A contribution to the study of periodic systems in the behavioral approach

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ABSTRACT. *In this paper we obtain new results on periodic kernel representations and propose a definition of "image" representation for periodic behaviors. Further, we characterize controllability and autonomicity in representation terms. We also show that the concept of free variable used in the time-invariant case cannot be carried over to the periodic case in a straightforward manner and introduce a new concept of variable freeness (P-periodic freeness). This allows us to define input/output structures for periodic behaviors.*

RÉSUMÉ. *Dans cet article nous présentons de nouveaux résultats sur la représentation des systèmes (comportements) périodiques par des équations aux différences aux coefficients périodiques; nous proposons aussi une définition de représentation d'image pour ces systèmes. Nous caractérisons également la commandabillité d'un système en termes de ses représentations. Ensuite, nous prouvons que le concept de variable libre utilisé dans le cas invariant au cours du temps ne peut pas être reporté de façon directe au cas périodique. Pour cette raison nous introduisons un nouveau concept de liberté d'une variable (liberté P-périodique) qui nous permet de définir des structures d'entrée-sortie pour des comportements périodiques.*

KEYWORDS: *Periodic systems; behaviors; representations.*

MOTS-CLÉS : *Systèmes périodiques; comportements; représentations.*

1. Introduction

In this paper we present new results on the behavioral theory of linear periodically time-varying systems based on the framework developed by Kuijper and Willems, see [KUI 97] and the references therein. This approach uses a technique known as *lifting*, that associates to each periodic behavior a time-invariant one. This allows to derive many results for periodic systems based on the existing ones for the time-invariant case. Using this technique, we present some further insights into (what we call) “kernel" and “image" representations. Moreover we study the structural properties of controllability and autonomicity and provide a characterization of these properties in terms of those representations. We show that, analogous to what happens for time-invariant systems, the correspondence between controllability and the existence of image representations also holds for periodic behaviors. However, in spite of the many formal resemblances, there are some fundamental differences between time-invariant and periodic behaviors. This is, for instance, the case with the relationship between free variables, controllability and autonomicity. Indeed, as we shall see, the usual definition of freeness used in the time-invariant case is not suitable for periodic systems. In order to overcome this complication we introduce a new concept of periodically free variable, which also allows us to define inputs and outputs in a periodic system.

2. Periodic behavioral systems

In the behavioral framework a dynamical system Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ as the time set, $\mathbb{W}$ as the signal space and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ as the behavior. Here we focus on the discrete-time case, that is, $\mathbb{T} = \mathbb{Z}$, assuming furthermore that our space of external variables is $\mathbb{W} = \mathbb{R}^q$ with $q \in \mathbb{Z}_+$.

Let the λ-shift

$$\sigma^\lambda : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}},$$

be defined by

$$\left(\sigma^\lambda w\right)(k) := w(k + \lambda).$$

Whereas the behavior of a time-invariant system is characterized by its invariance under the time shift, that is,

$$\sigma \mathfrak{B} = \mathfrak{B},$$

periodic behaviors are characterized by their invariance with respect to the P-shift ($P \in \mathbb{N}$), as stated in the next definition.

DEFINITION 2.1 [KUI 97] A system Σ is said to be P-periodic (with $P \in \mathbb{N}$) if its behavior $\mathfrak{B}$ satisfies $\sigma^P \mathfrak{B} = \mathfrak{B}$.

3. P-periodic kernel representations - *PPKR*

According to [KUI 97] and [WIL 91], a behavior $\mathfrak{B}$ is a σ^P-invariant linear closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$ (in the topology of point-wise convergence) if and only if it has a representation of the type

$$\left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk+t)=0,\quad t=1,\ldots,P,\ k\in\mathbb{Z},\tag{1}$$

where $R_t\in\mathbb{R}^{g_t\times q}\left[\xi,\xi^{-1}\right]$. Note that the Laurent-polynomial matrices R_t need not have the same number of rows (in fact we could even have some g_t equal to zero, meaning that the corresponding matrix R_t would be void and no restrictions were imposed at the time instants $Pk+t$). Analogously to the time-invariant case, although with some abuse of language, we refer to (1) as a *P-periodic kernel representation* (PPKR).

A common approach in dealing with periodic systems is to relate them with suitable time-invariant ones. Here, following [KUI 97], we associate with a P-periodic behavior $\mathfrak{B}\subset(\mathbb{R}^q)^{\mathbb{Z}}$ a time-invariant behavior $L\mathfrak{B}\subset\left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}}$, the *lifted-behavior*, defined by

$$L\mathfrak{B}=L(\mathfrak{B}):=\left\{\widetilde{w}\in\left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}}\mid\widetilde{w}=Lw,\ w\in\mathfrak{B}\right\},$$

where L is the linear map

$$L:(\mathbb{R}^q)^{\mathbb{Z}}\to\left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}},$$

defined by

$$(Lw)(k):=\begin{bmatrix}w(Pk+1)\\\vdots\\w(Pk+P)\end{bmatrix}.$$

Note that, since

$$\left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk+t)=\left(\left(\sigma^tR_t\left(\sigma,\sigma^{-1}\right)\right)w\right)(Pk),\ t=1,\ldots,P,\ k\in\mathbb{Z},$$

