the operator $L$. Normally the P.O.D.



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approximation is done on a signal for which $H$ is finite-dimensional. In that case one has that $L$ is just a matrix, and thus the norm always equals the largest eigenvalue. For infinite-dimensional Hilbert spaces $H$ the situation is more complicated. However, if $L$ has an (orthonormal) basis of eigenvectors, then the situation is similar to the finite-dimensional one. Note that since $L$ is self-adjoint it has a basis of eigenvectors if it is a compact operator. The following result is well-known if $H$ is finite-dimensional. The proof for the infinite-dimensional case is very similar, but since we shall use it later we present a proof here.

**Lemma 2.1** *If $L$ has a basis of eigenvectors, then $L$ is nuclear. Furthermore, if $\lambda_n$ denote the eigenvalues of $L$ with $\lambda_1 \geq \lambda_2 \geq \ldots$, then*

$$\min_{\|\phi\|=1, \alpha\in L^2(\mathbb{T})} \|X(\cdot) - \alpha(\cdot)\phi\|^2_{L^2(\mathbb{T},H)} = \sum_{n=2}^{\infty} \lambda_n.$$

**Proof** First note that the norm in $L^2(\mathbb{T}, H)$ is the combination of the norm on $H$ and on $L^2(\mathbb{T})$, i.e.,

$$\|F(\cdot)\|^2_{L^2(\mathbb{T},H)} = \left\| \|F(\cdot)\|^2_H \right\|^2_{L^2(\mathbb{T})}.$$

For fixed $\phi$ one can write

$$\begin{aligned} &\|X(\cdot) - \alpha(\cdot)\phi\|^2_{L^2(\mathbb{T},H)} \\ &= \|X(\cdot) - \langle X(\cdot), \phi\rangle\phi - \alpha(\cdot)\phi + \langle X(\cdot), \phi\rangle\phi\|^2_{L^2(\mathbb{T},H)} \\ &= \Big\| \big[\|X(\cdot) - \langle X(\cdot), \phi\rangle\phi\|^2_H \\ &\quad + \|\alpha(\cdot)\phi - \langle X(\cdot), \phi\rangle\phi\|^2_H\big] \Big\|^2_{L^2(\mathbb{T})} \\ &= \Big\| \big[\|X(\cdot) - \langle X(\cdot), \phi\rangle_H\phi\|^2_H \\ &\quad + |\alpha(\cdot) - \langle X(\cdot), \phi\rangle_H|^2\big] \Big\|^2_{L^2(\mathbb{T})}. \end{aligned}$$

Hence the optimal $\alpha$ is given by (6), and for this optimal function we have that

$$\begin{aligned} &\|X(\cdot) - \alpha_{\text{opt}}(\cdot)\phi\|^2_{L^2(\mathbb{T},H)} \\ &= \|X(\cdot) - \langle X(\cdot), \phi\rangle\phi\|^2_{L^2(\mathbb{T},H)} \\ &= \|X(\cdot)\|^2_{L^2(\mathbb{T},H)} - \|\langle X(\cdot), \phi\rangle_H\|^2_{L^2(\mathbb{T})} \\ &= \|X(\cdot)\|^2_{L^2(\mathbb{T},H)} - \langle \phi, L\phi\rangle_H. \end{aligned} \quad (7)$$

From this it is clear that the eigenfunction of $L$ corresponding to the largest eigenvalue is the optimal choice for $\phi$.

Since $X(t) \in H$, and since the eigenvector of $L$ form an orthonormal basis, we have that

$$\|X(t)\|^2_H = \sum_{n=1}^{\infty} |\langle X(t), \phi_n\rangle_H|^2$$

Thus

$$\begin{aligned} \|X(\cdot)\|^2_{L^2(\mathbb{T},H)} &= \sum_{n=1}^{\infty} \left\| |\langle X(\cdot), \phi_n\rangle_H|^2 \right\|^2_{L^2(\mathbb{T})} \\ &= \sum_{n=1}^{\infty} \langle \phi_n, L\phi_n\rangle_H \\ &= \sum_{n=1}^{\infty} \lambda_n. \end{aligned}$$

Since $X(\cdot) \in L^2(\mathbb{T}, H)$ we conclude that $L$ is nuclear. Furthermore, from (7), we conclude the assertion. ■

The above results are very general. Now we shall apply them to the impulse response of system (1). Hence we have that $\mathbb{T} = [0, \infty)$ and $X(t) = h(t) = T(t)b$. From Lemma 2.1 we know that we have to study the operator $L$. However, it is well-known that this operator is the unique solution of the Lyapunov equation

$$\langle L_{\text{cont}}x_1, A^*x_2\rangle + \langle A^*x_1, L_{\text{cont}}x_2\rangle = -\langle x_1, b\rangle\langle b, x_2\rangle \quad (8)$$

for all $x_1, x_2 \in D(A^*)$. Hence in order to obtain the P.O.D. of the impulse response, one needs to solve a Lyapunov equation. This result already appeared in Moore [3], and although this paper is frequently cited, the citation hardly ever refer to this result.

