The Complexity of Graph Contractions

Asaf Levin[1], Daniël Paulusma[2], and Gerhard J. Woeginger[2]

[1] Department of Statistics and Operations Research,
Tel Aviv University, Tel Aviv 69978, Israel.
levinas@post.tau.ac.il
[2] Faculty of Mathematical Sciences,
University of Twente, 7500 AE Enschede, The Netherlands.
{d.paulusma,g.j.woeginger}@math.utwente.nl

Abstract. For a fixed pattern graph H, let H-CONTRACTIBILITY denote the problem of deciding whether a given input graph is contractible to H. We continue a line of research that was started in 1987 by Brouwer & Veldman, and we determine the computational complexity of H-CONTRACTIBILITY for certain classes of pattern graphs. In particular, we pin-point the complexity for all graphs H with five vertices. Interestingly, in all cases that are known to be polynomially solvable, the pattern graph H has a dominating vertex, whereas in all cases that are known to be NP-complete, the pattern graph H does not have a dominating vertex.

1 Introduction

All graphs in this paper are undirected, finite, and simple. Let $G = (V, E)$ be a graph, and let $e = [u, v] \in E$ be an arbitrary edge. The *edge contraction* of edge e in G removes the two end-vertices u and v from G, and replaces them by a new vertex that is adjacent to precisely those vertices to which u or v were adjacent. The *edge deletion* of edge e removes e from E. The *edge subdivision* of e removes e from E, and introduces a new vertex that is adjacent to the two end-vertices u and v. A graph G is *contractible* to a graph H (G is H-contractible), if H can be obtained from G by a sequence of edge contractions. A graph G contains a graph H as a *minor*, if H can be obtained from G by a sequence of edge contractions and edge deletions. A graph G is a *subdivision* of a graph H, if G can be obtained from H by a sequence of edge subdivisions.

Now let $H = (V_H, E_H)$ be some fixed connected graph with vertex set $V_H = \{h_1, \ldots, h_k\}$. There is a number of natural and elementary algorithmic problems that check whether the structure of graph H shows up as a *pattern* within the structure of some input graph G:

– PROBLEM: H-MINOR CONTAINMENT
INSTANCE: A graph $G = (V, E)$.
QUESTION: Does G contain H as a minor?

H.L. Bodlaender (Ed.): WG 2003, LNCS 2880, pp. 322–333, 2003.

- PROBLEM: H-SUBDIVISION SUBGRAPH
  INSTANCE: A graph $G = (V, E)$.
  QUESTION: Does G contain a subgraph that is isomorphic to some subdivision of H?
- PROBLEM: ANCHORED H-SUBDIVISION SUBGRAPH
  INSTANCE: A graph $G = (V, E)$; k pairwise distinct vertices $v_1, \ldots, v_k$ in V.
  QUESTION: Does G contain a subgraph that is isomorphic to some subdivision of H, such that the isomorphism maps vertex v_i of the subgraph of G into vertex h_i of the subdivision of H, for $1 \leq i \leq k$?
- PROBLEM: H-CONTRACTIBILITY
  INSTANCE: A graph $G = (V, E)$.
  QUESTION: Is G contractible to H?

1.1 Known Results

A celebrated result by Robertson & Seymour [3] states that H-MINOR CONTAINMENT can be solved in polynomial time $O(|V|^3)$ for every *fixed* pattern graph H. In fact, [3] fully settles the complexity of the first three problems on our problem list above:

Proposition 1. *(Robertson & Seymour [3])*
For any fixed pattern graph H, the three problems H-MINOR CONTAINMENT, H-SUBDIVISION SUBGRAPH, and ANCHORED H-SUBDIVISION SUBGRAPH are polynomially solvable in polynomial time. ■

What about the fourth problem on our list, H-CONTRACTIBILITY? Perhaps surprisingly, there exist pattern graphs H for which this problem is NP-complete to decide! For instance, Brouwer & Veldman [1] have shown that P_4-CONTRACTIBILITY is NP-complete. The main result of [1] is the following.

