

# Improved Transport Capacity of the Hexagonal Lattice Network with Broadcast via Network Coding

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## Abstract

The capacity of a wireless lattice network with local interference is studied. In particular, the transport capacity under a multiple unicast traffic pattern is studied for a network of nodes placed on the hexagonal lattice. The communication model is that local interference allows reliable broadcast to all neighbors, but that simultaneous transmissions from multiple neighbors lead to a collision. If broadcast is not exploited and only point-to-point transmissions are used, the normalized transport capacity was previously shown to be between  $1/3$  and  $2/5$ . In this work it is demonstrated that by exploiting broadcast the normalized transport capacity of the hexagonal lattice is at least  $3/7$ .

## 1 Introduction

Network coding [1] in wireless networks provides a means to exploit *broadcast* [2–4] and achieve throughputs that are higher than possible based on point-to-point transmissions. We study the capacity of a network in which 1) nodes are located on the hexagonal lattice and 2) there is only local interference. In particular, we study the *transport capacity* in a multiple unicast setting. Our previous studies [5, 6] indicate that the capacity is at least  $2/5$  and at most  $6/7$ . The *contribution* of this work is an improved lower bound of  $3/7$ . If broadcast is not exploited and only point-to-point transmissions are used, the normalized transport capacity was previously shown [6] to be between  $1/3$  and  $2/5$ . Hence, the multiplicative factor of improvement obtained by exploiting broadcast is at least  $15/14$  and at most  $18/7$ .

The constructive scheme that is used to prove the new lower bound is based on the scheme given in [7, 8]. The aim of [7, 8] was to demonstrate the energy efficiency gains that can be obtained by using network coding. As a consequence, no attention was given to throughput and the presented network codes do not achieve high throughputs. The scheme that is presented in the current work uses the previous scheme as a building block and does achieve high throughput.

For an overview of related work on the benefits offered by exploiting broadcast and using network coding we refer the reader to [6, 8–10]

The outline of the remainder of this paper is as follows. The model is defined in Section 2 after which our main result is presented in Section 3. Finally, a constructive proof of this result is given in Section 4.

## 2 Model

### 2.1 Network topology

We consider a network with nodes located on the hexagonal lattice and edges between nearest neighbours. The size of the network is  $K \times K$ , where  $K = (L + 1)(L + 3)$  for some positive integer  $L$ . We index nodes with a vector  $u \in \mathbb{N}^2$ , where we write  $u = (u[1], u[2])$ . The location in  $\mathbb{R}^2$  of  $u$  is  $uG_\Lambda$ , with  $G_\Lambda = \begin{bmatrix} 1 & 0 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ . Now, we consider  $(V, E)$  with

$$V = \{u \in \mathbb{N}^2 \mid 0 \leq u[1] \leq K - 1, 0 \leq u[2] \leq K - 1\},$$

$$E = \{(u, v) \subset V \times V \mid \|(u[1] - v[1], u[2] - v[2])G_\Lambda\|_2 = 1\}.$$

Let  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$ .

### 2.2 Communication Model

Time is slotted and we consider half-duplex constraints, meaning that no node can simultaneously transmit and receive. Symbols are from the finite field  $\mathbb{F}_2$ , *i.e.*, we consider bits. Let  $\tilde{X}_n(u)$  and  $\tilde{Y}_n(u)$  be the symbols transmitted and received respectively, by node  $u$  in time slot  $n$ . Let  $N_u = \{v \in V \mid (v, u) \in E\}$  denote the neighbourhood of  $u$ . The channel output  $\tilde{Y}_n(v)$  depends only on the channel inputs of neighbouring nodes in the same time slot.

We use the communication model as defined in [6]. Broadcast is exploited, but interference leads to collisions. Hence the functional relation between channel inputs and outputs is that  $\tilde{Y}_n(u) = \tilde{X}_n(v)$ , iff  $u \in N_v$  and all nodes in  $u \cup N_u \setminus \{v\}$  remain silent. If half-duplex or interference constraints are not satisfied, the channel output is uniformly distributed and no information is obtained about the channel inputs, see [6] for details.

