Planar Graph Coloring with Forbidden Subgraphs: Why Trees and Paths Are Dangerous*

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Abstract. We consider the problem of coloring a planar graph with the minimum number of colors such that each color class avoids one or more forbidden graphs as subgraphs. We perform a detailed study of the computational complexity of this problem.

We present a complete picture for the case with a single forbidden connected (induced or non-induced) subgraph. The 2-coloring problem is NP-hard if the forbidden subgraph is a tree with at least two edges, and it is polynomially solvable in all other cases. The 3-coloring problem is NP-hard if the forbidden subgraph is a path, and it is polynomially solvable in all other cases. We also derive results for several forbidden sets of cycles.

Keywords: graph coloring; graph partitioning; forbidden subgraph; planar graph; computational complexity.

1 Introduction

We denote by $G = (V, E)$ a finite undirected and simple graph with $|V| = n$ vertices and $|E| = m$ edges. For any non-empty subset $W \subseteq V$, the subgraph of G induced by W is denoted by $G[W]$. A *clique* of G is a non-empty subset $C \subseteq V$ such that all the vertices of C are mutually adjacent. A non-empty subset $I \subseteq V$ is *independent* if no two of its elements are adjacent. An *r-coloring* of the vertices

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of G is a partition $V_1, V_2, \ldots, V_r$ of V; the r sets V_j are called the *color classes* of the r-coloring. An r-coloring is *proper* if every color class is an independent set. The *chromatic number* $\chi(G)$ is the minimum integer r for which a proper r-coloring exists.

Evidently, an r-coloring is proper if and only if for every color class V_j, the induced subgraph $G[V_j]$ does not contain a subgraph isomorphic to P_2. This observation leads to a number of interesting generalizations of the classical graph coloring concept. One such generalization was suggested by Harary [15]: Given a graph property π, a positive integer r, and a graph G, a π r-coloring of G is a (not necessarily proper) r-coloring in which every color class has property π. This generalization has been studied for the cases where the graph property π is being acyclic, or planar, or perfect, or a path of length at most k, or a clique of size at most k. We refer the reader to the work of Brown & Corneil [5], Chartrand *et al* [7,8], and Sachs [20] for more information on these variants.

In this paper, we will investigate graph colorings where the property π can be defined via some (maybe infinite) list of forbidden induced subgraphs. This naturally leads to the notion of $\mathcal{F}$-free colorings. Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be the set of so-called forbidden graphs. Throughout the paper we will assume that the set $\mathcal{F}$ is non-empty, and that all graphs in $\mathcal{F}$ are connected and contain at least one edge. For a graph G, a (not necessarily proper) r-coloring with color classes $V_1, V_2, \ldots, V_r$ is called *weakly $\mathcal{F}$–free*, if for all $1 \le j \le r$, the graph $G[V_j]$ does not contain any graph from $\mathcal{F}$ as an *induced* subgraph. Similarly, we say that an r-coloring is *strongly $\mathcal{F}$–free* if $G[V_j]$ does not contain any graph from $\mathcal{F}$ as an (induced or non-induced) subgraph. The smallest possible number of colors in a weakly (respectively, strongly) $\mathcal{F}$-free coloring of a graph G is called the *weakly* (respectively, *strongly*) *$\mathcal{F}$-free chromatic number*; it is denoted by $\chi^W(\mathcal{F}, G)$ (respectively, by $\chi^S(\mathcal{F}, G)$).

In the cases where $\mathcal{F} = \{F\}$ consists of a single graph F, we will sometimes simplify the notation and not write the curly brackets: We will write F-free short for $\{F\}$-free, $\chi^W(F, G)$ short for $\chi^W(\{F\}, G)$, and $\chi^S(F, G)$ short for $\chi^S(\{F\}, G)$. With this notation $\chi(G) = \chi^S(P_2, G) = \chi^W(P_2, G)$ holds for every graph G. Note that

$$\chi^W(\mathcal{F}, G) \;\le\; \chi^S(\mathcal{F}, G) \;\le\; \chi(G).$$

It is easy to construct examples where both inequalities are strict. For instance, for $\mathcal{F} = \{P_3\}$ (the path on three vertices) and $G = C_3$ (the cycle on three vertices) we have $\chi(G) = 3$, $\chi^S(P_3, G) = 2$, and $\chi^W(P_3, G) = 1$.

