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Caravan Awnings: a Geometrical Problem

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Abstract

Two questions regarding the design of caravan awnings were posed by a company. The company wishes to produce awnings with a pretty appearance. When an awning is attached to a caravan, some wrinkles could appear. We developed some methods to avoid the wrinkles. The problem is restricted to awnings which are made from one piece of cloth.

Keywords

Developable surfaces, caravan awning, minimal surface, optimal surface design.

4.1 Introduction

An interesting problem was posed by Gerjak B.V., a company which specializes in caravan awnings and tents. Mathematicians were asked to help the company to create an applicable mathematical model for designing good-looking caravan awnings.

A caravan awning is needed to protect campers from unpleasant weather conditions. The customers' expectation is to have good-looking awnings for their caravans. When it is attached to the front of the caravan, the awning should be taut and visually pleasing, so no wrinkles should appear. The back part of the awning is attached along the upper part of the caravan's body by a rail. This means that the back part of the awning follows the curved shape of the caravan's body, which is not triangular. In this report, we consider the classical caravan awning, i.e. the one with a triangular shape at the front. An example is given in Figure 4.1. This type of awning is preferred by most people. Because of the specific geometry of the awning's frame, i.e.

Figure 4.1: The classical caravan awning, with triangular shape at the front

the triangular shape at the front and the geometry of the upper part of the caravan's body at the back, it is not easy to design awning's patterns which suit very well to this kind of frame and the cloth could wrinkle.

The Gerjak company produces both tailor-made and ready-made awnings. A tailor-made awning is designed to fit onto one caravan only, whereas a ready-made awning is designed to fit onto more than one caravan.

This report is organized as follows: Mathematical preliminaries about developable surfaces are given in Section 4.2. A model for designing the tailor-made awning is discussed in Section 4.3, which also includes computer results for the awning design. In Section 4.4 we suggest an algorithm to make the best-possible ready-made awning. Finally, conclusions are given in the last section.

4.2 A brief theory of developable surfaces

The following theory is taken from [1]. It explains the notion of developable surfaces. In short, a developable surface can be laid-down in the two-dimensional plane, without splitting into parts, stretching, shrinking, or compressing. First of all, we introduce the concept of ruled surfaces. A ruled surface is a surface that can be constructed by straight lines and so admits a parameterization of the form

$$\mathbf{r}(s,t) = \mathbf{y}(s) + t\mathbf{z}(s), \tag{4.1}$$

where $\mathbf{r}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$; $t, s \in [0, 1]$. For each $s \in [0, 1]$, the line L_s which passes through $\mathbf{y}(s)$ and is parallel to $\mathbf{z}(s)$ is called the generator, and the curve $\mathbf{y}(s)$ is called the directrix of the surface $\mathbf{r}$.

It turns out that every developable surface is also a ruled surface. A developable surface is a ruled surface that has the additional property, the unit normal is constant

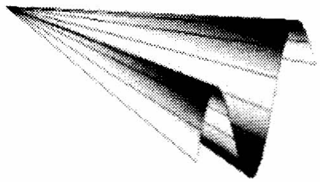

Figure 4.2: An example of a generalized cone

along every generator. Thus, the Gaussian curvature of a developable surface is identically equal to zero. In terms of $\mathbf{y}$ and $\mathbf{z}$, this means that the determinant vanishes:

$$|\mathbf{y}', \mathbf{z}, \mathbf{z}'| = 0. \tag{4.2}$$

Now we want to use the form (4.1) to construct a developable surface on a given close frame $\mathbf{x}_0$. Since a developable surface is also a ruled surface, there must be a line segment inside the surface from any point $\mathbf{x}_0(s)$ to some other point on the curve. We shall call this point $\mathbf{x}_0(\sigma(s))$. Now the parameterization of the surface takes the form

$$\mathbf{r}(s,t) = (1-t)\mathbf{x}_0(s) + t\mathbf{x}_0(\sigma(s)),$$

By substituting this to (4.2) we get

$$\left|\mathbf{y}', \mathbf{z}, \mathbf{z}'\right| = \frac{d\sigma}{ds}\left|\mathbf{x}_0(\sigma(s)) - \mathbf{x}_0(s), \mathbf{x}_0'(\sigma(s)), \mathbf{x}_0'(s)\right| = 0. \tag{4.3}$$

It is obvious that (4.3) is satisfied if σ is constant. The mapping $\sigma \equiv$ constant can be applied on any generalized cone. Therefore, every generalized cone is developable. See Figure 4.2 for an example of a generalized cone.

