Output Consensus for Networks of Non-identical Introspective Agents

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Abstract— In this paper, we consider three problems, namely, the output consensus problem, the model-reference output consensus problem, and the regulation of output consensus problem, for a network of non-identical right-invertible linear agents. The network provides each agent with a linear combination of multiple agents' outputs. We assume that all the agents are *introspective*, meaning that they have access to their own local measurements. Under this assumption, we then propose a distributed linear protocol to solve each problem for a broad class of network topologies, including not only Laplacian topologies for a directed graph which contains a directed spanning tree, but a wide family of asymmetric topologies.

I. INTRODUCTION

The problem of achieving *consensus* among agents in a network—that is, asymptotic agreement on the agents' state or output trajectories—has been extensively studied in recent years and has yielded some advances in e.g., sensor networking [8], [9], [7], [15] and autonomous vehicle control applications [14], [12], [11], [13].

Much of the attention has been devoted to achieving *state consensus* for a network of *identical* agents, where each agent has access to a linear combination of its own state relative to that of neighboring agents (e.g., [8], [9], [7], [12], [11], [21]). Roy, Saberi, and Herlugson [15] and Yang, Roy, Wan, and Saberi [29] considered the state consensus problem for more general network topologies. A more realistic scenario—that is, each agent receives a linear combination of its own partial-state output and that of neighboring agents—has been considered in [10], [22], [23], [6]. The results of [6] was expanded by [30] to more general network topologies.

Many of the results on the consensus problem are rooted in the seminal work of Wu and Chua [26], [27] in the circuit community, which gives conditions on a network topology for synchronization of coupled nonlinear oscillators.

A. Non-identical agents and output consensus

Recent activities in the consensus literature have been focused on achieving consensus for a network of *non-identical* agents. This problem is challenging and only some partial results are available, see for instance [4], [28], [1], [5], [25].

In a network of non-identical agents, the agents' states may have different dimensions. In this case, the state consensus is not even properly defined, and it is more natural to aim for *output consensus*—that is, asymptotic agreement on the agents' partial-state outputs. Chopra and Spong [1] studied the output consensus problem for weakly minimum-phase nonlinear systems of relative degree one, using a pre-feedback to create a single-integrator system with decoupled zero dynamics. Kim, Shim, and Seo [5] considered the output consensus problem for uncertain single-input single-output, minimum-phase linear systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output.

The design mentioned in Section I-A generally rely on some sort of self-knowledge that is separate from the information transmitted over the network. More specifically, the agents may know their own state, their own output, or their own state/output relative to that of the reference trajectory. We shall refer to agents that possess this type of self-knowledge as *introspective* agents. In this paper, we assume that all the agents in the network are introspective.

B. Contributions of this paper

In this paper we consider networks of non-identical, introspective, right-invertible linear agents, where each agent has access to its own local measurement, and receives a linear combination of multiple agents' outputs. We propose a distributed linear protocol to solve each of the three problems—that is, *output consensus problem*, *model-reference output consensus problem*, and *regulation of output consensus problem*—under a set of straightforward assumptions about the agents and the network topology. The assumptions about the agents are much more general than that made in [1], [5], and the assumption about the network topology includes not only Laplacian topologies for a directed graph which contains a directed spanning tree, but a wide family of asymmetric topologies.

C. Preliminaries and notations

Given a matrix $A \in \mathbb{R}^{m \times n}$, A^T denotes its transpose, and $\lambda_i(A)$ is its i'th eigenvalue. $A \in \mathbb{R}^{n \times n}$ is said to be Hurwitz stable if all its eigenvalues are in the open left-half plane. $\otimes$ denotes the Kronecker product between two matrices of appropriate dimensions. A and a matrix $B \in \mathbb{R}^{p \times q}$ is defined as the $\mathbb{R}^{mp \times nq}$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

where a_{ij} denotes element (i,j) of A. I_n denotes the identity matrix of dimension n, similarly, 0_n denotes the square matrix of dimension n with all zero elements; we sometimes drop the subscript if the dimension is clear in the context. When clear form the context, $\mathbf{1}$ denotes the column vector with all entries equal to one. For a set of vectors $x_1,\ldots,x_n$, we denote by $\operatorname{col}(x_1,\ldots,x_n)$ the column vector obtained by stacking the elements of $x_1,\ldots,x_n$.

In this paper, we will make use of the classical result on stabilizing a matrix by scaling. Let us recall Fisher and Fuller's result from [2].

Lemma 1: (Fisher and Fuller) Consider an $n \times n$ matrix G. If there exists a permutation matrix P such that all the leading principal minors of PGP^{-1} are nonzero, then there exists a diagonal matrix D such that the eigenvalues of DG are all in the open right-half complex plane (or, alternatively, in the open left-half complex plane).

Note that the proof of Lemma 1 given in [2] is constructive.

II. Problem Statement

We consider a network of n multiple-input multiple-output agents of the form

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \end{cases} \tag{1}$$

for $i \in \{1,\ldots,n\}$, where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^p$.

For the output consensus problem, our goal is to achieve agreement asymptotically on the agents' outputs; that is to ensure that $\lim_{t\to\infty}(y_i(t) - y_j(t)) = 0$ for all $i,j \in \{1,\ldots,n\}$.

Output consensus by itself does not impose any conditions on asymptotic behavior of the output of an individual agent beyond the fact that the outputs of all the agents have to be the same asymptotically.

In this paper we are, however, also interested in the model-reference output consensus problem. In this problem, our goal is not only to achieve output consensus, but to make the consensus trajectory follow the output of the reference system/virtual leader given by

$$\begin{cases} \dot{x}_r = A_r x_r + B_r u_r, \\ y_r = C_r x_r, \end{cases} \tag{2}$$

where $y_r \in \mathbb{R}^p$. That is, we wish to ensure that $\lim_{t\to\infty}(y_i(t) - y_r(t)) = 0$ for all $i \in \{1,\ldots,n\}$.