Proposition 2. *(Brouwer & Veldman [1])*
If H is a connected triangle-free graph other than a star, then H-CONTRACTIBILITY is NP-complete. If H is a star, then H-CONTRACTIBILITY is polynomially solvable. ■

Note that an equivalent way of stating Proposition 2 would be the following: H-CONTRACTIBILITY is NP-complete for every connected triangle-free graph H without a dominating vertex. H-CONTRACTIBILITY is polynomially for every connected triangle-free graph H with a dominating vertex. (A *dominating* vertex is a vertex that is adjacent to all other vertices.) Moreover, the paper [1] determines the complexity of H-CONTRACTIBILITY for all 'small' connected pattern graphs H: For $H = P_4$ and $H = C_4$, the problem is NP-complete (as implied by Proposition 2). For every other pattern graph H on at most four vertices, the problem is polynomially solvable.

The exact separating line between polynomially solvable cases and NP-complete cases of this problem (under P$\neq$NP) is unknown and unclear.

Brouwer & Veldman [1] write at the end of their paper that they expect the class of polynomially solvable cases to be very limited.

Watanabe, Ae & Nakamura [4] consider remotely related edge contraction problems where the goal is to find the minimum number of edge contractions that transform a given input graph G into a pattern from a certain given pattern class.

1.2 New Results

We follow the line of research that has been initiated by Brouwer & Veldman [1], and we classify the complexity of H-CONTRACTIBILITY for certain classes of pattern graphs that – in particular – contain all 'small' pattern graphs H with at most five vertices. Our results can be summarized as follows:

Theorem 1. *(Main result of the paper)*
Let H be a connected graph on at most five vertices. If H has a dominating vertex, then H-CONTRACTIBILITY is polynomially solvable. If H does not have a dominating vertex, then H-CONTRACTIBILITY is NP-complete.

It is difficult for us *not* to conjecture that the presence of a dominating vertex in the pattern graph H precisely separates the easy cases from the hard cases. However, we have no evidence for such a conjecture.

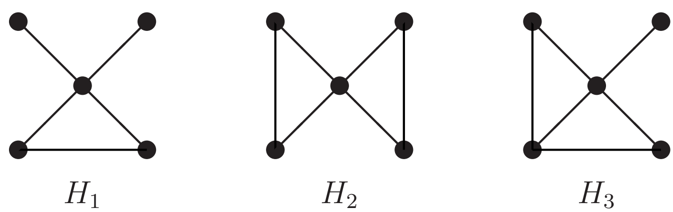

Fig. 1. The graphs $H_1 = H_1^*(2, 1, 0)$; $H_2 = H_1^*(0, 2, 0)$; and $H_3 = H_1^*(1, 0, 1)$.

There are fifteen graphs H on five vertices that are not covered by Proposition 2; these are exactly the connected graphs on five vertices that do contain a triangle; see Figures 1–6 for pictures of all these graphs. It turned out that ten of these fifteen graphs yield polynomially solvable H-CONTRACTIBILITY problems, whereas the other five of them yield NP-complete problems. Many of our results are actually more general: They do not only provide a specialized result for one particular five-vertex graph, but they do provide a result for an infinite family of pattern graphs, from which the result on the five-vertex graph falls out as a special case. Our main contributions may be summarized as follows:

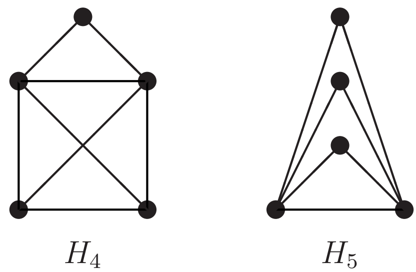

Fig. 2. The graphs $H_4 = H_2^*(1,1)$; and $H_5 = H_2^*(3,0)$.