1.1 Previous Results

The literature contains quite a number of papers on weakly and strongly $\mathcal{F}$-free colorings of graphs. The most general result is due to Achlioptas [1]: For any graph F with at least three vertices and for any $r \ge 2$, the problem of deciding whether a given input graph has a weakly F-free r-coloring is NP-hard.

The special case of weakly P_3-free colorings is known as the *subcoloring problem* in the literature. It has been studied by Broere & Mynhardt [4], by Albertson, Jamison, Hedetniemi & Locke [2], and by Fiala, Jansen, Le & Seidel [11].

Proposition 1. [Fiala, Jansen, Le & Seidel [11]]
Weakly P_3-free 2-coloring is NP-hard for triangle-free planar graphs.

A $(1,2)$-*subcoloring* of G is a partition of V_G into two sets S_1 and S_2 such that S_1 induces an independent set and S_2 induces a subgraph consisting of a matching and some (possibly no) isolated vertices. Le and Le [17] proved that recognizing $(1,2)$-subcolorable cubic graphs is NP-hard, even on triangle-free planar graphs.

The case of weakly P_4-free colorings has been investigated by Gimbel & Nešetřil [13] who study the problem of partitioning the vertex set of a graph into induced cographs. Since cographs are exactly the graphs without an induced P_4, the graph parameter studied in [13] equals the weakly P_4-free chromatic number of a graph. In [13] it is proved that the problems of deciding $\chi^W(P_4, G) \leq 2$, $\chi^W(P_4, G) = 3$, $\chi^W(P_4, G) \leq 3$ and $\chi^W(P_4, G) = 4$ all are NP-hard and/or coNP-hard for planar graphs. The work of Hoàng & Le [16] on weakly P_4-free 2-colorings was motivated by the Strong Perfect Graph Conjecture. Among other results, they show that weakly P_4-free 2-coloring is NP-hard for comparability graphs.

A notion that is closely related to strongly F-free r-coloring is the so-called *defective* graph coloring. A defective (k,d)-coloring of a graph is a k-coloring in which each color class induces a subgraph of maximum degree at most d. Defective colorings have been studied for instance by Archdeacon [3], by Cowen, Cowen & Woodall [10], and by Frick & Henning [12]. Cowen, Goddard & Jesurum [9] have shown that the defective $(3,1)$-coloring problem and the defective $(2,d)$-coloring problem for any $d \geq 1$ are NP-hard even for planar graphs. We observe that defective $(2,1)$-coloring is equivalent to strongly P_3-free 2-coloring, and that defective $(3,1)$-coloring is equivalent to strongly P_3-free 3-coloring.

Proposition 2. [Cowen, Goddard & Jesurum [9]]
(i) Strongly P_3-free 2-coloring is NP-hard for planar graphs.
(ii) Strongly P_3-free 3-coloring is NP-hard for planar graphs.

1.2 Our Results

We perform a complexity study of weakly and strongly $\mathcal{F}$-free coloring problems for *planar* graphs. By the Four Color Theorem (4CT), every planar graph G satisfies $\chi(G) \leq 4$. Consequently, every planar graph also satisfies $\chi^W(\mathcal{F}, G) \leq 4$ and $\chi^S(\mathcal{F}, G) \leq 4$, and we may concentrate on 2-colorings and on 3-colorings. For the case of a single forbidden subgraph, we obtain the following results for 2-colorings:

– If the forbidden (connected) subgraph F is not a tree, then *every* planar graph is strongly and hence also weakly F-free 2-colorable. Hence, the corresponding decision problems are trivially solvable.