Finally, we consider the regulation of output consensus problem, where the output of each agent has to track the same trajectory, generated by an arbitrary autonomous exosystem

$$\begin{cases} \dot{x}_r = A_r x_r, \\ y_r = C_r x_r, \end{cases} \tag{3}$$

where $x_r \in \mathbb{R}^r$ and $y_r \in \mathbb{R}^p$. That is we wish to ensure that $\lim_{t\to\infty}(y_i(t) - y_r(t)) = 0$ for all $i \in \{1,\ldots,n\}$.

Notice that the model-reference output consensus problem is more general than the regulation of output consensus problem, since we would get (3) if we choose $u_r = 0$ in (2). However, in order to solve the model-reference output consensus problem, we require stronger assumptions on (C_r, A_r, B_r) and that all the agents have to know u_r, which

will be seen explicitly in Section III-B. On the other hand, in the regulation of output consensus problem, we have hardly any restrictions on (C_r, A_r).

A. Available information

The agents are introspective, meaning that the agents have access to their own local information. Specifically, each agent has access to

$$z_i = C_i^m x_i. \tag{4}$$

Thus without any communications among agents provided by the network, we might still be able to simply asymptotically stabilize each individual agent and obtain output consensus with zero consensus trajectory, that is $\lim_{t\to\infty} y_i(t) = 0$. But to achieve nontrivial consensus behavior, each agent needs to have some information provided by the network. In particular, we assume that each agent i observes a linear combination of the outputs of multiple agents, that is,

$$\zeta_i = \sum_{j=1}^n g_{ij} y_j, \tag{5}$$

where $g_{ij} \in \mathbb{R}$ are scalars, referred as *observation weights*. Note that the observation weight g_{ij} represents the influence (through network communication) of each agent j's output on agent i's observation. We find it natural to assemble the weights g_{ij} into an $n \times n$ *network topology* $G = [g_{ij}]$.

In the model-reference output consensus and the regulation of output consensus problems, it is obvious that a non-empty subset of agents observes their outputs relative to that of the reference model given by (2), (3), respectively to ensure that the consensus trajectory tracks the reference trajectory. Specifically, let $\mathscr{I} \subset \{1,\ldots,n\}$ be such a subset. Then, each agent $i \in \{1,\ldots,n\}$ has access to the quantity

$$\psi_i = e_i(y_i - y_r), \quad e_i = \begin{cases} 1, & i \in \mathscr{I}, \\ 0, & i \notin \mathscr{I}. \end{cases} \tag{6}$$

In the model-reference output consensus problem, all the agents need to know the input u_r of the reference model.

B. Linear protocols

Having the data available for solving each problem, in this paper, we restrict our attention to linear protocols.

More specifically, for solving the output consensus problem, we consider the linear protocol of the form

$$\begin{cases} \dot{x}_i^c = \bar{A}_{c,i} x_i^c + \bar{B}_{c,i} \operatorname{col}(z_i, \zeta_i), \\ u_i = \bar{C}_{c,i} x_i^c + \bar{D}_{c,i} \operatorname{col}(z_i, \zeta_i), \end{cases} \tag{7}$$

for $i \in \{1,\ldots,n\}$, where $x_i^c \in \mathbb{R}^{q_i}$ is the state of agent i'th protocol.

For solving the model-reference output consensus problem, we consider the linear protocols of the form

$$\begin{cases} \dot{x}_i^c = \bar{A}_{c,i} x_i^c + \bar{B}_{c,i} \operatorname{col}(z_i, \zeta_i, \psi_i, u_r), \\ u_i = \bar{C}_{c,i} x_i^c + \bar{D}_{c,i} \operatorname{col}(z_i, \zeta_i, \psi_i, u_r), \end{cases} \tag{8}$$

for $i \in \{1,\ldots,n\}$, where $x_i^c \in \mathbb{R}^{q_i}$ is the state of agent i'th protocol.

For solving the regulation of output consensus problem, we consider the linear protocol of the form

$$\begin{cases} \dot{x}_i^c = \bar{A}_{c,i} x_i^c + \bar{B}_{c,i} \mathrm{col}(z_i, \xi_i, \psi_i), \\ u_i = \bar{C}_{c,i} x_i^c + \bar{D}_{c,i} \mathrm{col}(z_i, \xi_i, \psi_i), \end{cases} \quad (9)$$

for $i \in \{1, \ldots, n\}$, where $x_i^c \in \mathbb{R}^{q_i}$ is the state of agent i'th protocol.

C. Assumptions

We make the following assumption about the agents.

Assumption 1: For each $i \in \{1, \ldots, n\}$, assume that

1) (A_i, B_i) is stabilizable;
2) (C_i, A_i) is detectable;
3) (C_i, A_i, B_i) is right-invertible; and
4) (C_i^m, A_i) is detectable.

In the output consensus problem, we make the following assumption about the network topology.

Assumption 2: We assume that the network topology G has only one zero eigenvalue, with right eigenvector $\mathbf{1}$, and there exists a permutation matrix P such that all the leading principal minors of PGP^{-1}, of size less than n are nonzero.

In the model-reference output consensus problem and regulation of output consensus problem, we make the following assumption about the network topology.

Assumption 3: We assume that the network topology G has only one zero eigenvalue, with right eigenvector $\mathbf{1}$, and there exist a permutation matrix P and a constant $\alpha > 0$ such that all the leading principal minors of $P(G + \alpha E)P^{-1}$, where $E = \mathrm{diag}(e_1, \ldots, e_n)$, are nonzero.

Remark 1: As a special case, if G is a Laplacian matrix of a digraph which contains a directed spanning tree, then it is easy to check that Assumption 2 is always satisfied, and Assumption 3 is always satisfied by choosing $e_K = 1$ for the root agent K, and arbitrary $\alpha > 0$.

III. PROTOCOL DESIGN

In this section, we design a distributed protocol of the form (7), (8), and (9) to solve each of three problems formulated in Section II, respectively. The first stage of our design procedure of all three protocols is to design a protocol based on local measurement (4) to make all the agents substantially the same except for different exponentially decaying signals. This is shown in the following lemma.