In general, it is not possible to construct a developable surface inside a given awning's frame for which the second term in the product, $\left|\mathbf{x}_0(\sigma(s)) - \mathbf{x}_0(s), \mathbf{x}_0'(\sigma(s)), \mathbf{x}_0'(s)\right|$, is equal to zero. Therefore, we construct the awning by combining generalized cones.

4.3 Tailor-made awnings

Examples of developable caravan awnings which are constructed from three generalized cones are given in figures 4.3 and 4.4. For every chosen number of generalized cones, two basic designs can be chosen: the base of the left-most generalized cone may be placed either at the front or at the back of the awning. The difference is clear when Figure 4.3 and Figure 4.4 are compared.

This surface has area 8.8718

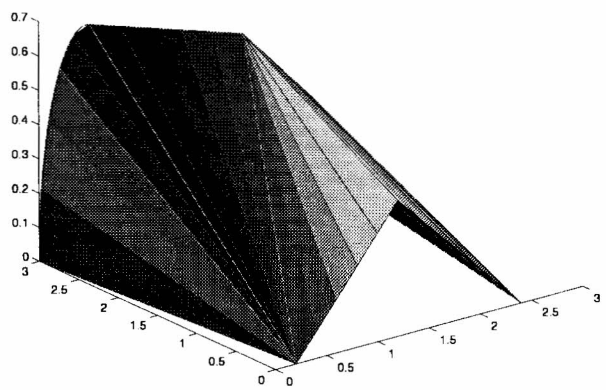

Figure 4.3: An example of a developable awning of type T_3' (constructed from three generalized cones)

This surface has area 8.8606

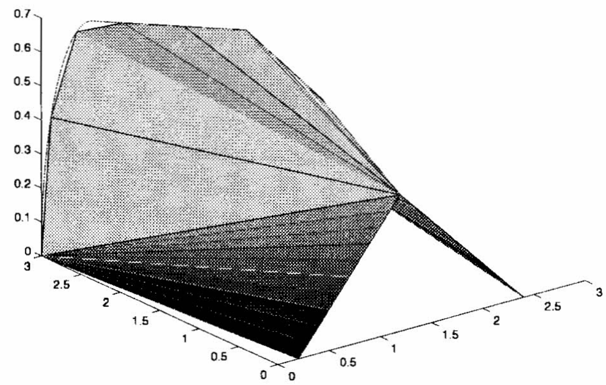

Figure 4.4: An example of a developable awning of type T_3 (constructed from three generalized cones)

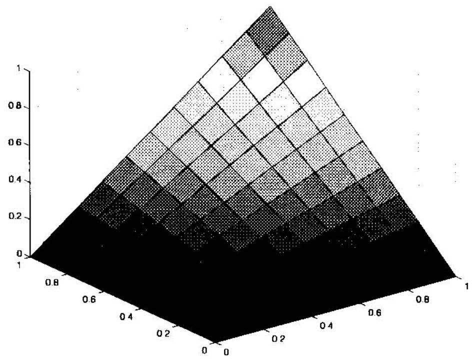

Figure 4.5: A nondevelopable surface given by $z = xy$.

This can be used to make a *classification* of all different designs, which we shall call T_N and T'_N. A design of type T_4, for example, consists of four generalized cones, and the left-most generalized cone has its base at the front. Analogously, a design of type T'_4 also consists of four generalized cones, but the left-most generalized cone has its base at the back.