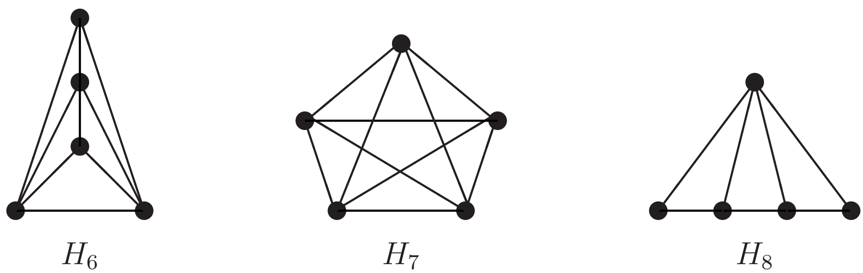

Fig. 3. The graphs $H_6 = H_3^*(2)$; $H_7 = K_5$; and $H_8 = P^+(4) = K_1 \bowtie P_4$.

(1) We analyze a class of cases where H contains one, two, or three dominating vertices, and where the set of non-dominating vertices induce a set of isolated vertices, isolated edges, and paths on three vertices. In Section 3, we prove that three subfamilies of this class yield polynomially solvable H-CONTRACTIBILITY problems. These classes contain the eight graphs H_1 thru H_8 on five vertices as depicted in Figures 1– 3.
Our structural results show that in case *some* H-contraction exists, then there also exists an H-contraction of a fairly primitive form. In our algorithmic results, we then enumerate all possibilities for these primitive pieces, and settle the remaining problems by applying the results of Robertson & Seymour [3].

(2) For the two five-vertex graphs H_9 and H_{10} as shown in Figure 4, we were not able to find 'straightforward' polynomial time algorithms. Our algorithms are based on lengthy (!) combinatorial investigations of potential contractions of an input graph to H_9 and H_{10}. They are not included in this extended abstract.

(3) In Section 4 we present a number of NP-completeness results. The NP-completeness proofs for the four (five-vertex) graphs $H_1^\#$, $H_2^\#$, $H_3^\#$, and $H_4^\#$ in Figure 5 are omitted in this extended abstract. They are done by reduction from HYPERGRAPH 2-COLORABILITY and they are inspired by a

similar NP-completeness argument of Brouwer & Veldman [1]. Theorem 5 and Theorem 6 present two generic NP-completeness constructions. As a special case, this yields NP-completeness of $H_5^{\#}$-CONTRACTIBILITY for the graph $H_5^{\#}$ in Figure 6.

The rest of this extended abstract contains the exact statements and some of the proofs for the above results.

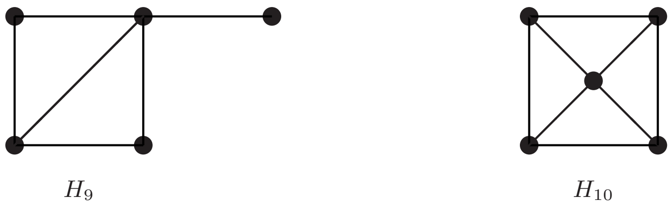

Fig. 4. The graphs H_9 and H_{10}.

2 Notations, Definitions, and Preliminaries

We denote by P_n the path on n vertices, by C_n the cycle on n vertices, and by K_n the complete graph on n vertices. For a subset $U \subset V$ we denote by $G[U]$ the induced subgraph of G over U; hence $G[U] = (U, E \cap U \times U)$.

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$, we denote their *join* by $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$, and their *disjoint union* by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. For the disjoint union $G \cup G \cup \cdots \cup G$ of k copies of the graph G, we write shortly kG_1; for $k = 0$ this yields the empty graph.

Consider a graph $G = (V, E)$ that is contractible to a graph $H = (V_H, E_H)$. An equivalent (and for our purposes more convenient) way of stating this fact is that

- for every vertex h in V_H, there is a corresponding connected subset $W(h) \subseteq V$ of vertices in G; we will sometimes say that $W(h)$ is the *witness* for vertex h.
- for every edge $e = [h_1, h_2] \in E_H$, there is a corresponding edge $W(e)$ in G that connects the vertex set $W(h_1)$ to the vertex set $W(h_2)$; we will sometimes say that this edge $W(e)$ in G is a *witness* for $[h_1, h_2]$.
- for every two vertices h_1, h_2 in H that are not connected by an edge in E_H, there are no edges between $W(h_1)$ and $W(h_2)$.