Lemma 2: Consider a network of agents (1) with the local measurement z_i given by (4). Let Assumption 1 be satisfied. Let n_q be the integer that is equal to the largest degree of all infinite zeros of (A_i, B_i, C_i), $i \in \{1, \ldots, n\}$. Given arbitrary matrices $A \in \mathbb{R}^{pn_q \times pn_q}$, $B \in \mathbb{R}^{pn_q \times p}$, and $C \in \mathbb{R}^{p \times pn_q}$ such that the system (C, A, B) has no invariant zeros, has uniform rank n_q, is invertible, and all the eigenvalues of A are in the closed left-half plane, then for each $i \in \{1, \ldots, n\}$, there exists a local output feedback of the following form

$$\begin{cases} \dot{p}_i = A_{c,i} p_i + B_{c,i} z_i + E_{c,i} \bar{u}_i, \\ u_i = C_{c,i} p_i + D_{c,i} \bar{u}_i, \end{cases} \quad (10)$$

where $\bar{u}_i \in \mathbb{R}^p$ is a new input, such that the resulting system (1) and (10) can be written as the following form

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + B(\bar{u}_i + \rho_i), \\ y_i = C\bar{x}_i, \end{cases} \quad (11)$$

where $\rho_i \in \mathbb{R}^p$ is given by dynamical equations

$$\begin{cases} \dot{\tilde{x}}_i = H_i \tilde{x}_i, \\ \rho_i = R_i \tilde{x}_i, \end{cases} \quad (12)$$

for some Hurwitz stable matrix H_i.

The proof of Lemma 2 is given in Appendix A.

Remark 2: Note that matrices A, B, and C in Lemma 2 can be chosen arbitrarily as long as the system (C, A, B) has no invariant zeros, has uniform rank n_q, is invertible, and all the eigenvalues of A are in the closed left-half plane. They play a role as design parameters. In this paper, we shall use this property in many locations for various purposes.

Lemma 2 shows that we can design a protocol based on the local measurement (4), to transform each non-identical agent model given by (1) into a new model given by (11). The new models are almost identical; that is, they are the same for each agent except for an exponentially decaying signal ρ_i. In the following three sections we shall first solve each of our three problems with respect to the new, almost-identical models given by (11), and then combine each result with Lemma 2 to solve each problem with respect to the agent models given by (1).

A. The output consensus problem

The following lemma gives conditions under which the output consensus problem for a network of agents (11) can be solved by using ξ_i given by (5).

Lemma 3: Consider a network n agents of the form (11). Let Assumption 2 be satisfied. Then there exists a protocol of the form

$$\begin{cases} \dot{q}_i = A_{c,i} q_i + B_{c,i} \xi_i, \\ \bar{u}_i = C_{c,i} q_i, \end{cases} \quad (13)$$

such that $\lim_{t \to \infty} (y_i(t) - y_j(t)) = 0$ for all $i, j \in \{1, \ldots, n\}$.

The proof of Lemma 3 is given in Appendix B.

Combining the results of Lemma 2 and Lemma 3, we obtain the following result.

Theorem 1: Consider n agents of the form (1). Let Assumptions 1 and 2 be satisfied. Then there exists a protocol of the form (7) that solves the output consensus problem, that is, $\lim_{t \to \infty} (y_i(t) - y_j(t)) = 0$ for all $i, j \in \{1, \ldots, n\}$.

Let us make several comments regarding Theorem 1.

- The network topology conditions given in Assumption 2 are satisfied for a broad class of matrices, including a Laplacian topology for a network which contains a directed spanning tree, and a class of matrices known as *D-semistable matrices*, which have a single eigenvalue at the origin with the corresponding right eigenvector $\mathbf{1}$. For the definition of D-semistability, please see [15], [3]. It is clear that D-semistable matrices includes a wide family of matrices with more general entry sign pattern

than the Laplacian matrix, and hence admits consensus control for a wider set of observation capabilities.

- The consensus trajectory is a linear combination of modes of the open loop system (11). Note that from Remark 2, we know that the eigenvalues of A can be assigned arbitrarily as long as all the eigenvalues of A are in the closed left-half plane. Therefore, the output consensus trajectory captures various types of consensus behavior. In order to avoid the trivial consensus, at least some of the eigenvalues of A need to be located on the imaginary axis.

B. The model-reference output consensus problem

Our focus so far has been on achieving output consensus, without regard to the particular consensus trajectory. In this section, we consider the model-reference output consensus problem.

To begin with, we make following assumption about the reference model/virtual leader given by (2).

Assumption 4: (C_r, A_r, B_r) has no invariant zeros, has uniform rank n_q, is invertible, and all the eigenvalues of A_r are in the closed left-half plane.

The conditions on (C_r, A_r, B_r) given in Assumption 4 are not as restrictive as it might appear. In fact, any trajectory $y_r \in \mathbb{R}^p$, which is n_q times differentiable can be generated by (2) with (C_r, A_r, B_r), which has no invariant zeros, has uniform rank n_q, is invertible, and all the eigenvalues of A_r are in the closed left-half plane. The most natural choice of reference model is p chains of integrators with each integrator of length n_q.

The following lemma gives conditions under which the model-reference output consensus problem for a network of agents (11) and the reference model given by (2) can be solved by using ζ_i given by (5) provided by the network, ψ_i given by (6) and the input to the reference model u_r.

Lemma 4: Consider n agents of the form (11) and the reference model (2). Let Assumptions 3 and 4 be satisfied. Then there exists a protocol of the form

$$\begin{cases} \dot{q}_i = A_{c,i} q_i + B_{c,i} \mathrm{col}(\zeta_i, \psi_i, u_r), \\ \bar{u}_i = C_{c,i} q_i + D_{c,i} \mathrm{col}(\zeta_i, \psi_i, u_r), \end{cases} \quad (14)$$

such that $\lim_{t \to \infty}(y_i(t) - y_r(t)) = 0$ for all $i \in \{1, \ldots, n\}$.