The two designs of type T_3 and T'_3 can only differ in the location of the point where all the three generalized cones meet. Therefore, the entire space of T_3 and T'_3-designs may be parameterized by one scalar. In general, the design-classes T_N and T'_N may be parameterized by $N - 2$ degrees of freedom.

The idea that an arbitrary number of generalized cones may be used for the construction of a developable surface leads to an interesting notion. As an example, we consider a surface given by $z = xy$ on $0 \leq x \leq 1$ and $0 \leq y \leq 1$. This surface is given in Figure 4.5. This closed frame can be described by the following closed curve in the three dimensional space:

$$(x(s), y(s), z(s)) = \begin{cases} (s, 0, 0) & , 0 \leq s \leq 1, \\ (1, s-1, s-1) & , 1 \leq s \leq 2, \\ (3-s, 1, 3-s) & , 2 \leq s \leq 3, \\ (0, 4-s, 0) & , 3 \leq s \leq 4, \end{cases}$$

in which $s \in [0, 4]$. This surface $z = xy$ is ruled, but not developable. Nevertheless, a developable surface which is constructed from a number of generalized cones can be fit into the frame. As the number of generalized cones increases, we notice that

- The distance between the developable surface and the surface $z = xy$ becomes arbitrarily small. This distance is defined as the $\mathbb{R}^2$−norm of the difference between the two surfaces.

- The two dimensional pattern of the developable surface seems to converge to some limit-shape.

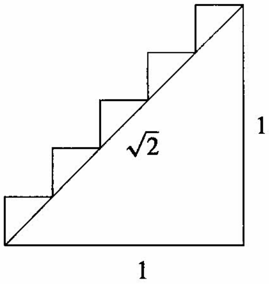

Figure 4.6: Approximation of a diagonal line with horizontal and vertical line segments

Remember that the surface $z = xy$ is non-developable! From this example we see that somehow, the limit of the series of developable surfaces is not developable.

Two more observations were made by doing numerical investigation:

- The area of the limit-shape is 4% larger than that of the surface $z = xy$.
- Adjacent triangles in the series of developable surfaces make angles which do not vanish in the limit.

The limit-surface is non-differentiable. The extra area of the surface is hidden by folding it back and forth many times. A simple analogy is shown in Figure 4.6. A diagonal line of length $\sqrt{2}$ can be approximated by a number of horizontal and vertical line segments. However, the total length of these segments will never converge to the length of the diagonal, because the ratio between this total length and the length of the diagonal line is always equal to $\sqrt{2}$.

Experimentally, using cardboard and paper for the close frame of the surface $z = xy$, it was confirmed that the constructed developable surface gives a wrinkly impression due to the excess area which had to be hidden by folding.

These ideas led to the following conclusions:

1. If a surface is larger than necessary, it gives a wrinkly appearance.
2. The smallest possible surface is probably optimal (i.e. the prettiest).

Some minimization procedures may be used to find the smallest possible awning pattern for each of the pattern classes which is defined in this section. In figures 4.3 and 4.4, we show some resulting designs for the caravan awning. In figures 4.7 and 4.8, we show the corresponding patterns of the awnings in figures 4.3 and 4.4, respectively.