the P-periodic kernel representation (1) can be written as

$$\left(R\left(\sigma,\sigma^{-1}\right)w\right)(Pk)=0,\quad k\in\mathbb{Z},\tag{2}$$

where

$$R\left(\xi,\xi^{-1}\right):=\begin{bmatrix}\xi R_1\left(\xi,\xi^{-1}\right)\\\xi^2R_2\left(\xi,\xi^{-1}\right)\\\vdots\\\xi^PR_P\left(\xi,\xi^{-1}\right)\end{bmatrix}\in\mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right],\tag{3}$$

with $g:=\sum_{t=1}^{P}g_t$. From now on we refer to the matrix $R\left(\xi,\xi^{-1}\right)$ as a *PPKR matrix* of the corresponding behavior.

Decomposing $R\left(\xi,\xi^{-1}\right)$ as

$$\begin{aligned} R\left(\xi,\xi^{-1}\right) &= \xi R^{1}\left(\xi^{P},\xi^{-P}\right)+\cdots+\xi^{P}R^{P}\left(\xi^{P},\xi^{-P}\right) \\ &= R^{L}\left(\xi^{P},\xi^{-P}\right)\Omega_{P,q}\left(\xi\right), \end{aligned} \tag{4}$$

with

$$\Omega_{P,q}\left(\xi\right):=\left[\begin{array}{ccc} \xi I_{q} & \cdots & \xi^{P}I_{q} \end{array}\right]^{T} \tag{5}$$

and

$$R^{L}\left(\xi,\xi^{-1}\right)=\left[\begin{array}{cccc} R^{1}\left(\xi,\xi^{-1}\right) & R^{2}\left(\xi,\xi^{-1}\right) & \cdots & R^{P}\left(\xi,\xi^{-1}\right) \end{array}\right], \tag{6}$$

and recalling the definition of the lifted trajectory Lw associated to w, (2) can be written as

$$\left(R^{L}\left(\sigma,\sigma^{-1}\right)\left(Lw\right)\right)\left(k\right)=0, \quad k\in\mathbb{Z}.$$

This allows us to conclude that the lifted behavior $L\mathfrak{B}$ is given by the *kernel representation*

$$L\mathfrak{B}=\left\{\widetilde{w}|\ R^{L}\left(\sigma,\sigma^{-1}\right)\widetilde{w}=0\right\}=\ker\ R^{L}\left(\sigma,\sigma^{-1}\right).$$

Taking into account that this reasoning can be reversed, we obtain the following result.

LEMMA 3.1 [KUI 97] A P-periodic behavior $\mathfrak{B}\subset\left(\mathbb{R}^{q}\right)^{\mathbb{Z}}$ is given by the kernel representation (1), that is,

$$\mathfrak{B}=\left\{w|\left(R_{t}\left(\sigma,\sigma^{-1}\right)w\right)\left(Pk+t\right)=0,\ t=1,\ldots,P,\ k\in\mathbb{Z}\right\}$$

if and only if the associated lifted behavior $L\mathfrak{B}$ is given by the kernel representation

$$L\mathfrak{B}=\left\{\widetilde{w}|R^{L}\left(\sigma,\sigma^{-1}\right)\widetilde{w}=0\right\},$$

where $R^{L}\left(\xi,\xi^{-1}\right)\in\mathbb{R}^{g\times Pq}\left[\xi,\xi^{-1}\right]$, $g=\sum_{t=1}^{P}g_{t}$, is given as in (6).

By using this one-to-one relation, some known results for the time-invariant case can be somehow mimicked into the P-periodic case. For instance, this happens with two important issues which are the questions of kernel representation equivalence and minimality.

THEOREM 3.2 [ALE 05] Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two P-periodic behaviors with representation matrices $R\left(\xi,\xi^{-1}\right)$ and $R'\left(\xi,\xi^{-1}\right)$, respectively. Then $\mathfrak{B}\subset\mathfrak{B}'$ if and only if there exists a Laurent-polynomial matrix $L\left(\xi,\xi^{-1}\right)$ such that

$$R'\left(\xi,\xi^{-1}\right)=L\left(\xi^{P},\xi^{-P}\right)R\left(\xi,\xi^{-1}\right).$$

This result can be proven as follows. Consider the time-invariant behaviors $L\mathfrak{B} = \ker R^L$ and $L\mathfrak{B}' = \ker R'^L$ associated with $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. Then $\mathfrak{B} \subset \mathfrak{B}'$ if and only if $L\mathfrak{B} \subset L\mathfrak{B}'$, meaning that there exists a Laurent-polynomial matrix $L\left(\xi, \xi^{-1}\right)$ such that $R'^L = LR^L$, which implies the desired relation between R' and R.

This theorem yields the following fundamental result, which is the counterpart for P-periodic behaviors of a similar result for time-invariant behaviors, [POL 98, Theorem 3.6.2].