The proof of Lemma 4 is given in Appendix C.

Combining the results of Lemma 2 and Lemma 4, we obtain the following result.

Theorem 2: Consider n agents of the form (1) and the reference model (2). Let Assumptions 1, 3, and 4 be satisfied. Then there exists a protocol of the form (8) that solves the model-reference output consensus problem, that is, $\lim_{t \to \infty}(y_i(t) - y_r(t)) = 0$ for all $i \in \{1, \ldots, n\}$.

C. The regulation of output consensus problem

The main disadvantage of the result in Section III-B is that the input to the reference model needs to be known by each agent. In many cases, we actually want the output of each agent to track a desired *a priori* specified trajectory, such as step or sinusoidal signals, generated by an arbitrary exosystem system given by (3). In this case we use ideas from output regulation [18].

To begin with, we make following classical assumption about the reference model given by (3).

Assumption 5: All the eigenvalues of A_r are on imaginary axis, and the pair (C_r, A_r) is detectable.

The following lemma gives conditions under which the regulation of output consensus problem for a network of agents (11) and the reference model given by (3) can be solved by using ζ_i given by (5) provided by the network, and ψ_i given by (6).

Lemma 5: Consider n agents of the form (11) and the reference model (3). Let Assumptions 3 and 5 be satisfied. Then there exists a protocol of the form

$$\begin{cases} \dot{q}_i = A_{c,i} q_i + B_{c,i} \mathrm{col}(\zeta_i, \psi_i), \\ \bar{u}_i = C_{c,i} q_i, \end{cases} \quad (15)$$

such that $\lim_{t \to \infty}(y_i(t) - y_r(t)) = 0$ for all $i \in \{1, \ldots, n\}$.

The proof of Lemma 5 is given in Appendix D.

In order to prove Lemma 5, we need following properties which can be guaranteed by our design.

1) Since the system (C, A, B) in Lemma 2 has no invariant zeros and is invertible, from [18], we know that the following equations with unknowns $\Pi \in \mathbb{R}^{pn_q \times r}$ and $\Gamma_i \in \mathbb{R}^{p \times r}$, commonly know as the *regulator equations*

$$\Pi A_r = A\Pi + B\Gamma, \quad (16a)$$
$$C_r = C\Pi, \quad (16b)$$

have a unique solution in terms of Π and Γ.

2) Since the matrix A in Lemma 2 is arbitrarily assignable as long as all the eigenvalues of A are in the closed left-half plane, we choose A such that A and A_r have no common eigenvalues.

Combining the results of Lemma 2 and Lemma 5, we obtain the following result.

Theorem 3: Consider n agents of the form (1) and the reference model (3). Let Assumptions 3 and 5 be satisfied. Then there exists a protocol of the form (9) that solves the regulation of output consensus problem, that is, $\lim_{t \to \infty}(y_i(t) - y_r(t)) = 0$ for all $i \in \{1, \ldots, n\}$.

REFERENCES

[1] N. CHOPRA AND W. SPONG, "Output synchronization of nonlinear systems with relative degree one", in Recent advances in learning and control, V.D. Blondel, S.P. Boyd, and H. Kimura, eds., vol. 371 of Lecture notes in control and information sciences, Springer Verlag, London, 2008, pp. 51–64.

[2] M.E. FISHER AND A.T. FULLER, "On the stabilization of matrices and the convergence of linear iterative processes", Proceedings of the Cambridge Philosophical Society, 54, 1958, pp. 417–425.

[3] D. HERSHKOWITZ, "Recent Directions in Matrix Stability", Linear Algebra and its Application, 171, 1 July 1992, pp. 161–186.

[4] D.J. HILL AND J. ZHAO, "Global synchronization of complex dynamical networks with non-identical nodes", in Proc. 47th CDC, Cancun, Mexico, 2008, pp. 817–822.

[5] H. KIM, H. SHIM, AND J.H. SEO, "Output consensus of heterogeneous uncertain linear multi-agent systems", IEEE Trans. Aut. Contr., 56(1), 2011, pp. 200–206.

[6] Z. LI, Z. DUAN, G. CHEN, AND L. HUANG, "Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint", IEEE Trans. Circ. & Syst.-I Regular papers, 57(1), 2010, pp. 213–224.

[7] R. OLFATI-SABER, J.A. FAX, AND R.M. MURRAY, "Consensus and cooperation in networked multi-agent systems", Proc. of the IEEE, 95(1), 2007, pp. 215–233.

[8] R. OLFATI-SABER AND R.M. MURRAY, "Agreement problems in networks with direct graphs and switching topology", in Proc. 42nd CDC, Maui, Hawaii, 2003, pp. 4126–4132.

[9] R. OLFATI-SABER AND R.M. MURRAY, "Consensus problems in networks of agents with switching topology and time-delays", IEEE Trans. Aut. Contr., 49(9), 2004, pp. 1520–1533.

[10] A.Y. POGROMSKY, G. SANTOBONI, AND H. NIJMEIJER, "Partial synchronization: from symmetry towards stability", Physica D, 172(1-4), 2002, pp. 65–87.

[11] W. REN AND E. ATKINS, "Distributed multi-vehicle coordinate control via local information", Int. J. Robust & Nonlinear Control, 17(10-11), 2007, pp. 1002–1033.

[12] W. REN, R. BEARD, AND E. ATKINS, "Information consensus in multivehicle cooperative control", IEEE Control Systems Magazine, 27(2), 2007, pp. 71–82.

[13] W. REN, K. L. MOORE, AND Y. CHEN, "High-order and model reference consensus algorithms in cooperative control of multi-vehicle systems", Journal of Dynamic Systems, Measurement, and Control, 129(5), 2007, pp. 678–688.

[14] S. ROY, A. SABERI, AND K. HERLUGSON, "Formation and alignment of distributed sensing agents with double-integrator dynamics", in Sensor Network Operations, S. Phoha, T.F. La Porta, and C. Griffin, eds., Wiley-IEEE Press, 2006.