– If the forbidden subgraph $F = P_2$, then F-free 2-coloring is equivalent to proper 2-coloring. It is well-known that this problem is polynomially solvable.
– If the forbidden subgraph is a tree T with at least two edges, then both weakly and strongly T-free 2-coloring are NP-hard for planar input graphs. Hence, these problems are intractable.

For 3-colorings with a single forbidden subgraph, we obtain the following results:

– If the forbidden (connected) subgraph F is not a path, then *every* planar graph is strongly and hence also weakly F-free 3-colorable. Hence, the corresponding decision problems are trivially solvable.
– For every path P with at least one edge, both weakly and strongly P-free 3-coloring are NP-hard for planar input graphs. Hence, these problems are intractable.

Moreover, we derive several results for 2-colorings with certain forbidden sets of cycles.

– For the forbidden set $\mathcal{F}_{345} = \{C_3, C_4, C_5\}$, weakly and strongly $\mathcal{F}_{345}$-free 2-coloring both are NP-hard for planar input graphs. Also for the forbidden set $\mathcal{F}_{cycle}$ of all cycles, weakly and strongly $\mathcal{F}_{cycle}$-free 2-coloring both are NP-hard for planar input graphs.
– For the forbidden set $\mathcal{F}_{odd}$ of all cycles of odd lengths, *every* planar graph is strongly and hence also weakly $\mathcal{F}_{odd}$-free 2-colorable.

2 The Machinery for Establishing NP-Hardness

Throughout this section, let $\mathcal{F}$ denote some fixed set of forbidden planar subgraphs. We assume that all graphs in $\mathcal{F}$ are connected and contain at least two edges. We will develop a generic NP-hardness proof for certain types of weakly and strongly $\mathcal{F}$-free 2-coloring problems. The crucial concept is the so-called *equalizer* gadget.

Definition 1. *(Equalizer)*
An (a, b)-equalizer for $\mathcal{F}$ is a planar graph $\mathcal{E}$ with two special vertices a and b that are called the contact points *of the equalizer. The contact points are non-adjacent, and they both lie on the outer face in some fixed planar embedding of $\mathcal{E}$. Moreover, the graph $\mathcal{E}$ has the following properties:*

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{E}$, the contact points a and b receive the same color.
(ii) There exists a strongly $\mathcal{F}$-free 2-coloring of $\mathcal{E}$ such that a and b receive the same color, whereas all of their neighbors receive the opposite color. Such a coloring is called a good 2*-coloring of $\mathcal{E}$.*

The following result is our (technical) main theorem. This theorem is going to generate a number of NP-hardness statements in the subsequent sections of the paper. We omit the proof of this theorem in this extended abstract.

Theorem 1. *(Technical main result of the paper)*
Let $\mathcal{F}$ be a set of planar graphs that all are connected and that all contain at least two edges. Assume that

- *$\mathcal{F}$ contains a graph on at least four vertices with a cut vertex, or a 2-connected graph with a planar embedding with at least five vertices on the outer face;*
- *there exists an (a, b)-equalizer for $\mathcal{F}$.*

Then deciding weakly $\mathcal{F}$-free 2-colorability and deciding strongly $\mathcal{F}$-free 2-colorability are NP-hard problems for planar input graphs.

3 Tree-Free 2-Colorings of Planar Graphs

The main result of this section will be an NP-hardness result for weakly and strongly T-free 2-coloring of planar graphs for the case where T is a tree with at least two edges (see Theorem 2). The proof of this result is based on an inductive argument over the number of edges in T. The following two auxiliary Lemmas 1 and 2 will be used to start the induction.

Lemma 1. *Let $K_{1,k}$ be the star with $k \geq 2$ leaves. Then it is NP-hard to decide whether a planar graph has a weakly (strongly) $K_{1,k}$-free 2-coloring.*

Proof. For $k = 2$, the statement for weakly $K_{1,k}$-free 2-colorings follows from Proposition 1, and the statement for strongly $K_{1,k}$-free 2-colorings follows from Proposition 2.(i). For $k \geq 3$, we apply Theorem 1. The first condition in this theorem is fulfilled, since for $k \geq 3$ the star $K_{1,k}$ is a graph on at least four vertices with a cut vertex. For the second condition, we construct an (a, b)-equalizer.