THEOREM 3.3 [ALE 05] Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two P-periodic behaviors with representation matrices $R\left(\xi, \xi^{-1}\right)$ and $R'\left(\xi, \xi^{-1}\right)$, respectively, possessing the same number of rows. Then $\mathfrak{B} = \mathfrak{B}'$ if and only if there exists a unimodular matrix $U\left(\xi, \xi^{-1}\right)$ such that

$$R'\left(\xi, \xi^{-1}\right) = U\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right).$$

As for the question of minimality, given a linear time-invariant system with behavior $\mathfrak{B}$ described by:

$$\left(R\left(\sigma, \sigma^{-1}\right) w\right)(k) = 0, \quad k \in \mathbb{Z}, \tag{7}$$

with $R\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times q}\left[\xi, \xi^{-1}\right]$, we say that the representation (7) is minimal if the number of rows of the matrix $R\left(\xi, \xi^{-1}\right)$ is minimal (among all the other representations of $\mathfrak{B}$). This is equivalent to say that $R\left(\xi, \xi^{-1}\right)$ has full row rank (over $\mathbb{R}\left[\xi, \xi^{-1}\right]$).

In the P-periodic case, we adopt the definition of minimality from the time invariant case.

DEFINITION 3.4 [ALE 05] A representation matrix $R \in \mathbb{R}^{g \times q}\left[\xi, \xi^{-1}\right]$ of a P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is said to be a minimal representation if for any other representation $R' \in \mathbb{R}^{g' \times q}\left[\xi, \xi^{-1}\right]$ of Σ, there holds $g \leq g'$.

It is not difficult to check that a representation $R\left(\xi, \xi^{-1}\right)$ of a P-periodic system Σ is minimal if and only if the same is true for the corresponding representation $R^L\left(\xi, \xi^{-1}\right)$ of the associated time-invariant lifted system Σ^L. Thus $R\left(\xi, \xi^{-1}\right)$ is minimal if and only if $R^L\left(\xi, \xi^{-1}\right)$ is full row rank over $\mathbb{R}\left[\xi, \xi^{-1}\right]$. The next lemma translates this in terms of the matrix $R\left(\xi, \xi^{-1}\right)$ itself.

LEMMA 3.5 [ALE 05] Let $R\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times q}\left[\xi, \xi^{-1}\right]$ be the representation matrix of a P-periodic system and consider the corresponding matrix $R^L\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times Pq}\left[\xi, \xi^{-1}\right]$ given by (4) and (6). Then, the following conditions are equivalent:

(i) $R^L\left(\xi, \xi^{-1}\right)$ has full row rank over $\mathbb{R}\left[\xi, \xi^{-1}\right]$;

(ii) $R\left(\xi,\xi^{-1}\right)$ has full row rank over $\mathbb{R}\left[\xi^{P},\xi^{-P}\right]$ (i.e., if $r\left(\xi^{P},\xi^{-P}\right) \in \mathbb{R}^{1\times g}\left[\xi^{P},\xi^{-P}\right]$ is such that $r\left(\xi^{P},\xi^{-P}\right)R\left(\xi,\xi^{-1}\right) = 0 \in \mathbb{R}^{1\times q}\left[\xi,\xi^{-1}\right]$, then $r\left(\xi^{P},\xi^{-P}\right)=0$).

Together with the previous considerations, this result yields the following characterization of minimality.

THEOREM 3.6 [ALE 05] Let $R\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right]$ be the representation matrix of a P-periodic system. Then $R\left(\xi,\xi^{-1}\right)$ is a minimal representation if and only if it has full row rank over $\mathbb{R}\left[\xi^{P},\xi^{-P}\right]$.

4. P-periodic image representations - *PPIR*

Image representations constitute an alternative system description in the time-invariant case. As a generalization of such representations we introduce here P-periodic image representations (*PPIR*).

DEFINITION 4.1 A behavior $\mathfrak{B}$ is said to have a *PPIR* if it can be described by equations of the form:

$$w\left(Pk+t\right) = \left(M_t\left(\sigma,\sigma^{-1}\right)v\right)\left(Pk+t\right),\ t=1,\ldots,P,\ k\in\mathbb{Z}, \tag{8}$$

where $w\in\left(\mathbb{R}^{q}\right)^{\mathbb{Z}}$ is the system variable and v is an auxiliary variable taking values in $\mathbb{R}^{\ell},\ \ell\in\mathbb{N}$.

Notice that (8) can be written as

$$\begin{bmatrix} w\left(Pk+1\right) \\ w\left(Pk+2\right) \\ \vdots \\ w\left(Pk+P\right) \end{bmatrix} = \left(M\left(\sigma,\sigma^{-1}\right)v\right)\left(Pk\right),\ k\in\mathbb{Z},$$

with

$$M\left(\xi,\xi^{-1}\right) := \begin{bmatrix} \xi M_1\left(\xi,\xi^{-1}\right) \\ \xi^2 M_2\left(\xi,\xi^{-1}\right) \\ \vdots \\ \xi^P M_P\left(\xi,\xi^{-1}\right) \end{bmatrix} \in\mathbb{R}^{Pq\times\ell}\left[\xi,\xi^{-1}\right];$$

we refer to this matrix as a *PPIR matrix*.
Consequently

$$\left(Lw\right)\left(k\right) = \left(M^{L}\left(\sigma,\sigma^{-1}\right)\left(Lv\right)\right)\left(k\right),\ k\in\mathbb{Z},$$

where M^{L} is such that

$$M\left(\xi,\xi^{-1}\right) = M^{L}\left(\xi^{P},\xi^{-P}\right)\Omega_{P,\ell}\left(\xi\right). \tag{9}$$

Therefore, if $\mathfrak{B}$ is a P-periodic behavior with *PPIR matrix* M, $L\mathfrak{B}$ is a time-invariant behavior with image representation M^L. It turns out that the converse also holds true, yielding the following result.