Hence we know which signal is the best approximation of the impulse response $h$. However, the time behavior of this signal, $\alpha(\cdot)$ will still be high-dimensional. In order to obtain a low dimensional time behavior, one normally projects the system (2) on the space spanned by $\phi$. By doing this, one obtains the system

$$\dot{\xi}(t) = A_{\text{pod}}\xi(t) + b_{\text{pod}}u(t) \quad (9)$$

where $A_{\text{pod}} = \langle A\phi, \phi\rangle$ and $b_{\text{pod}} = \langle b, \phi\rangle$. Thus $h(t)$ is approximated by $e^{A_{\text{pod}}t}b_{\text{pod}}\phi$. However, by doing so, one loses the errors estimates as given in Lemma 2.1. In the sequel we present another way for obtaining low order approximations. The advantage of these methods is that we do obtain error bounds.

## 3 $L^2$-Optimal first order approximations

In this section we show that it is possible to approximate $h(\cdot)$ by a first order time behavior on a one dimensional subspace of $Z$. The problem that we study is to find $C_a$ and $\lambda$ such that

$$\int_0^{\infty} \|h(t) - C_a e^{\lambda t}\|^2 dt \quad (10)$$

is minimized. It is easy to see that the expression in (10) is finite if and only if the real part of $\lambda$ is negative. Since we want that the approximated system is real, we assume that $\lambda$ is real and negative. Under this assumption, we write (10) as

$$\begin{aligned} &\int_0^{\infty} \|h(t) - C_a e^{\lambda t}\|^2 dt \\ &= \int_0^{\infty} \|h(t)\|^2 dt - 2\langle \hat{h}(-\lambda), C_a\rangle - \|C_a\|^2/2\lambda, \end{aligned} \quad (11)$$

where $\hat{h}$ is the Laplace transform of $h$. We minimize this expression for fixed $\lambda$, and obtain that the optimal $C_a$ is given by

$$C_a^{\mathrm{opt}} = -2\lambda \hat{h}(-\lambda). \quad (12)$$

With this optimal $C_a$, the expression in (11) becomes

$$\int_0^\infty \|h(t) - C_a e^{\lambda t}\|^2 dt = \int_0^\infty \|h(t)\|^2 dt + 2\lambda \|\hat{h}(-\lambda)\|^2. \quad (13)$$

Hence the optimization problem as described above become now to find that $\lambda \in (-\infty, 0)$ such that

$$G(\lambda) := \lambda \|\hat{h}(-\lambda)\|^2$$

is minimized.

## 4 Error bounds for the P.O.D. approximation

Before we can provide error bounds for the P.O.D. approximation we need to discuss some result for balanced approximation.

Consider the system (2), but with an output, i.e.,

$$\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = Cx(t). \quad (14)$$

Let $L_{\mathrm{cont}}$ again denote the solution to the Lyapunov equation (8), and let $L_{\mathrm{obs}}$ denote the (unique) solution of

$$\langle Ax_1, L_{\mathrm{obs}} x_2 \rangle + \langle L_{\mathrm{obs}} x_1, Ax_2 \rangle = -\langle Cx_1, Cx_2 \rangle, \quad (15)$$

for $x_1, x_2 \in D(A)$. We say that the system (14) is *balanced* if $L_{\mathrm{obs}} = L_{\mathrm{cont}} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, \ldots,)$. Normally the system will not be balanced. However, we can find a new realization $\Sigma(A^{\mathrm{bal}}, b^{\mathrm{bal}}, C^{\mathrm{bal}})$ of $C(sI-A)^{-1}b$ which is balanced. Truncating this balanced realization after the first state, and hence having a approximated system of dimension one, we have the error estimate

$$\|C(sI-A)^{-1}b - C_1^{\mathrm{bal}}(sI - A_1^{\mathrm{bal}})^{-1} b_1^{\mathrm{bal}}\|_\infty \leq 2 \sum_{n\geq 2} \sigma_n, \quad (16)$$