### 2.3 Transport Capacity

The traffic pattern that we consider is multiple unicast. For a set of  $M$  unicast sessions, let  $s(m)$  and  $r(m)$  denote the source and destination, respectively, of the  $m$ th session, and  $\mathcal{R}(m)$  its throughput. Our measure of interest is the *transport capacity* of a network which is defined as the maximum, over all configurations of unicast sessions on a given network and all possible transmission strategies, of  $\sum_{m=1}^M \text{dist}(s(m), r(m))\mathcal{R}(m)/|V|$ , where  $\text{dist}(s(m), r(m))$  is the number of hops on the shortest path from  $s(m)$  to  $r(m)$ . The transport capacity is the maximum number of bits  $\times$  hops per time slot, normalized by network size, that can be transported in the network.

Our interest is in the transport capacity of the hexagonal lattice in the limit of large number of nodes.

### 2.4 Notation

If  $f(x) = o(g(x))$ , then  $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$ . For integers  $a, b$  and  $p > 0$ ,  $a \equiv b \pmod{p}$  iff  $a - b$  is divisible by  $p$ .

### 3 Results

The main result presented in this work is the following lower bound on the normalized transport capacity.

**Theorem 1.** *The normalized transport capacity of the hexagonal lattice under half-duplex constraints and reliable broadcast to all neighbors is at least  $3/7$ .*

This bound provides an improvement over the earlier bound of  $2/5$  given in [6]. Some other results from [6] are an upper bound of  $6/7$  of on the transport capacity as well as bounds on the capacity in case no broadcast is available. More precisely, if all transmissions in the network are point-to-point, the transport capacity is lower bounded by  $1/3$  and upper bounded by  $2/5$ . Note that this upper bound corresponds to the previous lower bound in the presence of broadcasting. Hence, previously, it was not possible to claim that exploiting broadcast would improve transport capacity. From Theorem 1 it follows that exploiting broadcast does improve the transport capacity. Combining all known bounds gives that exploiting broadcast leads to a multiplicative improvement factor of at least  $15/14$  and at most  $18/7$ .

### 4 Achievable Strategy

The achievable strategy is based on the construction that was first given in [7]. An improved presentation of this construction is given in [8]. The problem addressed in the above mentioned works is that of minimizing energy consumption without imposing any constraints on the achieved throughput. The elements of the energy efficient scheme are as follows:

1. Carefull placement of unicast sessions. The sessions have the property that all sources and destinations are located at the border of the network. (A similar set of sessions will be considered in the construction below, see Figure 4.)
2. Operation in rounds of multiple time slots.
3. In each round:
  - (a) Nodes in the interior transmit once,
  - (b) Nodes on the boundary transmit twice,
  - (c) One symbol is decoded from each session in the network.

Observe, that since all interior nodes transmit only once per round, the above construction achieves good energy efficiency; the fraction of nodes that transmit twice is approaching zero in the limit of a large network.

In this section we present a new constructive scheme that uses the existing construction as a building block. The new scheme achieves normalized transport capacity  $3/7$ , hence proving Theorem 1. Remember that the network is of size  $K \times K$ , where  $K = (L + 1)(L + 3)$  for a positive integer  $L$ . The elements of the strategy are:

1. Construct subsets of the network of size  $(L + 2) \times (L + 2)$ . The number of such subsets is  $L^3$ . This implies overlap among the subsets.
2. In each subset of size  $(L + 2) \times (L + 2)$  construct  $4L + 1$  sessions, such that the total length of all sessions in a subset is  $3L(L + 1)$ .
3. Operate in rounds of  $7(L + 4)$  time slots.
4. In each round deliver, for each session, one symbol at each receiver.