[15] S. ROY, A. SABERI, AND K. HERLUGSON, "A control-theoretic perspective on the design of distributed agreement protocols", Int. J. Robust & Nonlinear Control, 17(10-11), 2007, pp. 1034–1066.

[16] A. SABERI, P.V. KOKOTOVIC, AND H.J. SUSSMANN, "Global stabilization of partially linear composite systems", SIAM J. Contr. & Opt., 28(6), 1990, pp. 1491–1503.

[17] A. SABERI AND P. SANNUTI, "Squaring down by static and dynamic compensators", IEEE Trans. Aut. Contr., 33(4), 1988, pp. 358–365.

[18] A. SABERI, A.A. STOORVOGEL, AND P. SANNUTI, Control of linear systems with regulation and input constraints, Communication and Control Engineering Series, Springer Verlag, Berlin, 2000.

[19] P. SANNUTI AND A. SABERI, "Special coordinate basis for multivariable linear systems – finite and infinite zero structure, squaring down and decoupling", Int. J. Contr., 45(5), 1987, pp. 1655–1704.

[20] J.H. SEO, H. SHIM, AND J. BACK, "Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach", Automatica, 45(11), 2009, pp. 2659–2664.

[21] S.E. TUNA, "LQR-based coupling gain for synchronization of linear systems", Available: arXiv:0801.3390v1, 2008.

[22] S.E. TUNA, "Synchronizing linear systems via partial-state coupling", Automatica, 44(8), 2008, pp. 2179–2184.

[23] S.E. TUNA, "Conditions for synchronizability in arrays of coupled linear systems", IEEE Trans. Aut. Contr., 55(10), 2009, pp. 2416–2420.

[24] X. WANG, A.A. STOORVOGEL, A. SABERI, H.F. GRIP, AND P. SANNUTI, "H_2 and H_∞ low-gain theory", in American Control Conference, San Francisco, CA, 2011, pp. 4463–4469.

[25] P. WIELAND, R. SEPULCHRE, AND F. ALLGÖWER, "An internal model principle is necessary and sufficient for linear output synchronization", Automatica, 47(5), 2011, pp. 1068–1074.

[26] C.W. WU AND L.O. CHUA, "Application of graph theory to the synchronization in an array of coupled nonlinear oscillators", IEEE Trans. Circ. & Syst.-I Fundamental theory and applications, 42(8), 1995, pp. 494–497.

[27] C.W. WU AND L.O. CHUA, "Application of Kronecker products to the analysis of systems with uniform linear coupling", IEEE Trans. Circ. & Syst.-I Fundamental theory and applications, 42(10), 1995, pp. 775–778.

[28] J. XIANG AND G. CHEN, "On the V-stability of complex dynamical networks", Automatica, 43(6), 2007, pp. 1049–1057.

[29] T. YANG, S. ROY, Y. WAN, AND A. SABERI, "Constructing consensus controllers for networks with identical general linear agents", Int. J. Robust & Nonlinear Control, 21(11), 2011, pp. 1237–1256.

[30] T. YANG, A.A. STOORVOGEL, AND A. SABERI, "Consensus for multi-agent systems — synchronization and regulation for complex networks", in American Control Conference, San Francisco, CA, 2011, pp. 5312–5317.

APPENDIX

A. Proof of Lemma 2

1) We design a *squaring-down* precompensator

$$\begin{cases} \dot{x}_i^1 = A_i^1 x_i^1 + B_i^1 v_i, \\ u_i = C_i^1 x_i^1 + D_i^1 v_i, \end{cases} \qquad (17)$$

for each agent $i \in \{1, \ldots, n\}$, where x_i^1 is the internal state of the precompensator, and $v_i \in \mathbb{R}^p$, such that the resulting system has an equal number of inputs and outputs and is invertible. The design procedure was developed in [17]. Notice that the design introduces additional zeros which can be chosen to be stable.

2) We design a *rank-equalizing* precompensator

$$\begin{cases} \dot{x}_i^2 = A_i^2 x_i^2 + B_i^2 w_i, \\ v_i = C_i^2 x_i^2 + D_i^2 w_i, \end{cases} \qquad (18)$$

for each agent $i \in \{1, \ldots, n\}$, where x_i^2 is the internal state of the precompensator, and $w_i \in \mathbb{R}^p$, such that the resulting system has uniform rank and, moreover, the infinite-zero structure of the resulting system are identical (that is, all of them have relative degree n_q—an integer equal to the largest degree of all infinite zeros of (C_i, A_i, B_i)). The design procedure was developed in [16]. Notice in order to use the above design procedure, all the agents need to know the integer n_q.

3) We design an observer to estimate $x_{i,a}$ and $x_{i,q}$ based on $\tilde{z}_i = \text{col}(z_i, x_i^1, x_i^2)$ and then choose a pre-feedback such that the resulting system is given by (11).

In order to proceed this stage, let us first consider an arbitrary system (C, A, B) such that the system (C, A, B) has no invariant zeros, has uniform rank n_q and is invertible. From [19], we know that for each $i \in \{1, \ldots, n\}$, there exist a matrix $\bar{A} \in \mathbb{R}^{p \times pn_q}$, a nonsingular matrix $M \in \mathbb{R}^{p \times p}$ and a nonsingular state transformation Γ_{i3} such that $\hat{x}_i = \Gamma_{i3} \bar{x}_i$, and with this state transformation, (11) can be transformed into

$$\begin{cases} \dot{\hat{x}}_i = \begin{bmatrix} 0 & I_{p(n_q-1)} \\ \bar{A} \end{bmatrix} \hat{x}_i + \begin{bmatrix} 0 \\ M \end{bmatrix} (\bar{u}_i + \rho_i), \\ y_i = \begin{bmatrix} I_p & 0 \end{bmatrix} \hat{x}_i. \end{cases} \qquad (19)$$