THEOREM 4.2 A P-periodic behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ has a *PPIR* if and only if the associated lifted behavior $L\mathfrak{B}$ has an image representation.

5. Controllability

Loosely speaking, a behavior $\mathfrak{B}$ is said to be *controllable* if the past of every trajectory in $\mathfrak{B}$ can be concatenated with the future of an arbitrary trajectory in this behavior. More concretely,

DEFINITION 5.1 [POL 98] A behavior $\mathfrak{B}$ is said to be *controllable* if for all $w_1,\ w_2 \in \mathfrak{B}$ and $k_0 \in \mathbb{Z}$, there exists $k_1 \geq 0$ and $w \in \mathfrak{B}$ such that $w(k) = w_1(k)$, for $k \leq k_0$, and $w(k) = w_2(k)$, for $k > k_0 + k_1$.

As stated in the next theorem, the controllability of a P-periodic behavior is equivalent to the controllability of its associated lifted system.

THEOREM 5.2 [ALE 05] A P-periodic behavior $\mathfrak{B}$ is controllable if and only if the corresponding lifted behavior $L\mathfrak{B}$ is controllable.

From Theorem 5.2, together with the characterization of behavioral controllability given in [WIL 91, Theorem V.2], it is possible to characterize the controllability of P-periodic systems.

PROPOSITION 5.3 [ALE 05] Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system, represented by (1), with representation matrix R as in (3). Then Σ is controllable if and only if the corresponding matrix R^L (see (4) and (6)) is such that $R^L\left(\lambda, \lambda^{-1}\right)$ has constant rank over $\mathbb{C}\backslash\{0\}$.

In case the matrix $R^L\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times Pq}\left[\xi, \xi^{-1}\right]$ has full row rank, the condition that $R^L\left(\lambda, \lambda^{-1}\right)$ has constant rank over $\mathbb{C}\backslash\{0\}$ is equivalent to say that $R^L\left(\xi, \xi^{-1}\right)$ is left-prime, i.e, all its left divisors are unimodular matrices in $\mathbb{R}^{g \times g}\left[\xi, \xi^{-1}\right]$. It turns out that the left-primeness of $R^L\left(\xi, \xi^{-1}\right)$ can be related with the following primeness property for $R\left(\xi, \xi^{-1}\right)$.

DEFINITION 5.4 [ALE 05] A Laurent-polynomial matrix $R\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times q}\left[\xi, \xi^{-1}\right]$ with full row rank over $\mathbb{R}\left[\xi^P, \xi^{-P}\right]$ is said to be left-prime over $\mathbb{R}\left[\xi^P, \xi^{-P}\right]$, or simply P-left-prime, if whenever it is factored as

$$R\left(\xi, \xi^{-1}\right) = D\left(\xi^P, \xi^{-P}\right) \overline{R}\left(\xi, \xi^{-1}\right),$$

with $D\left(\xi^P,\xi^{-P}\right) \in \mathbb{R}^{g\times g}\left[\xi^P,\xi^{-P}\right]$, then the factor $D\left(\xi^P,\xi^{-P}\right)$ and, equivalently, $D\left(\xi,\xi^{-1}\right)$, are unimodular (over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$ and $\mathbb{R}\left[\xi,\xi^{-1}\right]$, respectively).

LEMMA 5.5 [ALE 05] Let $P \in \mathbb{N}$ and $R\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right]$ have full row rank over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$. Consider the associated matrix $R^L\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times Pq}\left[\xi,\xi^{-1}\right]$ according to the decomposition (4). Then, the following conditions are equivalent:

(i) $R^L\left(\xi,\xi^{-1}\right)$ is left-prime;

(ii) $R\left(\xi,\xi^{-1}\right)$ is P-left-prime.

This leads to the following direct characterization of controllability.

THEOREM 5.6 [ALE 05] A P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ with *PPKR* is controllable if and only if its minimal *PPKR* matrices $R\left(\xi,\xi^{-1}\right)$ are P-left-prime.

Since, for time-invariant behaviors, there is an equivalence between behavioral controllability and the existence of image representations (see [POL 98]), Theorem 4.2, together with Theorem 5.2, allows to prove the following result.

THEOREM 5.7 A P-periodic behavior $\mathfrak{B}$ has a *PPIR* if and only if it is controllable.

Combining Theorems 5.6 and 5.7 we can state that:

THEOREM 5.8 Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system with *PPKR*. Then the following are equivalent:

(i) $\mathfrak{B}$ is controllable;

(ii) all the minimal *PPKR* of $\mathfrak{B}$ are P-left-prime;

(iii) $\mathfrak{B}$ has a *PPIR*.