If for every $h \in V_H$, we contract the vertices in $W(h)$ to a single vertex, then we end up with the graph H. Note that in general, these witness sets $W(h)$

and witness edges $W(e)$ are not uniquely defined (since there may be many different sequences of contractions that lead from G to H). In our polynomial time algorithms, we will explore the structure of the witnesses, and often prove that there exists *at least one* witness with certain 'strong' and 'nice' properties.

Proposition 3. *For any fixed integer k, the following problem can be solved in polynomial time: Given a graph $G = (V, E)$ with up to k vertices that are colored by $\ell \leq k$ colors, can this coloring be extended to an ℓ-coloring of the whole vertex set V such that every color class induces a connected subgraph?*

Proof. Consider some fixed color c, and let $S_c \subseteq V$ be the set of all vertices that are pre-colored by color c. Any solution to the problem must contain a mono-chromatic tree T in color c whose leaf-set coincides with S_c. Such a tree T has at most k branch-vertices (that is, vertices of degree three or higher), and there is only a fixed number of different topologies for connecting the branch-vertices and the leaves in S_c to each other.

The strategy is as follows: For every color c, we guess the branch-vertices for a mono-chromatic tree with leaf-set S_c, and we also guess the topology of this tree. Since k is a fixed constant, this altogether only yields a polynomial number of guesses for all trees for all colors (where the degree of the polynomial depends on k). Then we are left with a special case of the ANCHORED H-SUBDIVISION SUBGRAPH problem that can be solved in polynomial time according to Proposition 1. ■

3 Some Simple Polynomially Solvable Cases

Consider a connected graph $G = (V, E)$ with a cut-vertex v, and let $C_1, \ldots, C_k$ denote the connected components of $G - v$. For $1 \leq i \leq k$ we say that the vertex subset $V - C_i$ is *induced* by the cut-vertex v,

For non-negative integers a, b, c, we let $H_1^*(a, b, c)$ be the graph $K_1 \bowtie (aK_1 \cup bK_2 \cup cP_3)$, $H_2^*(a, b)$ be the graph $K_2 \bowtie (aK_1 \cup bK_2)$, and $H_3^*(a)$ be the graph $K_3 \bowtie aK_1$.

Lemma 1. *Let u be the dominating vertex in $H_1^*(a, b, c)$. If a graph $G = (V, E)$ is contractible to $H_1^*(a, b, c)$, then there exists a witness structure with the following property: For every vertex $h \neq u$ in $H_1^*(a, b, c)$, the witness set $W(h)$ is either induced by some cut-vertex, or it consists of a single vertex.*

Proof. Consider a witness structure W for G with respect to $H_1^*(a, b, c)$ that maximizes the cardinality of $W(u)$. Let x, y, z be three vertices in $H_1^*(a, b, c) - u$ that induce a P_3 with edges $[x, y]$ and $[y, z]$. We will only show that the desired property holds for $W(x)$, $W(y)$, and $W(z)$. The arguments for the other cases are similar, but simpler.

Suppose for the sake of contradiction that there are two distinct vertices $x_1, x_2 \in W(x)$ that both have neighbors in $W(y)$. Let $[u_1, x_3] \in E$ be a witness edge for $[u, x] \in E_H$. Consider a tree T within $W(x)$ with the minimum number of edges that connects x_1, x_2, x_3 to each other. Then we could move vertex x_3 (and part of this tree T, and possibly some other vertices) from $W(x)$ to $W(u)$ while keeping the witness structure intact. This would increase the cardinality of $W(u)$. This contradiction shows that $W(x)$ contains exactly one vertex x_1 with neighbors in $W(y)$. Next, suppose that there is some vertex $x_4 \in W(x)$ with $x_4 \neq x_1$, such that x_4 has an edge to $W(u)$. Then we could move x_4 (and part of a path from x_4 to x_1 within $W(x)$, and perhaps some other vertices) from $W(x)$ to $W(u)$. This second contradiction shows that x_1 is the unique vertex in $W(x)$ with neighbors outside $W(x)$. A symmetric argument shows that $W(z)$ contains a unique vertex z_1 with neighbors outside $W(z)$. Hence, $W(x)$ and $W(z)$ both are of the desired form.

