

A new treatment of commutation-errors in large-eddy simulation

A.W., Vreman and B.J., Geurts

Faculty of Mathematical Sciences, University of Twente
 P.O. Box 217, 7500 AE Enschede, The Netherlands

Preprint of the work published in Advances in turbulence IX, Eds IP Castro, PE Hancock, TG Thomas (CIMNE, Barcelona 2002), p199-202.

In the traditional formulation of the large-eddy equations the spatial filter operation is assumed to be of convolution-type, i.e., $\bar{u}_i = G * u_i$ where G is the filter-kernel. These filters commute with spatial differentiation ($\overline{\partial_i u_j} = \partial_i \bar{u}_j$) and necessarily imply a constant filter-width Δ . The commonly adopted decomposition of the convective term corresponding to convolution filters is

$$\overline{\partial_j(u_i u_j)} = \partial_j(\bar{u}_i \bar{u}_j) + \partial_j \tau_{ij} \quad \text{with} \quad \tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \quad (1)$$

in which the well-known turbulent stress-tensor τ_{ij} is introduced which represents the subgrid closure problem.

Turbulent flow in complex geometries, such as frequently arise in technological applications, are usually statistically stationary and characterized by strong spatial variations in local turbulence levels. Comparably quiescent, laminar regions may be found next to regions in which the flow displays intense turbulent fluctuations. In order to simulate these problems efficiently a filter-width which adapts to these variations offers appealing advantages [1]. However, the use of spatially non-uniform filter-widths implies that the filter-operation no longer commutes with the derivative operators ($\overline{\partial_i u_j} \neq \partial_i \bar{u}_j$). A common approach to deal with these situations implies the interchange of the filter with differentiation and consequently the introduction of additional commutation-errors [1], [2]. The corresponding decomposition of the convective term then reads:

$$\overline{\partial_j(u_i u_j)} = \partial_j(\bar{u}_i \bar{u}_j) + \partial_j \tau_{ij} + \left(\overline{\partial_j(u_i u_j)} - \partial_j(\bar{u}_i \bar{u}_j) \right) \quad (2)$$

where the last term is the commutation-error which represents an additional subgrid closure problem that needs to be considered separately.

We propose an alternative, more direct treatment of large-eddy simulation based on spatially non-uniform filtering. Instead of actually interchanging filtering as in equation 2 we directly consider the spatially filtered derivative operator. Specifically we define $D_j(\cdot) = \overline{\partial_j(\cdot)}$ and assume a top-hat filter with a volume

of eight grid-cells at the collocated locations, i.e. at grid point (i, j, k) the filter width is given by $\Delta_1 = x_1^{i+1} - x_1^{i-1}$, $\Delta_2 = x_2^{j+1} - x_2^{j-1}$, and $\Delta_3 = x_3^{k+1} - x_3^{k-1}$, where $x_m^{i,j,k}$ denotes the collocated grid in the x_m direction. For technical convenience we assume the grid to be orthogonal, but the method can also be formulated for more complicated grids.

For filtered-derivatives in e.g. the x_1 -direction we can write without any approximation

$$(D_1 f)_{i,j,k} = \frac{1}{\Delta_1 \Delta_2 \Delta_3} \int_{x_3^{k-1}}^{x_3^{k+1}} \int_{x_2^{j-1}}^{x_2^{j+1}} \int_{x_1^{i-1}}^{x_1^{i+1}} \partial_1 f(\mathbf{x}) dx_1 dx_2 dx_3 \quad (3)$$

$$= \frac{1}{\Delta_1 \Delta_2 \Delta_3} (F_{i+1} - F_{i-1}) \quad (4)$$

The symbol F_i denotes the surface integral of the quantity f at the volume face $x_1 = x_1^i$. Next the surface integrals are numerically approximated with, e.g., the trapezoidal rule, which is directly applicable on a non-uniform grid. In this formulation we do not assume commutation of the filtering with partial differentiation, but treat the combination of the filter and spatial derivative directly. The approach is related to Schumann's finite volume method [3], but we use a larger volume and do not assume that a filtered quantity is constant within the volume.

The closure problem in this formulation can be expressed as

$$D_j(u_i u_j) = D_j(\bar{u}_i \bar{u}_j) + D_j a_{ij} \quad \text{with} \quad a_{ij} = u_i u_j - \bar{u}_i \bar{u}_j \quad (5)$$

where a_{ij} is the new subgrid stress-tensor which corresponds to the combined filtered-derivative operation. It is not perceived to be a problem that this definition of the turbulent stress does not satisfy Galilean invariance for general filters, because the non-uniformly filtered Navier-Stokes equations are not Galilean invariant either [2].

A model for the new subgrid stress-tensor a_{ij} will be treated with a new dynamic procedure. The standard dynamic procedure for the subgrid modeling problem either assumes that the model-coefficient C is constant on the so-called test-filter level denoted by $\widehat{(\cdot)}$ or a difficult integral equation needs to be solved [2], [4]. In contrast, the new subgrid stress-tensor a_{ij} satisfies a new 'Germano-type' identity

$$A_{ij} - a_{ij} = B_{ij} \quad (6)$$

where $A_{ij} = u_i u_j - \widehat{\bar{u}_i \bar{u}_j}$ is the new subgrid-stress on the $\widehat{(\cdot)}$ level and $B_{ij} = \bar{u}_i \bar{u}_j - \widehat{\bar{u}_i \bar{u}_j}$ is the new resolved contribution. The essential difference with the traditional Germano-identity is that the contribution of a_{ij} in equation 6 does not require test-filtering.