6. Autonomicity

Autonomicity is the opposite of controllability. Indeed, whereas in a controllable behavior the future of a trajectory is independent of its past, in an autonomous behavior every trajectory is uniquely determined by its past.

DEFINITION 6.1 [WIL 91] A behavior $\mathfrak{B}$ is said to be autonomous if for all $k_0 \in \mathbb{Z}$ and all $w_1, w_2 \in \mathfrak{B}$

$$w_1(k) = w_2(k) \text{ for } k < k_0 \quad \Rightarrow \quad w_1 = w_2.$$

Similar to what is the case with controllability, the autonomicity of $\mathfrak{B}$ and of $L\mathfrak{B}$ are one-to-one related.

THEOREM 6.2 [KUI 97] Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system. Then $\mathfrak{B}$ is autonomous if and only if $L\mathfrak{B}$ is autonomous.

Taking into account the characterization of autonomicity for time-invariant behaviors given in [POL 98], the following result is trivially obtained.

COROLLARY 6.3 Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system with a *PPKR* and a representation matrix R. Then $\mathfrak{B}$ is autonomous if and only if the corresponding representation matrix of the associated lifted system, R^L, has full column rank (fcr).

7. Free variables

Given a behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$, a component w_i of the system variable w is said to be *free* if for all $\alpha \in \mathbb{R}^{\mathbb{Z}}$ there exist a trajectory $w^* \in \mathfrak{B}$ such that $w_i^*(k) = \alpha(k)\,,\ k \in \mathbb{Z}$. This means that w_i is not restricted by the system laws.

The existence or absence of free variables is related, in the time-invariant case, to properties as controllability and autonomicity: a non-trivial time-invariant controllable behavior must have free variables; on the other hand the absence of free variables is equivalent to autonomicity, [POL 98]. As the next examples show, this no longer holds in the P-periodic case.

Example 1 Consider the 2-periodic behavior $\mathfrak{B}$ with *PPKR*

$$R\left(\xi, \xi^{-1}\right) = \xi - 1,$$

i.e., described by

$$w\,(2k+1) = w\,(2k)\,,\ k \in \mathbb{Z}.$$

Since

$$R\left(\xi, \xi^{-1}\right) = \xi - 1 = \left[\begin{array}{cc} 1 & -\xi^{-2} \end{array}\right] \left[\begin{array}{c} \xi \\ \xi^2 \end{array}\right],$$

its associated lifted behavior $L\mathfrak{B}$ is described by the kernel representation

$$\left(R^L\left(\sigma, \sigma^{-1}\right) \left[\begin{array}{c} \widetilde{w}_1 \\ \widetilde{w}_2 \end{array}\right]\right)(k) = 0,\ k \in \mathbb{Z},$$

where

$$R^L\left(\xi, \xi^{-1}\right) = \left[\begin{array}{cc} 1 & -\xi^{-1} \end{array}\right].$$

It is also possible to describe this lifted behavior in terms of an image representation, namely

$$L\mathfrak{B} = \operatorname{im} \begin{bmatrix} 1 \\ \sigma \end{bmatrix}.$$

In order to achieve the decomposition (9) we use the fact that $L\mathfrak{B}$ can also be given as

$$L\mathfrak{B} = \operatorname{im} M^L\left(\sigma, \sigma^{-1}\right),$$

with $M^L\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{2\times 2\ell}\left[\xi, \xi^{-1}\right]$, such that

$$M^L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} 1 & 0 \\ \xi & 0 \end{bmatrix}.$$

Therefore the original 2-periodic behavior has a *PPIR matrix* M given by

$$\begin{aligned} M\left(\xi, \xi^{-1}\right) &= \begin{bmatrix} 1 & 0 \\ \xi^2 & 0 \end{bmatrix} \Omega_{2,1}(\xi) \\ &= \begin{bmatrix} 1 & 0 \\ \xi^2 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \xi^2 \end{bmatrix} \\ &= \begin{bmatrix} \xi \\ \xi^3 \end{bmatrix}, \end{aligned}$$

that is, the 2-periodic behavior $\mathfrak{B}$ allows the *PPIR*

$$\begin{bmatrix} w(2k+1) \\ w(2k+2) \end{bmatrix} = \left(M\left(\sigma, \sigma^{-1}\right) v\right)(2k), \ k \in \mathbb{Z} \tag{10}$$

and is consequently controllable. However $\mathfrak{B}$ has no free variables, since the values of w on each even time instant and its consecutive one must coincide.

Example 2 Let $\mathfrak{B} \subset \mathbb{R}^{\mathbb{Z}}$ be the 2-periodic behavior described by $w(2k) = 0, \ k \in \mathbb{Z}$. Clearly the only system variable w is not free, since it is required to be zero on even time instants. However, $\mathfrak{B}$ is not autonomous. Indeed fixing the values of $w(k)$ for $k \leq 0$ does not yield a unique trajectory, since the values of $w(2k+1), \ k > 0$ can still be chosen freely. Thus the absence of free variables does not imply autonomicity.

The analysis of these examples suggests that a different notion of free variable should be considered in the P-periodic case.