Let us turn to $W(y)$. Consider a vertex $y_1 \in W(y)$ that has a neighbor in $W(u)$. First we discuss the case where y_1 is a cut-vertex of the subgraph induced by $W(y)$. Let $C_1, \ldots, C_k$ denote the connected components induced by $W(y)-y_1$. If some component, say C_1, is adjacent to both x_1 and z_1, then we could redefine $W(y) := C_1$ and merge all the other components together with y_1 into $W(u)$; a contradiction. Hence, we may assume that every component is adjacent to at most one of the two vertices x_1 and z_1. If x_1 has a neighbor $y_2 \in W(y)$ with $y_2 \neq y_1$, then we could redefine $W(x) := \{y_2\}$ and merge x_1 (and the rest of $W(x)$) into $W(u)$; a contradiction to the choice of $W(u)$. A symmetric argument shows that the only neighbor of z_1 in $W(y)$ is y_1. But now we are done: If $W(u)$ has a neighbor $y_3 \in W(y)$ with $y_3 \neq y_1$, then we may merge y_3 (and maybe some other vertices in $W(y)$) into $W(u)$. If $W(u)$ does not have any other neighbor but y_1 in $W(y)$, then all the edges between $W(x) \cup W(u) \cup W(z)$ and $W(y)$ are incident to vertex y_1. Hence, the witness set $W(y)$ is either induced by the cut-vertex y_1, or it consists of the single vertex y_1.

In the remaining case, vertex y_1 is not a cut-vertex of the subgraph induced by $W(y)$. If x_1 and z_1 both have neighbors other than y_1 in $W(y)$, then we could simply move y_1 into $W(u)$, and arrive at a contradiction. If y_1 is the unique neighbor of x_1 in $W(y)$ and if z_1 also has another neighbor $y_4 \in W(y)$, then we could redefine $W(z) := \{y_4\}$ and merge the old $W(z)$ into $W(u)$. A symmetric arguments settles the case where y_1 is the unique neighbor of z_1, but not the unique neighbor of x_1. Finally, the sub-case where y_1 is the unique neighbor of both x_1 and z_1 in $W(y)$ can be handled similarly as in the previous paragraph. ■

Lemma 2. *Let u_1 and u_2 be the two dominating vertices in $H_2^*(a,b)$. If a graph $G=(V,E)$ is contractible to $H_2^*(a,b)$, then there exists a witness structure with the following property:*

1. *For every vertex x in $H_2^*(a,b) - \{u_1, u_2\}$ that is only connected to u_1 and u_2, the witness set $W(x)$ is either induced by some cut-vertex or it consists of a single vertex.*
2. *For any pair y and z of adjacent vertices in $H_2^*(a,b) - \{u_1, u_2\}$, there exist two vertices $y_1 \in W(y)$ and $z_1 \in W(z)$, such that: $W(y)$ contains y_1 and (in case y_1 is a cut-vertex) a vertex subset induced by y_1. $W(z)$ contains*

z_1 and (in case z_1 is a cut-vertex) a vertex subset induced by z_1. Moreover if $\{y_1, z_1\}$ forms a cut-set of G, then with one exception, all components of $G - \{y_1, z_1\}$ that are adjacent to y_1 and z_1 belong to $W(y) \cup W(z)$.

Proof. Omitted in this extended abstract. ■

Lemma 3. *Let u_1, u_2, u_3 be the three dominating vertices in $H_3^*(a)$. If a graph $G = (V, E)$ is contractible to $H_3^*(a)$, then there exists a witness structure with the following property: For every vertex h in $H_3^*(a) - \{u_1, u_2, u_3\}$, the witness set $W(h)$ is either induced by some cut-vertex, or it consists of a single vertex.*

Proof. Omitted in this extended abstract. ■

Theorem 2. *For any fixed non-negative integers a, b, c, contractibility to $H_1^*(a, b, c)$, to $H_2^*(a, b)$, and to $H_3^*(a)$ can be decided in polynomial time.*

Proof. The proof combines the statements in Lemmas 1–3 with a lot of guessing and with an application of Proposition 3. In the following, we will use the same notation as in the statements of these lemmas.