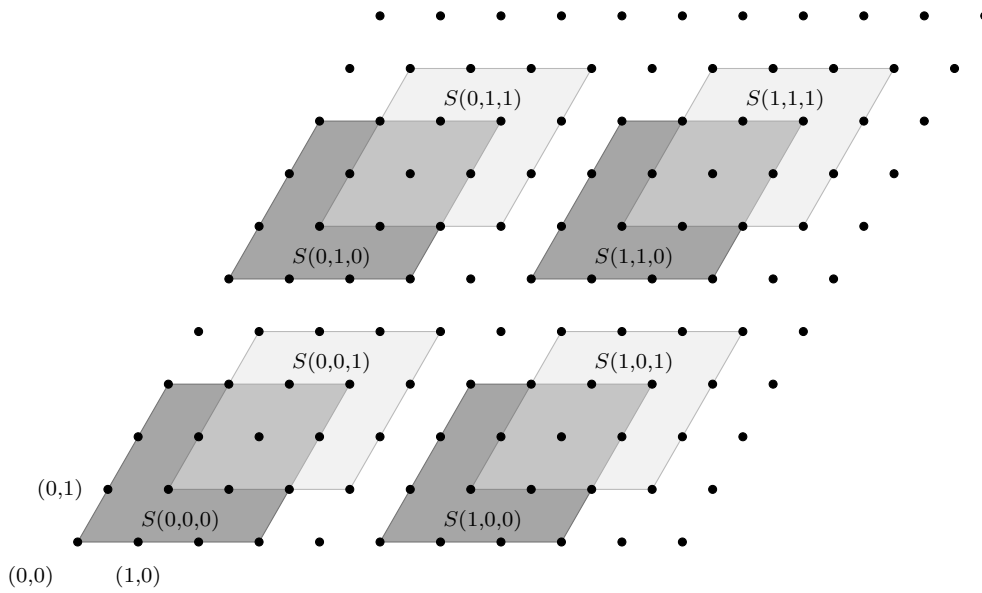


Figure 1: The sets  $S(\alpha_h, \alpha_v, \alpha_o)$  for  $0 \leq \alpha_h, \alpha_v, \alpha_o \leq L - 1$ . ( $L = 2$ )

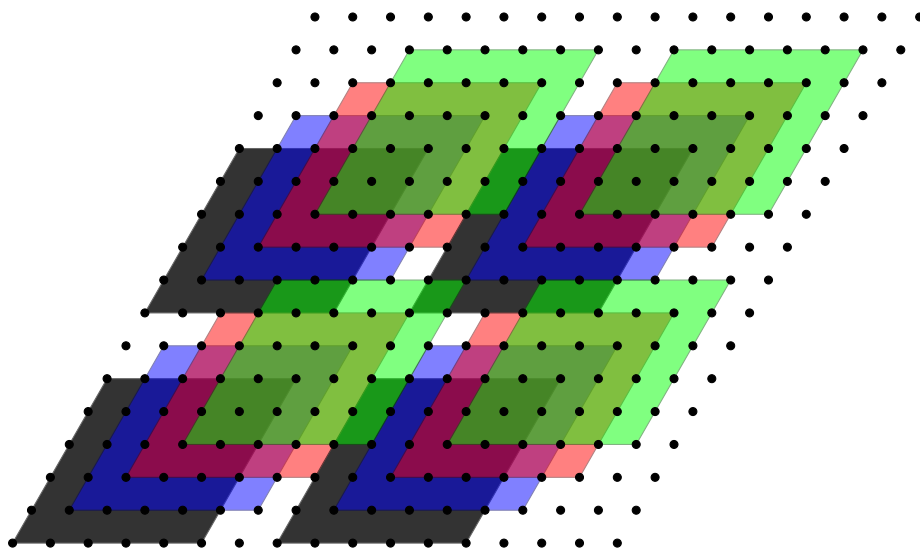


Figure 2: For  $L = 4$ , the sets  $S(\alpha_h, \alpha_v, \alpha_o)$  for  $0 \leq \alpha_h, \alpha_v \leq 1$  and  $0 \leq \alpha_o \leq L - 1$ . (Only part of the network is shown.)

*Proof of Theorem 1.* Observe that based on the above strategy, the details of which are given below, the achieved number of bit×hops per time slot is

$$\frac{L^3 3L(L+1)}{7(L+4)} = \frac{3}{7}L^4 + o(L^4), \quad (1)$$

which normalized by the network size of  $L^4 + o(L^4)$  is approaching 3/7 in the limit of large  $L$ .  $\square$

The different elements of the strategy are addressed in detail in subsequent subsections.