DEFINITION 7.1 Let $\mathfrak{B} \subset \left(\mathbb{R}^q\right)^{\mathbb{Z}}$ be a behavior in q variables. The *i*th system variable $w_i, \ i \in \{1, \ldots, q\}$, is said to be P-periodically free with offset t or t-P-periodically free, for $t = 1, \ldots, P$, if $w_i(Pk+t), \ k \in \mathbb{Z}$, is not restricted by the behavior. More precisely, if for all $\alpha \in \mathbb{R}^{\mathbb{Z}}$, there exists $w^* \in \mathfrak{B}$ such that its *i*th-component satisfies

$$w_i^*(Pk+t) = \alpha(k), \quad k \in \mathbb{Z}.$$

Moreover, w_i is said to be P-periodically free if it is P-periodically free with offset t for some $t = 1, \ldots, P$.

This definition yields the usual notion of free variable for time-invariant behaviors, if one regards time-invariance as 1-periodicity.

As a direct consequence of this definition we can state the following result:

PROPOSITION 7.2 Given a P-periodic behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$, the *i*th system variable $w_i,\ i \in \{1, \ldots, q\}$, is t-P-periodically free (in $\mathfrak{B}$) if and only if $(Lw)_{(t-1)q+i}$ is free in $L\mathfrak{B}$.

Now, a controllable P-periodic behavior must have P-periodically free variables.

Example 3 As we have seen, the variable w in Example 1 is not free. However, this variable is 2-periodically free. Recall that the associated lifted behavior $L\mathfrak{B}$ is described by

$$\left(R^L\left(\sigma, \sigma^{-1}\right)\begin{bmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \end{bmatrix}\right)(k) = 0,\ k \in \mathbb{Z},$$

or, equivalently,

$$\widetilde{w}_1\left(k\right) = \widetilde{w}_2\left(k-1\right),\ k \in \mathbb{Z},$$

showing that either $\widetilde{w}_1$ or $\widetilde{w}_2$ are free in $L\mathfrak{B}$. Thus w is 2-periodically free since it is 2-periodically free with offsets $t = 1$ or $t = 2$.

Moreover, the following characterization of autonomicity in terms of P-periodically free variables holds.

THEOREM 7.3 Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system. Then $\mathfrak{B}$ is autonomous if and only if $\mathfrak{B}$ has no P-periodically free variables.

Example 4 As we have seen, although behavior $\mathfrak{B}$ in Example 2 is not autonomous, the system variable w is not free. Notice that however w is 2-periodically free since in this case we have

$$R\left(\xi, \xi^{-1}\right) = 1 = \begin{bmatrix} 0 & \xi^{-2} \end{bmatrix}\begin{bmatrix} \xi \\ \xi^2 \end{bmatrix},$$

which leads to

$$R^L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} 0 & \xi^{-1} \end{bmatrix}.$$

Therefore the associated lifted behavior $L\mathfrak{B}$ is described by

$$\left(R^L\left(\sigma, \sigma^{-1}\right)\begin{bmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \end{bmatrix}\right)(k) = 0,\ k \in \mathbb{Z},$$

equivalently,

$$\widetilde{w}_2\left(k-1\right) = 0,\ k \in \mathbb{Z},$$

or still

$$\widetilde{w}_2\left(k\right)=0,\ k\in\mathbb{Z}.$$

Thus $\widetilde{w}_1$ is free and w is 2-periodically free since it is 2-periodically free with offset $t=1$.

The notion of P-periodic freeness plays an important role in the definition of input/output structures for periodic behaviors. This implies considering simultaneously free components in the system variable. However, in a P-periodic behavior, one has to take into account that such components may be P-periodically free with different offsets. This is illustrated in the following example.

Example 5 Let $\mathfrak{B}\subset\left(\mathbb{R}^2\right)^{\mathbb{Z}}$ be the 3-periodic behavior given by the equations

$$w_2\left(3k+1\right)=w_2\left(3k+2\right)=w_1\left(3k+3\right)=0,\quad k\in\mathbb{Z}.$$

Clearly the values of $w_1\left(3k+1\right)$, $w_1\left(3k+2\right)$ and $w_2\left(3k+3\right)$ ($k\in\mathbb{Z}$) are free, i.e., w_1 is 3-periodically free with offsets 1 and 2, and w_2 is 3-periodically free with offset 3. Note further, that none of the variables is free at all the possible offsets $t=1,2,3$.

This can be put in a more compact form by saying that (w_1,w_1,w_2) is $(1,2,3)$-3-periodically free. Note that, in this case, the freeness in the system cannot be assigned to one of the two system variables alone. Therefore, neither w_1 nor w_2 can be taken as an “input", in the classical, time-invariant sense. This suggests to use an alternative approach.