For $H_1^*(a, b, c)$, we may guess for each of the $a + b + c$ witness sets $W(h)$ its unique element or its crucial cut-vertex. There are only $O(n^{a+b+c})$ possibilities for that. This fully defines the witness sets $W(h)$ with $h \neq u$. All remaining vertices are put into $W(u)$. It is easy to check in polynomial time, whether the guessed structure yields a feasible witness structure.

For $H_2^*(a, b)$, we guess for every vertex x (that has u_1 and u_2 as its only neighbors) the unique element or the crucial cut-vertex for $W(x)$. For every pair y and z (of adjacent vertices in $H_2^*(a, b) - \{u_1, u_2\}$), we guess the crucial vertices y_1 and y_2, and we also guess the component of $G - \{y_1, z_1\}$ that contains $W(u_1) \cup W(u_2)$. Finally, we guess two neighbors of y_1 in $W(u_1)$ and in $W(u_2)$, and two neighbors of z_1 in $W(u_1)$ and in $W(u_2)$. There are $O(n^{a+3b+1})$ possibilities for all these guesses. The guesses fully specify the witness sets for the non-dominating vertices in $H_2^*(a, b)$. They also specify $W(u_1) \cup W(u_2)$, and they specify a (fixed constant) number of vertices in $W(u_1)$ respectively $W(u_2)$ that result from the above neighbor guesses. Checking feasibility of this witness structure boils down to checking whether there exists a partition (= 2-coloring) of the vertex set $W(u_1) \cup W(u_2)$ into two connected sets $W(u_1)$ and $W(u_2)$ that respects the assignment of the guessed vertices. But that's just a special case of the problem in Proposition 3 with two colors.

For $H_3^*(a)$, we guess for every non-dominating vertex h the crucial vertex h_1 in G that specifies $W(h)$. Moreover, we guess three neighbors of h_1 in $W(u_1)$, $W(u_2)$, and $W(u_3)$. There are $O(n^{4a})$ possibilities for all these guesses. It remains to check whether $W(u_1) \cup W(u_2) \cup W(u_3)$ can be divided into three connected sets $W(u_1)$, $W(u_2)$, $W(u_3)$ that contain the appropriate guessed vertices. This is a special case of the problem in Proposition 3 with three colors. ■

Proposition 4. *(Robertson & Seymour [3])*
For any fixed integer $a \geq 1$, contractibility to the complete graph K_a can be decided in polynomial time.

Proof. A trivial consequence of Proposition 1, since K_a-MINOR CONTAINMENT and K_a–CONTRACTIBILITY are the same problem. ∎

Theorem 3. *For any fixed integer $a \geq 2$, contractibility to $P^+(a) := K_1 \bowtie P_a$ can be decided in polynomial time.*

Proof. Omitted in this extended abstract. ∎

Now let us apply the results of this section to the five-vertex graphs depicted in Figures 1– 3. The first three graphs H_1, H_2, and H_3 have a single dominating vertex and fall into the classes $H_1^*(*, *, *)$. The graphs H_4 and H_5 have two dominating vertices, and fall into the classes $H_2^*(*, *)$. The graph H_6 has a dominating triangle and falls into one of the classes $H_3^*(*)$. Hence, contractibility to these six graphs can be decided in polynomial time according to Theorem 2.

Graph H_7 is the clique on five vertices, and hence by Proposition 4 also contractibility to H_7 can be decided in polynomial time. Finally, graph H_8 equals $K_1 \bowtie P_4$, and hence contractibility to H_8 is polynomial by Theorem 3.

4 The NP-Complete Cases

Let $H^\#$ be the graph on 5 vertices v, w, x, y, z with edges $[w, x], [x, y], [y, z]$, and $[v, x]$. Let $H_1^\#$ be the graph $H^\#$ with the edge $[v, w]$, $H_2^\#$ be the graph $H^\#$ with the edge $[v, y]$, $H_3^\#$ be the graph $H^\#$ with edges $[v, w], [v, y]$, and $H_4^\#$ be the graph $H^\#$ with edges $[v, w], [v, y], [w, z]$. See Figure 5 for some pictures.