where $\|\cdot\|_\infty$ denotes the $\mathcal{H}_\infty$-norm, i.e., the supremum over the imaginary axis. Note that a small $\mathcal{H}_\infty$-error is normally needed if one wants to do the controller design for (14) on basis of an approximation. We show that the P.O.D. approximation which we obtained in Section 2 can be seen as a special balanced realization, and hence we obtain an estimate on the $\mathcal{H}_\infty$-error. In order to simplify the argument, we assume that $A$ satisfies

$$A^* + A = -C^* C, \quad (17)$$

where $C$ satisfies for some $\alpha > 0$

$$\|Cx\| \geq \alpha \|x\| \quad (18)$$

for all $x \in D(A)$. Furthermore, we assume that $L_{\mathrm{cont}}$ is compact. Let $\{\phi_n\}$ be the eigenvectors of $L_{\mathrm{cont}}$, and let $\lambda_n$ denote the eigenvalues, numbered in decreasing order. Thus $\phi_1 = \phi$.

Since $L_{\mathrm{cont}}$ is compact and self-adjoint, we know that there exists a bounded, unitary operator $S$ such that $Se_n = \phi_n$, where $\{e_n\}$ is the standard basis of $H$. Using this and equations (8) and (17), it is not hard to see that $\Sigma(S^{-1} L_{\mathrm{cont}}^{-1/4} A L_{\mathrm{cont}}^{1/4} S, S^{-1} L_{\mathrm{cont}}^{-1/4} b, C L_{\mathrm{cont}}^{1/4} S)$ is a balanced realization of $\Sigma(A, b, C)$. Furthermore, the $\sigma_n$'s are given as the square root of $\lambda_n$'s. Thus we know an $\mathcal{H}_\infty$-error estimate, but we still need to find the balanced truncation.

By definition of $C_1^{\mathrm{bal}}$ we have that

$$\begin{aligned} C_1^{\mathrm{bal}} &= C^{\mathrm{bal}} e_1 = C L_{\mathrm{cont}}^{1/4} S e_1 = C L_{\mathrm{cont}}^{1/4} \phi_1 \\ &= C \lambda_1^{1/4} \phi_1 = \lambda_1^{1/4} C\phi. \end{aligned} \quad (19)$$

Similarly, one can show that

$$A_1^{\mathrm{bal}} = \langle A\phi, \phi \rangle, \qquad b_1^{\mathrm{bal}} = \lambda_1^{-1/4} \langle b, \phi \rangle, \quad (20)$$

The differential equation corresponding to the truncated balanced realization is given as

$$\dot{x}(t) = A_1^{\mathrm{bal}} x(t) + b_1^{\mathrm{bal}} u(t), \quad y(t) = C_1^{\mathrm{bal}} x(t)$$

Note that by (9) $A_1^{\mathrm{bal}} = A_{\mathrm{pod}}$, and $b_1^{\mathrm{bal}} = \lambda_1^{-1/4} b_{\mathrm{pod}}$. Hence using (16) and (18)–(20) we have the following error estimate for (9).

$$\begin{aligned} &\alpha \|(sI-A)^{-1}b - (sI - A_{\mathrm{pod}})^{-1} b_{\mathrm{pod}} \phi\|_\infty \\ &\quad \leq \|C(sI-A)^{-1}b - C(sI - A_{\mathrm{pod}})^{-1} b_{\mathrm{pod}} \phi\|_\infty \\ &\quad = \|C(sI-A)^{-1}b - \lambda_1^{-1/4} C_1^{\mathrm{bal}} (sI - A_{\mathrm{bal}})^{-1} \lambda_1^{1/4} b_{\mathrm{bal}}\|_\infty \\ &\quad = \|C(sI-A)^{-1}b - C_1^{\mathrm{bal}} (sI - A_1^{\mathrm{bal}})^{-1} b_1^{\mathrm{bal}}\|_\infty \\ &\quad \leq 2 \sum_{n \geq 2} \lambda_n^{1/2}. \end{aligned} \quad (21)$$

This seems to be the first error bound for the P.O.D. system approximation.

We would like to remark that the conditions (17) and (18) can be replaced by the condition that $T(t)$ generates a strict contraction, or by the condition that there exists an output operator $C$ such that $\Sigma(A, b, C)$ is exactly observable. In the latter case one may need a similarity transformation changing the constant 2 in (21).

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