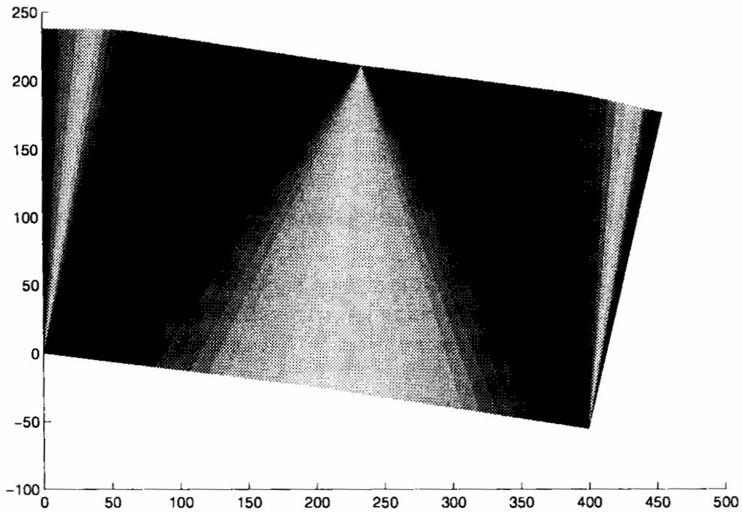

Figure 4.7: The 2D pattern of the awning in Figure 4.3

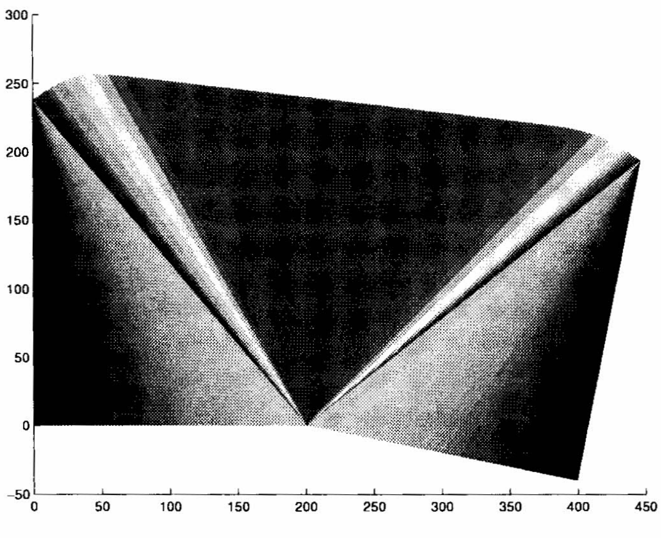

Figure 4.8: The 2D pattern of the awning in Figure 4.4

4.4 Ready-made awnings

In the previous section, we presented a method to construct an awning which fits nicely to one caravan. In this section, we present some methods to design a ready-made awning. The objective in designing a ready-made awning is to construct the best awning for some selections of caravans. Since it is not possible to make an awning that fits perfectly on each of those caravans, we only design an awning which fits fairly well.

There are two possible approaches to design ready-made awnings:

1. First of all, we choose one caravan as a representative of the entire selection of caravans. After that we design an awning which fits perfectly well on that chosen caravan. This idea is presented in Subsection 4.4.1.

2. Minimizing the average excess area of the awning's cloth, when it is attached to each of the caravans. This idea is presented in Subsection 4.4.2.

The first approach turns out to be much simpler than the second one. However, the first approach does not in any way guarantee that the solution will be good (and definitely not optimal), whereas the second approach does.

4.4.1 The average input approach

In Figure 4.9, we give a description of the general awning's frame by a finite number of parameters. The representative frame can be obtained by combining these parameters for a number of caravans to a (weighted) average. The weights can be determined from the relative importance of each of the caravan models, for example: the selling percentage.

The awning which fits optimally on the representative frame is also expected to fit nicely on each of the other caravans. However, this design cannot be called optimal by any meaningful sense of the word. In the following subsection, we will discuss a different approach, which leads into the construction of an awning which fits optimally, in some sense of the word. Due to the limitation of time, the authors were not able to investigate it in great mathematical detail.

4.4.2 The minimal wrinkle approach

In this subsection, we investigate the possibility of constructing the optimal ready-made awning for a given set of caravans. Each caravan is numbered from 1 to N. The construction is done by minimizing the average wrinkle on the awning's cloth. In order to do so, it is clear that we need to measure or to quantify the wrinkle.

First of all, we make the following assumptions.