Theorem 4. *H-CONTRACTIBILITY is NP-complete if H is $H_1^\#$, $H_2^\#$, $H_3^\#$, or $H_4^\#$.*

Proof. Omitted in this extended abstract. ∎

We now will now describe two families of pattern graphs for which the corresponding contractibility problem is NP-complete. The (five-vertex) graph $H_5^\#$ in Figure 6 belongs to the family that is analyzed in Theorem 6; $H_5^\#$ is the last one of the fifteen five-vertex graphs that are not covered by Proposition 2.

For a graph $G = (V, E)$ with $x \in V$ and $e = [x, z] \in E$ let G_{xy} denote the graph G with a new vertex y and edge $[x, y]$, and let G_{ey} denote the graph G with a new vertex y and edges $[x, y]$ and $[y, z]$.

Theorem 5. *Let H be a 2-connected graph. If H-CONTRACTIBILITY is NP-complete, then H_{xy}-CONTRACTIBILITY is NP-complete for all $x \in V_H$.*

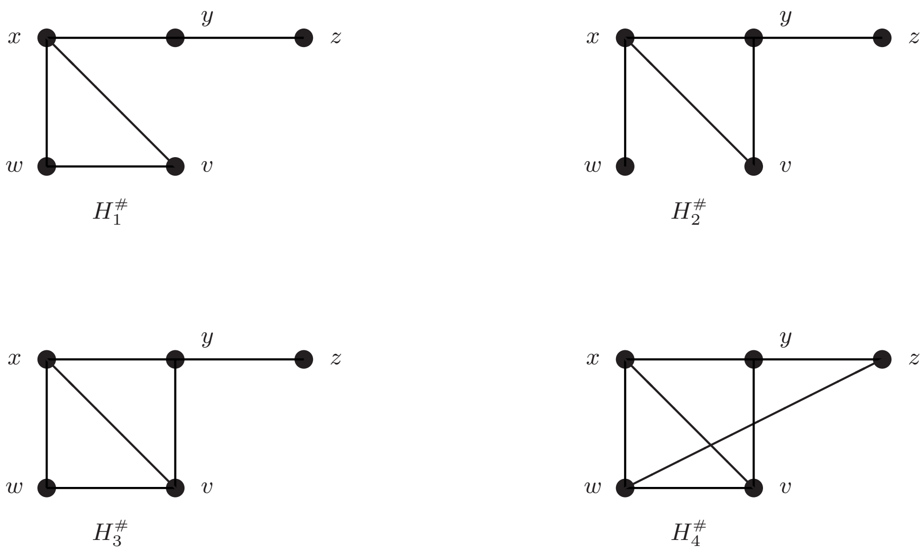

Fig. 5. The graphs $H_1^{\#}$, $H_2^{\#}$, $H_3^{\#}$, and $H_4^{\#}$ that yield NP-complete contractibility problems.

Proof. Given an instance graph G of H-CONTRACTIBILITY we construct a graph G' as follows. Let $V_G = \{r_1, \ldots, r_m\}$. First we make m disjoint copies G^i of G. We insert an extra vertex s, and for $1 \leq i \leq m$ we connect the vertex r_i of the i-th copy to s by an edge.

Our claim is that G is contractible to H if and only if G' is contractible to H_{xy} for $x \in V_H$.

If G is contractible to H, then we define $W_{G'}(y) := V_{G' \setminus G^j}$, where G^j is chosen such that r_j is a vertex in $W_{G^j}(x)$. Clearly, $W_{G'}(y)$ is connected, and G' is contractible to H_{xy}.

Conversely, suppose G' is contractible to H_{xy}. Suppose s is in $W_{G'}(v)$ for some $v \in V_H \cup y$. First assume that $v \neq x$. If vertices of more than one copy G^j of G are contained in other witness sets than $W_{G'}(v)$, then clearly $v \neq y$ but $v \neq x$ would be a cutvertex of H.

Hence, only vertices of G^j are in $V_{G'} \backslash W_{G'}(v)$. Now we remove all other copies G^i $(i \neq j)$ from G'. It is straightforward to see that the remaining part of $W_{G'}(v)$ still induces a connected subgraph.