First, we model a_{ij} with a dynamic eddy-viscosity model, given by $m_{ij} = -C \Delta^2 |S(\bar{\mathbf{u}})| S_{ij}(\bar{\mathbf{u}})$. Substituting m_{ij} in equation 6 gives rise to a set of equations for C without the assumption that the model-coefficient is constant

on the scale of the test-filter. The model-coefficient C is obtained with the standard least-squares approach,

$$C = \frac{\langle M_{ij} B_{ij} \rangle}{\langle M_{ij} M_{ij} \rangle} \quad \text{with} \quad M_{ij} = -C(\kappa\Delta)^2 |S(\widehat{\mathbf{u}})| S_{ij}(\widehat{\mathbf{u}}) + C\Delta^2 |S(\overline{\mathbf{u}})| S_{ij}(\overline{\mathbf{u}}). \quad (7)$$

Note that there is no test-filter operation in the second term of M_{ij} . To avoid that the filter operation crosses solid boundaries, it is appropriate to choose the test-filter operation equal to the basic filter operation adopted in equation 3, i.e. $\widehat{(\cdot)} = \overline{(\cdot)}$, which implies $\kappa = \sqrt{2}$ [4].

In addition to the new dynamic eddy-viscosity model (M1), a new dynamic mixed model (M2) can be formulated,

$$m_{ij} = \left(\overline{u_i u_j} - \overline{u_i} \overline{u_j} \right) - C\Delta^2 |S(\overline{\mathbf{u}})| S_{ij}(\overline{\mathbf{u}}), \quad (8)$$

where the term between brackets is the similarity approximation of a_{ij} and C is obtained by substituting m_{ij} in equation 6.

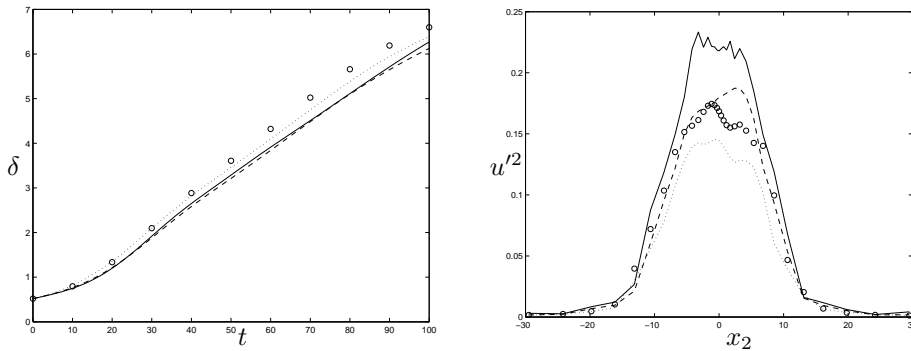


Figure 1: Momentum thickness (left) and streamwise Reynolds stress profile at $t = 70$ (right). Filtered DNS (o), LES with no model (solid), M1 (dashed) and M2 (dash-dotted).

Large-eddy simulations with the new models M1 and M2 are performed for the temporal mixing layer, the flow documented in [4]. The 32^3 -grid is stretched in the normal direction, clustered in the central region and the largest and smallest grid-spacing differ with a factor 17. These simulations are compared with a coarse-grid simulation with no subgrid-model (M0) and with a non-uniformly filtered DNS, originally performed on a uniform 192^3 -grid.

The large-eddy equations are solved on a collocated grid in a compressible formulation at low Mach number, such that compressibility effects are unimportant. With respect to spatial discretization the operator D_j is applied to $\overline{u_i u_j}$ and also to the pressure, whereas the derivatives in the viscous and subgrid-terms are treated with smaller control volumes [4]. Several numerical approximations

for the surface integrations in equation 3 and the top-hat filter operations that explicitly occur in the models can be applied. For example, the trapezoidal rule with uniform weights (A, second order), the trapezoidal rule with non-uniform weights (B, second order) and an integration over the approximate parabolic function (C, third order). For the present grid, the differences between the results for these methods are very small, therefore only results for method A are presented.

The results shown in figure 1 indicate that the models M1 and M2 are improvements when compared to M0. The jagged structure of the profile obtained with M0 in the right figure indicates that the resolved field contains an excessive amount of small scales. The models M1 and M2 have about the correct dissipative behaviour.

Summary

An efficient extension of large-eddy simulation to strongly non-uniform turbulent flow involves the use of filters which do not commute with spatial derivatives. In particular, these filters have a spatially varying filter-width in order to adapt to variations in local turbulence levels. We proposed a new treatment for the commutation-problem in which the spatial filtering of the convective term is directly combined with the derivative-operator rather than interchanged with this operator. The latter would lead to additional commutation-errors. This re-formulation gives rise to an adaptation of the remaining subgrid modeling problem which is considered in a new dynamic context. Unlike the traditional dynamic procedure, the new procedure does not require the assumption that the model coefficient is slowly varying on the test-filter scale. The new approach was tested on a moderately stretched grid and provided satisfactory results when the new dynamic models were employed.

References

- [1] B.J. Geurts, A.W. Vreman and J.G.M. Kuerten. Proceedings *74th Fluid Dynamics Panel and Symposium on Application of DNS and LES to transition and turbulence, Crete*, AGARD Conf. Proceedings 551:51, 1994.
- [2] S. Ghosal. Mathematical and physical constraints on LES of turbulence. *AIAA J.*, 37:425, 1999.
- [3] U. Schumann. Subgrid scale model for finite difference simulations of turbulent flows in plane channels and annuli. *J. Comp. Phys.*, 18:465, 1975.
- [4] A.W. Vreman, B.J. Geurts and J.G.M. Kuerten. Comparison of subgrid-models in large eddy simulation of the temporal mixing layer. *J.Fluid Mech.*, 339:357, 1997