The equalizer is the complete bipartite graph $K_{2,2k-1}$ with bipartitions I, $|I| = 2k - 1$, and $\{a, b\}$. This graph satisfies Definition 1.(i): In any 2-coloring, at least k of the vertices in I receive the same color, say color 0. If a and b are colored differently, then one of them is colored 0. This yields an induced monochromatic $K_{1,k}$. A good coloring as required in Definition 1.(ii) results from coloring a and b by the same color, and all vertices in I by the opposite color.

For $1 \leq k \leq m$, a *double-star* $X_{k,m}$ is the tree of the following form: $X_{k,m}$ has $k+m+2$ vertices. There are two adjacent central vertices y_1 and y_2. Vertex y_1 is adjacent to k leaves, and y_2 is adjacent to m leaves. In other words, the double-star $X_{k,m}$ results from adding an edge between the two central vertices of the stars $K_{1,k}$ and $K_{1,m}$. Note that $X_{1,1}$ is isomorphic to the path P_4.

Lemma 2. *Let $X_{k,m}$ be a double star with $1 \leq k \leq m$. Then it is NP-hard to decide whether a planar graph has a weakly (strongly) $X_{k,m}$-free 2-coloring.*

Proof. We apply Theorem 1. The first condition in this theorem is fulfilled, since $X_{k,m}$ is a graph on at least four vertices with a cut vertex. For the second condition, we will construct an (a,b)-equalizer.

The (a,b)-equalizer $\mathcal{E} = (V', E')$ consists of $2m+k-1$ independent copies (V^i, E^i) of the double star $X_{k,m}$ where $1 \le i \le 2m+k-1$. Moreover, there are five special vertices a, b, v_1, v_2, and v_3. We define

$$V' = \{v_1, v_2, v_3, a, b\} \;\cup \bigcup_{1 \le i \le 2m+k-1} V^i \qquad \text{and}$$

$$\begin{aligned} E' = \{v_i v_j : 1 \le i,j \le 3\} \;\cup\; av_3 \;\cup\; bv_3 \;\cup \\ \bigcup_{1 \le i \le 2m+k-1} E^i \;\cup \\ \bigcup_{1 \le i \le m} \{v_1 v \colon v \in V^i\} \;\cup \\ \bigcup_{m+1 \le i \le 2m} \{v_2 v \colon v \in V^i\} \;\cup \\ \bigcup_{2m+1 \le i \le 2m+k-1} \{v_3 v \colon v \in V^i\}\,. \end{aligned}$$

We claim that every 2-coloring of $\mathcal{E}$ with a and b colored in different colors contains a monochromatic induced copy of $X_{k,m}$: Consider some weakly $X_{k,m}$-free coloring of $\mathcal{E}$. Then each copy (V^i, E^i) of $X_{k,m}$ must have at least one vertex that is colored 0 and at least one vertex that is colored 1. If v_1 and v_2 had the same color, then together with appropriate vertices in V^i, $1 \le i \le 2m$, they would form a monochromatic copy of $X_{k,m}$. Hence, we may assume by symmetry that v_1 is colored 1, that v_2 is colored 0, and that v_3 is colored 0. Suppose for the sake of contradiction that a and b are colored differently. Then one of them would be colored 0, and there would be a monochromatic copy of $X_{k,m}$ with center vertices v_3 and v_2. Thus $\mathcal{E}$ satisfies property (i) in Definition 1.

To show that also property (ii) in Definition 1 is satisfied, we construct a good 2-coloring: The vertices a, b, v_1 are colored 0, and v_2 and v_3 are colored 1. In every set V^i with $1 \le i \le m$, one vertex is colored 0 and all other vertices are colored 1. In every set V^i with $m+1 \le i \le 2m+k-1$, one vertex is colored 1 and all other vertices are colored 0.