Thus, without loss of generality, we can assume that (C, A, B) in (11) are given by

$$A = \begin{bmatrix} 0 & I_{p(n_q-1)} \\ \bar{A} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M \end{bmatrix}, \quad C = \begin{bmatrix} I_p & 0 \end{bmatrix}. \qquad (20)$$

Now, let us present the resulting system with two precompensators in the special coordinate basis (SCB) [19]. From [19], [17], [16], we know that for each $i \in \{1, \ldots, n\}$, there exist nonsingular transformations Γ_{i1} and Γ_{i2}, such that $\text{col}(x_i, x_i^1, x_i^2) = \Gamma_{i1} \text{col}(x_{i,a}, x_{i,q})$ and $w_i = \Gamma_{i2} \bar{w}_i$, where $x_{i,a}$ represents the zero dynamics of the resulting system and has dimension $n_{i,a}$, and $x_{i,q}$ represents the infinite-zero dynamics of the resulting system, and has dimension pn_q. Using the

above, we can find the following state space representation of the resulting system with input $\tilde{w}_i$ and output y_i

$$\begin{cases} \dot{x}_{i,q} = \hat{A}x_{i,q} + \hat{B}(\tilde{w}_i + D_{i,a}x_{i,a}), \\ y_i = \hat{C}x_{i,q}, \end{cases} \tag{21}$$

where

$$\hat{A} = \begin{bmatrix} \begin{bmatrix} 0 & I_{p(n_q-1)} \end{bmatrix} \\ D_i \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} I_p & 0 \end{bmatrix},$$

for some matrices $D_i \in \mathbb{R}^{p \times pn_q}$ and $D_{i,a} \in \mathbb{R}^{p \times n_{i,a}}$.

Note that the information $\bar{z}_i$ is available for agent i since the local information z_i is available for agent i and the internal states x_i^1 and x_i^2 are always available for agent i. Moreover, since $\bar{z}_i$ is detectable which follows directly from the detectability of the pair (C_i^m, A_i), we can always design an local observer such that the observer error $\tilde{x}_i = \text{col}(x_{i,a}, x_{i,q}) - \text{col}(\hat{x}_{i,a}, \hat{x}_{i,q})$ satisfies $\dot{\tilde{x}}_i = H_i\tilde{x}_i$, for some Hurwitz stable H_i.

Now, let us choose the following pre-feedback

$$\tilde{w}_i = M\bar{u}_i - D_{i,a}\hat{x}_{i,a} + \bar{A}\hat{x}_{i,q} - D_i\hat{x}_{i,q}, \tag{22}$$

where $\hat{x}_{i,a}$ and $\hat{x}_{i,q}$ are observer estimates, and $\bar{u}_i \in \mathbb{R}^p$ is a new input. Defining $R_i = M^{-1}\begin{bmatrix} D_{i,a} & D_i - \bar{A} \end{bmatrix}$, $\rho_i = R_i\tilde{x}_i$, and $x_{i,q} = \bar{x}_i$, we then obtain (11) by applying the pre-feedback (22) to (21).

Notice that the above three stages can be represented by (10) by defining $p_i = \text{col}(x_i^1, x_i^2, \hat{x}_{i,a}, \hat{x}_{i,q})$. The above three stages can be captured in Fig. 1.
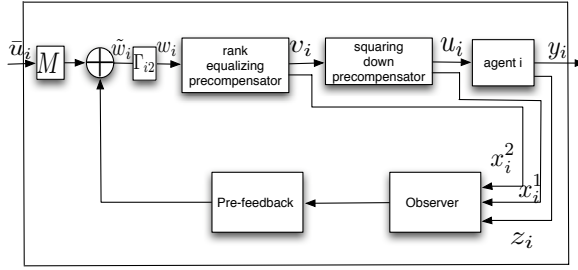

Fig. 1.   First three stages of the design

B. Proof of Lemma 3

Since G satisfies Assumption 2, according to the proof of classical result of Fisher and Fuller (quoted as Lemma 1 above) given in [2], we can design a diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ such that all the eigenvalues of DG are in the closed right-half plane, and that DG has only one eigenvalue at origin, with right eigenvector $\mathbf{1}$. Next, we define the matrix $\overline{DG}$ as the $(n-1) \times (n-1)$ matrix obtained by removing the last row and last column from the matrix $DG - d_n\mathbf{1}g_n^{\mathrm{T}}$, where g_n^{T} is the last row of G, and we define $\tau = \min_{i=1,\ldots,n-1} \text{Re}(\lambda_i(\overline{DG}))$. It is easy to see that all the eigenvalues of $\overline{DG}$ are the nonzero eigenvalues of DG, therefore, $\tau > 0$. Let $P(\varepsilon) = P^{\mathrm{T}}(\varepsilon) > 0$ be the unique solution of the algebraic Riccati equation

$$A^{\mathrm{T}}P(\varepsilon) + P(\varepsilon)A - \tau P(\varepsilon)BB^{\mathrm{T}}P(\varepsilon) + \varepsilon I_{pn_q} = 0. \tag{23}$$

Next let us choose a particular protocol of the form (13) as

$$\begin{cases} \dot{q}_i = (A - KC - BB^{\mathrm{T}}P(\varepsilon))q_i + d_iK\zeta_i, \\ \bar{u}_i = -B^{\mathrm{T}}P(\varepsilon)q_i, \end{cases} \tag{24}$$

where K is a matrix such that $A - KC$ is Hurwitz stable, and ε is sufficiently small.

Let us define relative-state variables as $\bar{p}_i = \bar{x}_i - \bar{x}_n$, and $\bar{q}_i = q_i - q_n$ for all $i \in \{1, \ldots, n-1\}$. We then define $\bar{p} = \text{col}(\bar{p}_1, \ldots, \bar{p}_{n-1})$, $\bar{q} = \text{col}(\bar{q}_1, \ldots, \bar{q}_{n-1})$, and $\xi = \text{col}(\bar{p}, \bar{q}, \tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}_n)$.