## 4.1 Definition of subsets

Let  $S_0 = \{u \in V | 0 \leq u[1] \leq L+1, 0 \leq u[2] \leq L+1\}$  and

$$\begin{aligned} I_0 &= \{u \in S_0 | 1 \leq u[1] \leq L, 1 \leq u[2] \leq L\}, \\ L_0 &= \{u \in S_0 | u[1] = 0, 1 \leq u[2] \leq L\}, \\ R_0 &= \{u \in S_0 | u[1] = L+1, 1 \leq u[2] \leq L\}, \\ T_0 &= \{u \in S_0 | 1 \leq u[1] \leq L, u[2] = L+1\}, \\ B_0 &= \{u \in S_0 | 1 \leq u[1] \leq L, u[2] = 0\}, \\ C_0 &= L_0 \cup R_0 \cup T_0 \cup B_0, \end{aligned} \quad (2)$$

*i.e.*, we have a set  $S_0$  of size  $(L+2) \times (L+2)$ , its interior, its left, right, top and bottom boundary and its complete boundary (excluding corners). We denote by  $u_l \in L_0$ ,  $u_r \in R_0$ ,  $u_t \in T_0$ ,  $u_b \in B_0$  and  $u_i \in I_0$ , arbitrary nodes in these sets.

We consider  $L^3$  different subsets of  $V$  constructed from  $S_0$  by using an offset. The new subsets are indexed by the three parameters  $\alpha_h, \alpha_v, \alpha_o$ , each chosen from the set  $\{0, 1, \dots, L-1\}$ . The subsets are defined as

$$S(\alpha_h, \alpha_v, \alpha_o) = S_0 + \alpha_h e_1(L+3) + \alpha_v e_2(L+3) + \alpha_o(e_1 + e_2). \quad (3)$$

For  $L=2$  the sets  $S(\alpha_h, \alpha_v, 0)$  are depicted in Figure 1, some of the sets for  $L=4$  are given in Figure 2. Let  $I(\alpha_h, \alpha_v, \alpha_o)$  and the other components of subset  $S(\alpha_h, \alpha_v, \alpha_o)$  be defined according to (2).

The subsets have been defined in such a way, that even though each node is part of a number of subsets that grows with  $L$ , it is at the border of at most 4 such subsets.

**Lemma 1.** *For any  $u \in V$*

$$|\{(\alpha_h, \alpha_v, \alpha_o) | u \in C(\alpha_h, \alpha_v, \alpha_o)\}| \leq 4. \quad (4)$$

*Proof.* If  $u \in L(\alpha_h, \alpha_v, \alpha_o)$ , then there is no  $(\tilde{\alpha}_h, \tilde{\alpha}_v, \tilde{\alpha}_o) \neq (\alpha_h, \alpha_v, \alpha_o)$  for which  $u \in L(\tilde{\alpha}_h, \tilde{\alpha}_v, \tilde{\alpha}_o)$ . The same holds for the other border types.  $\square$

## 4.2 Scheduling

We use rounds of  $7(L+4)$  time slots, counting time slots in a round from 0 to  $7(L+4)-1$ . The schedule consists of two parts. The first part assigns to each time slot a set of nodes that are scheduled to transmit in that time slot. The second part assigns to a scheduled node a set of parameters  $\alpha_h, \alpha_v$  and  $\alpha_o$ , *i.e.*, a set  $S(\alpha_h, \alpha_v, \alpha_o)$ .

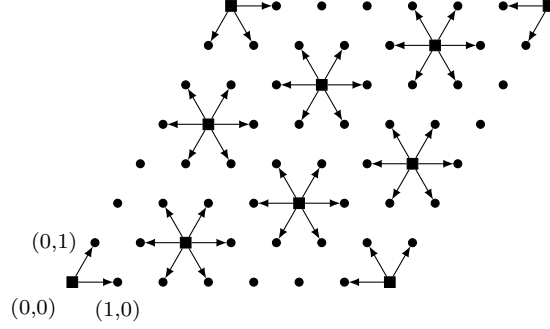


Figure 3: Schedule: Square nodes are scheduled to transmit in the 0-th time slot of a round.

In the  $i$ th time slot of a round node  $u$  is scheduled to transmit iff

$$u[1] - 2u[2] \equiv i \pmod{7}. \quad (5)$$

The schedule is illustrated in Figure 3. It has the property that in each time slot, no node in the network has more than one scheduled neighbour. Also, in each round, each node is scheduled  $L + 4$  times.