Using the operator $\Omega_{P,q}$ introduced in section 3, we have that

$$\Omega_{3,2}\left(\sigma\right)\left(w\right)=\begin{bmatrix}\sigma w_1\\ \sigma w_2\\ \sigma^2 w_1\\ \sigma^2 w_2\\ \sigma^3 w_1\\ \sigma^3 w_2\end{bmatrix}.$$

Thus

$$\begin{aligned}w_1\left(3k+1\right)&=\left(\Omega_{3,2}\left(\sigma\right)w\right)_1\left(3k\right)\\ w_1\left(3k+2\right)&=\left(\Omega_{3,2}\left(\sigma\right)w\right)_3\left(3k\right)\\ w_2\left(3k+3\right)&=\left(\Omega_{3,2}\left(\sigma\right)w\right)_6\left(3k\right).\end{aligned}$$

where the sub-indices correspond to the components of $\Omega_{3,2}\left(\sigma\right)w$. Now

$$u=\left(\left(\Omega_{3,2}\left(\sigma\right)w\right)_1,\left(\Omega_{3,2}\left(\sigma\right)w\right)_3,\left(\Omega_{3,2}\left(\sigma\right)w\right)_6\right)$$

is a free set of variables of $\Omega_{3,2}(\sigma)\,w$, since $u\,(3k)$ can be chosen freely for all $k \in \mathbb{Z}$, i.e., given $\alpha \in \left(\mathbb{R}^3\right)^{\mathbb{Z}}$, there exists $w^* \in \mathfrak{B}$, such that,

$$u^*\,(3k) = \left(\Omega_{3,2}(\sigma)\,w^*\right)(3k) = \alpha\,(k)\,,\ k \in \mathbb{Z}.$$

Moreover, u is a *maximally* free set of variables, in the sense that once u is fixed (say, $u\,(3k) = 0$, $k \in \mathbb{Z}$) no other free components are left in $\Omega_{3,2}(\sigma)\,w$. Therefore, we call u a *P-periodic input* of $\mathfrak{B}$. The complementary components of $\Omega_{3,2}(\sigma)\,w$,

$$y = \left(\left(\Omega_{3,2}(\sigma)\,w\right)_2, \left(\Omega_{3,2}(\sigma)\,w\right)_4, \left(\Omega_{3,2}(\sigma)\,w\right)_5\right),$$

constitute the corresponding P-periodic output.

In the general case, given a P-periodic behavior $\mathfrak{B}$ with variable w, a choice of (possibly repeated) components of w, $\left(w_{i_1} \cdots w_{i_m}\right)^T$, $i_r \in \{1, \ldots, q\}$ for $r = 1, \ldots, m$, is said to be $(t_1, \ldots, t_m)$-*P-periodically free*, $t_r \in \{1, \ldots, P\}$ for $r = 1, \ldots, m$, if for all $\alpha_r \in \mathbb{R}^{\mathbb{Z}}$, there exists $w^* \in \mathfrak{B}$ such that its i_rth-component satisfies

$$w^*_{i_r}\,(Pk + t_r) = \alpha_r\,(k)\,, \quad k \in \mathbb{Z}.$$

Note that $\left(w_{i_1} \cdots w_{i_m}\right)^T$ is $(t_1, \ldots, t_m)$-*P-periodically free* if and only if

$$u = \left(\left(\Omega_{P,q}(\sigma)\,w\right)_{(t_1-1)q+i_1}, \ldots, \left(\Omega_{P,q}(\sigma)\,w\right)_{(t_m-1)q+i_m}\right)$$

is a free set of variables of $\Omega_{P,q}(\sigma)\,w$, with $\Omega_{P,q}(\xi)$ defined as in (5).

DEFINITION 7.4 Given a P-periodic behavior $\mathfrak{B} \subset \left(\mathbb{R}^q\right)^{\mathbb{Z}}$ with variable $w = \left(w_1 \cdots w_q\right)^T$, a choice of components

$$u = \left(\left(\Omega_{P,q}(\sigma)\,w\right)_{\ell_1}, \ldots, \left(\Omega_{P,q}(\sigma)\,w\right)_{\ell_m}\right)$$

of $\Omega_{P,q}(\sigma)\,w$ is said to be a P-periodic input of $\mathfrak{B}$ if u is a maximally free set of variables of $\Omega_{P,q}(\sigma)\,w$ in the following sense:

(i) u is free, i.e., $\forall \alpha \in \left(\mathbb{R}^m\right)^{\mathbb{Z}}\ \exists w^* \in \mathfrak{B}$ s.t.

$$\begin{aligned} u^*\,(Pk) &= \left(\left(\Omega_{P,q}(\sigma)\,w^*\right)_{\ell_1}, \ldots, \left(\Omega_{P,q}(\sigma)\,w^*\right)_{\ell_m}\right)(Pk) \\ &= \alpha\,(k)\,,\ k \in \mathbb{Z}; \end{aligned}$$

(ii) The set of trajectories

$$\left\{\left(\Omega_{P,q}(\sigma)\,w\right)(Pk)\,,\ w \in \mathfrak{B} :\ u\,(Pk) = 0\right\}$$

has no free variables.

A choice of components y of $\Omega_{P,q}(\sigma)\,w$ is said to be a P-periodic output of $\mathfrak{B}$ if (u,y) is a partition of the components of $\Omega_{P,q}(\sigma)\,w$. Finally, an input-output structure for $\mathfrak{B}$ is defined as a partition (u,y) of the components of $\Omega_{P,q}(\sigma)\,w$, such that u is an input and y is an output.