With just a little bit algebra, from (5), (11), (12) and (24), we obtain the dynamical equations of the state variable ξ as

$$\dot{\xi} = \begin{bmatrix} I_{n-1} \otimes A & -I_{n-1} \otimes (BB^{\mathrm{T}}P(\varepsilon)) & Q_1 & Q_2 \\ (\overline{DG}) \otimes (KC) & I_{n-1} \otimes (A - KC - BB^{\mathrm{T}}P(\varepsilon)) & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & H_n \end{bmatrix} \xi, \tag{25}$$

where $Q_1 = \text{blkdiag}(BR_1, \ldots, BR_{n-1})$, $Q_2 = -\mathbf{1}_{n-1} \otimes (BR_n)$, and $Q_3 = \text{blkdiag}(H_1, \ldots, H_{n-1})$.

It is then easy to see that the output consensus problem is solved if the relative-state system (25) is asymptotically stable, that is, all the eigenvalues of the matrix

$$\begin{bmatrix} I_{n-1} \otimes A & -I_{n-1} \otimes (BB^{\mathrm{T}}P(\varepsilon)) \\ (\overline{DG}) \otimes (KC) & I_{n-1} \otimes (A - KC - BB^{\mathrm{T}}P(\varepsilon)) \end{bmatrix}$$

are in the open left-half plane due to the upper block-triangular form of the system matrix of (25) and the fact that $H_1, \ldots, H_n$ are Hurwitz stable.

With just a little bit algebra, we can show that the above matrix is similar to $I_{n-1} \otimes \tilde{A} + (\overline{DG}) \otimes (\tilde{B}\tilde{C})$, where

$$\tilde{A} = \begin{bmatrix} A & -BB^{\mathrm{T}}P(\varepsilon) \\ 0 & A - KC - BB^{\mathrm{T}}P(\varepsilon) \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ K \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{26}$$

Following the methodology of [27], we define U such that $J = U^{-1}\overline{DG}U$, where J is the Jordan canonical form of $\overline{DG}$. Then, using the matrix $U \otimes I_{n-1}$ to perform a similarity transform of the matrix $I_{n-1} \otimes \tilde{A} + (\overline{DG}) \otimes (\tilde{B}\tilde{C})$, we obtain

$$(U^{-1} \otimes I)[I \otimes \tilde{A} + (\overline{DG}) \otimes (\tilde{B}\tilde{C})](U \otimes I) = I \otimes \tilde{A} + J \otimes (\tilde{B}\tilde{C}).$$

The resulting matrix is upper block-triangular, we see that its eigenvalues are the union of the eigenvalues of $\tilde{A} + \lambda_i\tilde{B}\tilde{C}$ for each λ_i of the matrix $\overline{DG}$.

From [20, Theorem 4], we see that all the eigenvalues of $\tilde{A} + \lambda_i\tilde{B}\tilde{C}$ are in the open left-half plane, and thus relative-state dynamics (25) is asymptotically stable. Hence, the output consensus is achieved.

C. Proof of Lemma 4

Let us first note that from Lemma 2, we know that the system (C, A, B) is arbitrarily assignable as long as the system (C, A, B) has no invariant zeros, has uniform rank n_q, is invertible, and all the eigenvalues of A are in the closed left-half plane. Therefore, we choose $(C, A, B) = (C_r, A_r, B_r)$.

Given the matrix E, let α be such that Assumption 3 is satisfied. According to the proof of classical result of Fisher and Fuller (quoted as Lemma 1 above) given in [2], we can design a matrix $D = \text{diag}(d_1, \ldots, d_n)$ such that all the eigenvalues of $D(G + \alpha E)$ are in the open right-half plane,

and we define $\tau = \min_{i=1,\dots,n} \mathrm{Re}(\lambda_i(D(G+\alpha E))) > 0$. Let $P(\varepsilon) = P^{\mathrm{T}}(\varepsilon) > 0$ be the unique solution of (23). Let us choose a particular protocol of the form (14) as

$$\begin{cases} \dot{q}_i = (A - KC - BB^{\mathrm{T}}P(\varepsilon))q_i + d_i K\zeta_i + \alpha d_i K\psi_i, \\ \bar{u}_i = -B^{\mathrm{T}}P(\varepsilon)q_i + u_r, \end{cases} \quad (27)$$

where K is a matrix such that $A - KC$ is Hurwitz stable, and ε is sufficiently small.

Note that the protocol state q_i is an estimate of $\bar{x}_i - x_r$. Let us define $r_i = \bar{x}_i - x_r$, $r = \mathrm{col}(r_1,\dots,r_n)$, $q = \mathrm{col}(q_1,\dots,q_n)$, $\tilde{x} = \mathrm{col}(\tilde{x}_1,\dots,\tilde{x}_n)$ and $\xi = \mathrm{col}(r,q,\tilde{x})$. Note that $\sum_{j=1}^n g_{ij}C\bar{x}_j = \sum_{j=1}^n g_{ij}C(\bar{x}_j - x_r)$ since $G\mathbf{1} = 0$, we then obtain

$$\dot{\xi} = \begin{bmatrix} I_n \otimes A & -I_n \otimes (BB^{\mathrm{T}}P(\varepsilon)) & Q_4 \\ D(G+\alpha E)\otimes(KC) & I_n \otimes (A - KC - BB^{\mathrm{T}}P(\varepsilon)) & 0 \\ 0 & 0 & Q_5 \end{bmatrix}\xi,$$
$$(28)$$

where $Q_4 = \mathrm{blkdiag}(BR_1,\dots,BR_n)$, and $Q_5 = \mathrm{blkdiag}(H_1,\dots,H_n)$.

Following the same lines as the proof of Lemma 3, we can easily show that the closed-loop system (28) is asymptotically stable, and thus the model-reference output consensus is achieved.