Now we are ready to prove the main result of this section.

Theorem 2. *Let T be a tree with at least two edges. Then it is NP-hard to decide whether a planar input graph G has a weakly (strongly) T-free 2-coloring.*

Proof. By induction on the number ℓ of edges in T. If T has $\ell = 2$ edges, then $T = K_{1,2}$, and NP-hardness follows by Lemma 1. If T has $\ell \ge 3$ edges, then we consider the so-called *shaved* tree T^* of T that results from T by removing all the leaves. If the shaved tree T^* is a single vertex, then T is a star, and

NP-hardness follows by Lemma 1. If the shaved tree T^* is a single edge, then T is a double star, and NP-hardness follows by Lemma 2.

Hence, it remains to settle the case where the shaved tree T^* contains at least two edges. In this case we know from the induction hypothesis that weakly (strongly) T^*-free 2-coloring is NP-hard. Consider an arbitrary planar input graph G^* for weakly (strongly) T^*-free 2-coloring. To complete the NP-hardness proof, we will construct in polynomial time a planar graph G that has a weakly (strongly) T-free 2-coloring if and only if G^* has a weakly (strongly) T^*-free 2-coloring: Let Δ be the maximum vertex degree of T. For every vertex v in G^*, we create Δ independent copies $T_1(v), \dots, T_\Delta(v)$ of T, and we connect v to all vertices of all these copies.

Assume first that G^* is weakly (strongly) T^*-free 2-colorable. We extend this coloring to a weakly (strongly) T-free coloring of G by taking a proper 2-coloring of every subgraph $T_i(v)$ in G. It can be verified that this extended coloring for G does not contain any monochromatic copy of T.

Now assume that G is weakly (strongly) T-free 2-colorable, and let c be such a 2-coloring. Every subgraph $T_i(v)$ in G must meet both colors. This implies that every vertex v in the subgraph G^* of G has at least Δ neighbors of color 0 and at least Δ neighbors of color 1 in the subgraphs $T_i(v)$. This implies that the restriction of the coloring c to the subgraph G^* is a weakly (strongly) T^*-free 2-coloring. This concludes the proof of the theorem.

4 Cycle-Free 2-Colorings of Planar Graphs

In the previous sections we have shown that for every tree F with $|E(F)| \geq 2$, the problem of deciding whether a given planar graph has a weakly (strongly) F-free 2-coloring is NP-hard. If the forbidden tree F is a P_2, then F-free 2-coloring is equivalent to proper 2-coloring, and hence the corresponding problem is polynomially solvable.

We now turn to the case in which F is not a tree and hence contains a cycle (we assume F is connected).

If F contains an odd cycle, then the Four Color Theorem (4CT) shows that any planar graph G has a weakly (strongly) F-free 2-coloring: a proper 4-coloring of G partitions V_G into two sets S_1 and S_2 each inducing a bipartite graph. Coloring all the vertices of S_i by color i yields a weakly (strongly) F-free 2-coloring of G. If we extend the set of forbidden cycles to all cycles of odd length, denoted by $\mathcal{F}_{odd}$, then the converse is also true: In any $\mathcal{F}_{odd}$-free 2-coloring of G both monochromatic subgraphs of G are bipartite, yielding a 4-coloring of G. To summarize we obtain the following.

Lemma 3. *The statement* "$\chi^S(\mathcal{F}_{odd}, G) \leq 2$ for every planar graph G" *is equivalent to the 4CT.*

In case F is just the triangle C_3, one can avoid using the heavy 4CT machinery to prove that for every planar graph G $\chi^S(C_3, G) \leq 2$ by applying a result due to Burstein [6]. We omit the details.

If F contains no triangles, a result of Thomassen [21] can be applied. He proved that the vertex set of any planar graph can be partitioned into two sets each of which induces a subgraph with no cycles of length exceeding 3. Hence every planar graph is weakly (strongly) $\mathcal{F}_{\geq 4}$-free 2-colorable, where $\mathcal{F}_{\geq 4}$ denotes the set of all cycles of length exceeding 3. The following theorem summarizes the above observations.