Suppose $v = y$. Since the degree of y in H_{xy} is exactly one, we remove s and move the remaining vertices of $W_{G'}(y)$ to $W_{G'}(x)$. This way we can contract $G = G^j$ to H.

If $v \neq y$, then besides s also r_j is in $W_{G'}(v)$. Otherwise $W_{G'}(v)$ would be equal to $\{s\}$, and v would have degree one implying that H is not 2-connected. Then we can remove s from $W_{G'}(v)$ and the remaining set $W_{G'}(v)$ induces a connected subgraph. By moving $W_{G'}(y)$ to $W_{G'}(x)$ the graph $G = G^j$ is contractible to H.

The case $v = x$ can be solved using similar arguments as above. ■

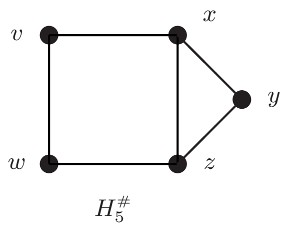

Fig. 6. The graph $H_5^\#$.

Theorem 6. *Let H be a 2-connected graph that does not have any vertex v with exactly two neighbors w_1, w_2 such that $[w_1, w_2] \in E_H$. If H-CONTRACTIBILITY is NP-complete, then H_{ey}-CONTRACTIBILITY is NP-complete for all $e \in E_H$.*

Proof. Given an instance graph G of H-CONTRACTIBILITY we construct a graph G' as follows. Let $E_G = \{e_1, \ldots, e_n\}$. First we make n disjoint copies G^i of G. We insert an extra vertex s, and for $1 \leq i \leq n$ we connect the end points of e_i of the i-th copy to s by an edge.

Our claim is that G is contractible to H if and only if G' is contractible to H_{ey} for $e = [x, z] \in V_E$.

If G is contractible to H, then an edge $e_j \in E_G$ exists that has one of its end points in $W_{G^j}(x)$ and the other one in $W_{G^j}(z)$. We define $W_{G'}(y) := V_{G' \backslash G^j}$. Clearly, $W_{G'}(y)$ is connected, and G' is contractible to H_{ey}.

Conversely, suppose G' is contractible to H_{ey}. Suppose s is in $W_{G'}(v)$ for some $v \in V_H \cup y$. Because H is 2-connected, also H_{ey} does not have any cutvertices. If vertices of more than one copy G^j of G are contained in other witness sets than $W_{G'}(v)$, then v would be a cutvertex of H_{ey}.

Hence, only vertices of G^j are in $V_{G'} \backslash W_{G'}(v)$. Now we remove all other copies G^i $(i \neq j)$ from G'. It is straightforward to see that the remaining part of $W_{G'}(v)$ still induces a connected subgraph.

Suppose $v = y$. We remove s and move the remaining vertices of $W_{G'}(y)$ to $W_{G'}(x)$. We can contract $G = G^j$ to H this way, because the only neighbors of y in H_{ey} are x and z and $[x, z]$ is an edge in H.

If $v \neq y$, then besides s at least one end point of e_j is in $W_{G'}(v)$. Otherwise $W_{G'}(v)$ would be equal to $\{s\}$, and v would have degree one in H or exactly two neighbors w_1 and w_2 with $[w_1, w_2] \in E_H$. Then we can remove s from $W_{G'}(v)$ and the remaining set $W_{G'}(v)$ induces a connected subgraph. By moving $W_{G'}(y)$ to $W_{G'}(x)$, we see that the graph $G = G^j$ is contractible to H. ■

References

1. A.E. Brouwer and H.J. Veldman (1987). Contractibility and NP-completeness. *Journal of Graph Theory 11*, 71–79.
2. M.R. Garey and D.S. Johnson *Computers and Intractability.* W.H. Freeman and Co., New York, 1979.
3. N. Robertson and P.D. Seymour (1995). Graph minors. XIII. The disjoint paths problem. *Journal of Combinatorial Theory, Series B 63*, 65–110.
4. T. Watanabe, T. Ae, and A. Nakamura (1981). On the NP-hardness of edge-deletion and edge-contraction problems. *Discrete Applied Mathematics 6*, 63–78.