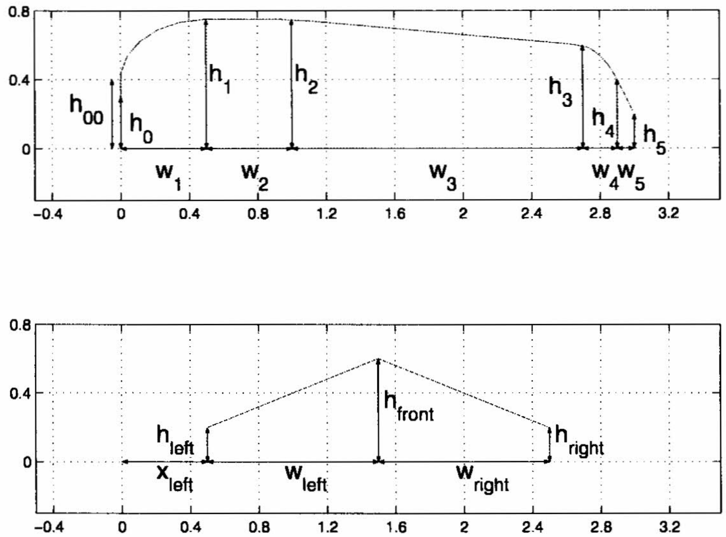

Figure 4.9: Parameterization of the awning's frame

- If a flat surface $D \subset \mathbb{R}^2$ is the two-dimensional pattern of a three-dimensional object E which is nicely attached on a certain close frame, then a larger flat surface $D' \supset D$ can be forced to fit in that close frame. An experiment was done by using cardboard and paper. It was not easy to hide the excess area by folding.

- When the two-dimensional cloth is attached on the close frame, the *excess area,* i.e., the area difference between D' and D, appears as wrinkles on the three-dimensional object. Therefore the quantity $\|D'\| - \|D\|$ is a suitable measure for wrinkling.

Since the notions of 'a wrinkle' and 'a visually pleasing surface' are not mathematically defined, this assumption can be verified nor falsified. This way of quantifying the wrinkle has the following consequences:

- The shape of the pattern is not considered. Two patterns are defined to be equally wrinkled if they have the same excess area. Nevertheless in reality, after we attach them to the close frame, it is possible that one pattern looks better than the other.

- If one pattern has twice as much excess area as the other pattern, then we say that the first pattern is twice wrinkled than the other one.

- It does not matter where the wrinkle is located (e.g., at the back, side, middle or front part of the awning).

The consequences above show the difficulties in defining 'a visually pleasing surface' mathematically. This notion is abstract and subject to personal feelings.

Another aspect of setting up an awning is the fact that the owner of the awning has freedom in attaching the awning to the caravan. Although the shape of the caravan is fixed, he may shift the awning cloth from left to right, adjust the length of the tent and awning poles and the place where the poles are put. In fact, this action of setting up the awning/tent determines the parameters x_{left}, h_{00}, h_5 and w_5, h_{left}, h_{front}, h_{right} w_{left}, w_{right} and the depth of the tent as given in Figure 4.9. These free parameters may be adjusted in order to minimize the wrinkles.

We summarize the process of setting up a given cloth D on the i–th caravan as follows: The frame which the owner creates by setting his free parameters define another pattern D_i. This ideal pattern must be smaller than the available cloth, D. In other words $D_i \subset D$. If this is not the case, the cloth D will not fit in the frame. The excess area $\|D\| - \|D_i\|$ measures the wrinkles. We assume that this adjustment process leads to a unique final result, namely D_i^*. This optimal setup D_i^* is a function of the available tent sheet,

$$D_i^* = D_i^*(D) \tag{4.4}$$

The basic structure of the minimization procedure for the wrinkle is relatively simple. The optimal pattern, namely D^*, fits optimally on all caravans, $1, \dots, N$. This means:

$$\sum_{i=1}^{N} w_i |D^* - D_i^*(D^*)| = \min_{D \in V} \sum_{i=1}^{N} w_i |D - D_i^*(D)|^2 \tag{4.5}$$

In this formula, V is the space of all possible awning's patterns, and the each weight, w_i, is due to the relative importance of each caravan.

4.5 Concluding remarks

As a follow-up of the industrial week, the results have been used to answer one of the question which is asked by the tent/awning manufacturing company. Some examples of the awning patterns were produced by using computer graphics. These examples have been used to make a few adjustments to the designs which are actually made by the company. Further ideas about the optimal shapes for ready-made awnings were developed, but they were beyond the manufacturer's needs.

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