D. Proof of Lemma 5

Let us first expand the exosystem (3) as

$$\begin{cases} \dot{x}_r = A_r x_r, \\ \dot{\omega} = A\omega + B\Gamma x_r, \\ y_r(t) = C_r x_r(t), \end{cases}$$

with $\omega(0) = \Pi x_r(0)$, where Π and Γ are the unique solution of regulator equation (16). It is then easy to verify that $\omega(t) = \Pi x_r(t)$ for all t and thus $y_r(t) = C_r x_r(t) = C\Pi x_r(t) = C\omega(t)$. The reason behind this expansion is that the exosystem also contains a target for the state of the individual agents.

Next, we expand each individual agent (11) as

$$\begin{cases} \dot{x}_{r,i} = A_r x_{r,i} + u_{i,1}, \\ \dot{\bar{x}}_i = A\bar{x}_i + B\Gamma x_{r,i} + Bu_{i,2} + B\rho_i, \\ y_i(t) = C\bar{x}_i(t), \end{cases}$$

where we have used $\bar{u}_i = \Gamma x_{r,i} + u_{i,2}$. Note that the first state equation for $x_{r,i}$ is basically part of our consensus protocol.

Let us define $x_{f,i} = \mathrm{col}(x_{r,i}, \bar{x}_i)$, $x_{f,r} = \mathrm{col}(x_r, \omega)$, $u_{f,i} = \mathrm{col}(u_{i,1}, u_{i,2})$, $u_{f,r} = 0$, and

$$A_f = \begin{bmatrix} A_r & 0 \\ B\Gamma & A \end{bmatrix}, \quad B_f = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}, \quad C_f = \begin{bmatrix} 0 & C \end{bmatrix}. \quad (29)$$

We then obtain the expanded reference model

$$\begin{cases} \dot{x}_{f,r} = A_f x_{f,r} + B_f u_{f,r}, \\ y_r = C_f x_{f,r}, \end{cases} \quad (30)$$

where $u_{f,r} = 0$ and n individual agents

$$\begin{cases} \dot{x}_{f,i} = A_f x_{f,i} + B_f(u_{f,i} + \begin{bmatrix} 0 \\ \rho_i \end{bmatrix}), \\ y_i = C_f x_{f,i}. \end{cases} \quad (31)$$

Notice that (C_f, A_f) is detectable since A and A_r have no common eigenvalues while (C,A) and $(C\Pi, A_r)$ are detectable. This follows from the fact that

$$\begin{bmatrix} I & 0 \\ -\Pi & I \end{bmatrix} A_f \begin{bmatrix} I & 0 \\ \Pi & I \end{bmatrix} = \begin{bmatrix} A_r & 0 \\ 0 & A \end{bmatrix}, \quad C_f \begin{bmatrix} I & 0 \\ \Pi & I \end{bmatrix} = \begin{bmatrix} C\Pi & C \end{bmatrix}.$$

Given the matrix E, let α be such that Assumption 3 is satisfied. According to the proof of classical result of Fisher and Fuller (quoted as Lemma 1 above) given in [2], we can design a diagonal matrix $D = \mathrm{diag}(d_1,\dots,d_n)$ such that all the eigenvalues of $D(G+\alpha E)$ are in the open right-half plane. Define $\tau = \min_{i=1,\dots,n} \mathrm{Re}\,\lambda_i(D(G+\alpha E)) > 0$. Let $P(\varepsilon) = P^{\mathrm{T}}(\varepsilon) > 0$ be the unique solution of the algebraic Riccati equation

$$A_f P(\varepsilon) + P(\varepsilon)A_f^{\mathrm{T}} - \tau P(\varepsilon)B_f^{\mathrm{T}}B_f P(\varepsilon) + \varepsilon I_{pn_p+r} = 0. \quad (32)$$

Notice that we need that the unique solution $P(\varepsilon)$ of (32) satisfies $\lim_{\varepsilon\to 0} P(\varepsilon) = 0$. In order to achieve this, it is required that all the eigenvalues of A_f are in the closed left-half plane from [24, Theorem 3, Lemma 4]. This is satisfied due to the lower block-triangular form of A_f, and the facts that all the eigenvalues of A_r are on imaginary axis and all the eigenvalues of A are in the closed left-half plane. For each $i \in \{1,\dots,n\}$, let us choose the following protocol

$$\begin{cases} \dot{q}_i = (A_f - K_f C_f - B_f B_f^{\mathrm{T}}P(\varepsilon))q_i + d_i K_f\zeta_i + \alpha d_i K_f\psi_i, \\ u_{f,i} = -B_f^{\mathrm{T}}P(\varepsilon)q_i, \end{cases} \quad (33)$$

where K_f is a matrix such that $A_f - K_f C_f$ is Hurwitz stable and ε is sufficiently small.

Note that the protocol state q_i is an estimate of $x_{e,i} = x_{f,i} - x_{f,r}$. Let us define $x_e = \mathrm{col}(x_{e,1},\dots,x_{e,n})$, $q = \mathrm{col}(q_1,\dots,q_n)$, $\tilde{x} = \mathrm{col}(\tilde{x}_1,\dots,\tilde{x}_n)$, and $\xi = \mathrm{col}(x_e, p, \tilde{x})$.

With some algebra, from (30), (31), and (33), we obtain the closed loop system

$$\dot{\xi} = \begin{bmatrix} I_n \otimes A_f & -I_n \otimes (B_f B_f^{\mathrm{T}}P(\varepsilon)) & Q_6 \\ (D(G+\alpha E))\otimes(K_f C_f) & I_n \otimes (A_f - K_f C_f - B_f B_f^{\mathrm{T}}P(\varepsilon)) & 0 \\ 0 & 0 & Q_5 \end{bmatrix}\xi,$$
$$(34)$$

where $Q_6 = (I_n \otimes \begin{bmatrix} 0 \\ B \end{bmatrix})\mathrm{blkdiag}(R_1,\dots,R_n)$.

Following the same lines as the proof of Lemma 3, we can easily show that the closed loop system (34) is asymptotically stable. The above argument shows that there exists a protocol of the form (15) that solves the regulation of output consensus for multi-agent system (11).