Theorem 3. *If the forbidden connected subgraph F is not a tree, then every planar graph G is strongly and hence also weakly F-free 2-colorable.*

The picture changes if one forbids several cycles.

Theorem 4. *Let $\mathcal{F}_{345} = \{C_3, C_4, C_5\}$ be the set of cycles of lengths three, four, and five. Then the problem of deciding whether a given planar graph has a weakly (strongly) $\mathcal{F}_{345}$-free 2-coloring is NP-hard.*

We omit the proof of the theorem in the extended abstract.

Recently Kaiser & Škrekovski announce the proof of $\chi^W(\mathcal{F}, G) \leq 2$ for $\mathcal{F} = \{C_3, C_4\}$ and every planar graph G.

5 3-Colorings of Planar Graphs

A *linear forest* is a disjoint union of paths and isolated vertices. The following result was proved independently in [14] and [19]:

Proposition 3. [Goddard [14] and Poh [19]]
Every planar graph G has a partition of its vertex set into three subsets such that every subset induces a linear forest.

This result immediately implies that if a connected graph F is not a path, then $\chi^W(F, G) \leq 3$ and $\chi^S(F, G) \leq 3$ hold for *all* planar graphs G. Hence, these coloring problems are trivially solvable in polynomial time.

We now turn to the remaining cases of F-free 3-coloring for planar graphs where the forbidden graph F is a path. We start with a technical lemma that will yield a gadget for the NP-hardness argument.

Lemma 4. *For every $k \geq 2$, there exists an outer-planar graph Y_k that satisfies the following properties.*

(i) Y_k is not weakly P_k-free 2-colorable.
(ii) There exists a strongly P_k-free 3-coloring of Y_k, in which one of the colors is only used on an independent set of vertices.

We omit the proof of the lemma here.

Theorem 5. *For any path P_k with $k \geq 2$, it is NP-hard to decide whether a planar input graph G has a weakly (strongly) P_k-free 3-coloring.*

Proof. We will use induction on k. The basic cases are $k = 2$ and $k = 3$. For $k = 2$, weakly and strongly P_2-free 3-coloring is equivalent to proper 3-coloring which is well-known to be NP-hard for planar graphs.

Next, consider the case $k = 3$. Proposition 2.(ii) yields NP-hardness of strongly P_3-free 3-coloring for planar graphs. For weakly P_3-free 3-coloring, we sketch a reduction from proper 3-coloring of planar graphs. As a gadget, we use the outer-planar graph Z depicted in Figure 1. The crucial property of Z is that it does not allow a weakly P_3-free 2-coloring, as is easily checked. Now consider an arbitrary planar graph G. From G we construct the planar graph G': For every vertex v in G, create a copy $Z(v)$ of Z, and add all possible edges between v and $Z(v)$. It can be verified that $\chi(G) \leq 3$ if and only if $\chi^W(P_3, G') \leq 3$.

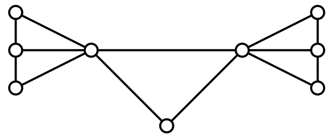

Fig. 1. The graph Z in the proof of Theorem 5.

For $k \geq 4$, we will give a reduction from weakly (strongly) P_{k-2}-free 3-coloring to weakly (strongly) P_k-free 3-coloring. Consider an arbitrary planar graph G, and construct the following planar graph G': For every vertex v in G, create a copy $Y_k(v)$ of the graph Y_k from Lemma 4, and add all possible edges between v and $Y_k(v)$. Since Y_k is outer-planar, the new graph G' is planar. If G has a weakly (strongly) P_{k-2}-free 3-coloring, then this can be extended to a weakly (strongly) P_k-free 3-coloring of G' by coloring the subgraphs $Y_k(v)$ according to Lemma 4.(ii). And if G' has a weakly (strongly) P_k-free 3-coloring, then by Lemma 4.(i) this induces a weakly (strongly) P_{k-2}-free 3-coloring for G.

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