Since

$$(\Omega_{P,q}(\sigma)\,w)\,(Pk) = (Lw)\,(k)\,,\ k \in \mathbb{Z},$$

it is obvious that:

PROPOSITION 7.5 $\widetilde{u} = \left((Lw)_{\ell_1},\ldots,(Lw)_{\ell_m}\right)$ is an input of the time-invariant behavior $L\mathfrak{B}$ if and only if $u = \left((\Omega_{P,q}(\sigma)\,w)_{\ell_1},\ldots,(\Omega_{P,q}(\sigma)\,w)_{\ell_m}\right)$ is a P-periodic input of $\mathfrak{B}$.

Taking into account the relationship between the P-periodically free variables of a P-periodic behavior and the free variables of its associated lifted system, it is now possible to define input/output structures in the periodic case based on the available results for time-invariant systems. This leads to the following theorem.

THEOREM 7.6 Every P-periodic behavior $\mathfrak{B}$ admits an input/output structure.

Example 6 Consider the 3-periodic behavior $\mathfrak{B}$ with *PPKR matrix*

$$R\left(\xi,\xi^{-1}\right) = \begin{bmatrix} \xi^2-1 & \xi \\ \xi-\xi^2 & \xi^3 \\ \xi^{-3} & 1 \\ 2\xi^3-\xi & \xi^3-\xi^2 \end{bmatrix}.$$

Its associated lifted system has also a kernel representation, that is,

$$L\mathfrak{B} = \ker R^L\left(\sigma,\sigma^{-1}\right),$$

with

$$R^L\left(\xi,\xi^{-1}\right) = \begin{bmatrix} 0 & 1 & 1 & 0 & -\xi^{-1} & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \xi^{-2} & \xi^{-1} \\ -1 & 0 & 0 & -1 & 2 & 1 \end{bmatrix}.$$

Letting

$$\widetilde{R}^L\left(\xi,\xi^{-1}\right)=R^L\left(\xi,\xi^{-1}\right)\begin{bmatrix}0&0&0&0&1&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{bmatrix}$$

$$=\left[\begin{array}{cccc|cc}1&1&0&-\xi^{-1}&0&0\\0&-1&0&0&1&1\\0&0&0&\xi^{-2}&0&\xi^{-1}\\0&0&-1&2&-1&1\end{array}\right]$$

$$=:\left[\begin{array}{c|c}P\left(\xi,\xi^{-1}\right)&-Q\left(\xi,\xi^{-1}\right)\end{array}\right],$$

the lifted system can be represented as

$$\left(P\left(\sigma,\sigma^{-1}\right)\begin{bmatrix}(Lw)_2\\(Lw)_3\\(Lw)_4\\(Lw)_5\end{bmatrix}\right)(k)=\left(Q\left(\sigma,\sigma^{-1}\right)\begin{bmatrix}(Lw)_1\\(Lw)_6\end{bmatrix}\right)(k),\ k\in\mathbb{Z}.$$

Since $\det P\left(\xi,\xi^{-1}\right)\neq 0$, $\widetilde{u}:=\begin{bmatrix}Lw_1 & Lw_6\end{bmatrix}^T$ is an input in $L\mathfrak{B}$ and, consequently

$$u=\left(\left(\Omega_{3,2}\left(\sigma\right)w\right)_1,\left(\Omega_{3,2}\left(\sigma\right)w\right)_6\right)$$

is a 3-periodic input for $\mathfrak{B}$.

8. Conclusions

In the sequel of the work carried out in [KUI 97] and [ALE 05], we have considered P-periodic systems within the framework of the behavioral approach. We analyzed some properties of P-periodic kernel representations, such as equivalence and minimality. Moreover, at the level of system theoretic properties, we have obtained further results on controllability and autonomicity. We defined a new type of representations, P-periodic image representations (*PPIR*), that generalize time-invariant image representations. Further, we have introduced a new concept of P-periodic free variables and analyzed the relationship between the existence of such variables and controllability and autonomicity. Related to our notion of freeness, we defined the concept of P-periodic input, as well as input/output structures in P-periodic systems. In our opinion, these preliminary results will play an important role in other contexts, such as for instance the study of control problems for P-periodic behaviors.

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9. References

[ALE 05] ALEIXO J., POLDERMAN J., ROCHA P., "Further results on periodically time-varying behavioral systems", *Proceedings of the 44th IEEE Conference on Decision & Control, and the European Control Conference - CDC-ECC'05*, Seville, Spain, 2005, p. 808–813.

[KUI 97] KUIJPER M., WILLEMS J., "A behavioral framework for periodically time-varying systems", *Proceedings of the IEEE 36th Conference on Decision & Control*, vol. 3, San Diego, California USA, 1997, p. 2013-2016.

[POL 98] POLDERMAN J. W., WILLEMS J. C., *Introduction to Mathematical Systems Theory: A Behavioral Approach*, vol. 26 of *Texts in Applied Mathematics*, Springer-Verlag, New York, 1998.

[WIL 91] WILLEMS J., "Paradigms and Puzzles in the Theory of Dynamical Systems", *IEEE Transactions on Automatic Control*, vol. 36, num. 3, 1991, p. 259–294.