Next, for each node  $u$  we assign its  $L + 4$  scheduled slots to triples  $(\alpha_h, \alpha_v, \alpha_o)$ . This is done by considering all triples  $(\alpha_h, \alpha_v, \alpha_o)$ . If  $u \in I(\alpha_h, \alpha_v, \alpha_o)$  one slot is assigned to  $(\alpha_h, \alpha_v, \alpha_o)$ , if  $u \in C(\alpha_h, \alpha_v, \alpha_o)$ , two time slots are assigned to  $(\alpha_h, \alpha_v, \alpha_o)$ . By Lemma 1,  $L + 4$  slots suffice.

Equivalently, we can think of the above procedure of assigning scheduled transmissions to different subsets  $S(\alpha_h, \alpha_v, \alpha_o)$ . The corresponding received symbols are assigned to a subset  $S(\alpha_h, \alpha_v, \alpha_o)$  accordingly. The resulting schedule is such that for each  $S(\alpha_h, \alpha_v, \alpha_o)$ :

1. All nodes in  $I(\alpha_h, \alpha_v, \alpha_o)$  are scheduled to transmit once.
2. All nodes in  $C(\alpha_h, \alpha_v, \alpha_o)$  are scheduled to transmit twice.
3. All transmissions are successfully received by all neighbours.
4. No node in  $S(\alpha_h, \alpha_v, \alpha_o)$  is receiving from nodes outside  $S(\alpha_h, \alpha_v, \alpha_o)$ .

From these observations it follows that the schedule completely decouples the different subsets. The same set of unicast sessions (not considering offsets) and the same network code can be used in all subsets. Therefore, we will consider only  $S_0$  in the remainder.

### 4.3 Sessions

In  $S_0$  we construct  $4L + 1$  unicast sessions. These sessions are denoted by  $m^1(i)$ ,  $m^2(j)$  and  $m^3(k)$ , where  $1 \leq i \leq L$ ,  $1 \leq j \leq L$  and  $1 \leq k \leq 2L + 1$ . Session  $m^d(i)$ ,  $d \in \{1, 2, 3\}$ , has the sequence of message symbols  $m_0^d(i), m_1^d(i), m_2^d(i), \dots$  to be transferred. In the achievable strategy session  $m^3(L + 1)$  will be given throughput zero. Hence, we put all its message symbols to zero, *i.e.*,  $m_t^3(L + 1) = 0$  for all  $t$ .

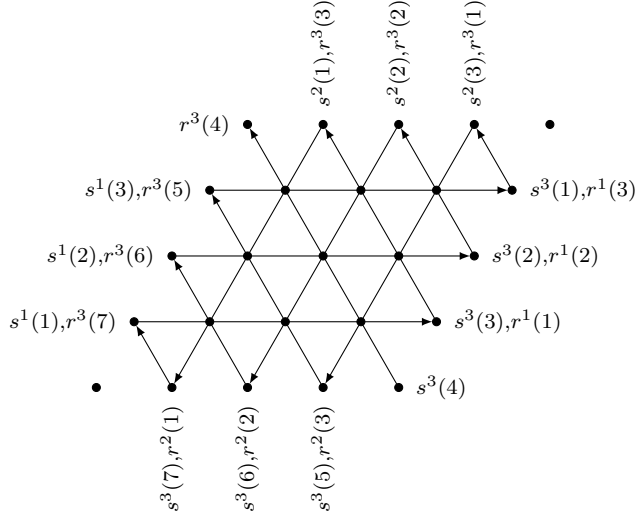


Figure 4: Placement of unicast sessions in  $S_0$ . ( $L = 3$ )

The source and destination of session  $m^d(i)$  are denoted by  $s^d(i)$  and  $r^d(i)$  respectively. Sources and destinations are positioned as depicted in Figure 4, *i.e.*, as follows

$$\begin{aligned}
s^1(i) &= (0, i), & r^1(i) &= (L + 1, i), \\
s^2(j) &= (j, L + 1), & r^2(j) &= (j, 0), \\
s^3(k) &= (L + 1 + \min\{0, L + 1 - k\}, \max\{0, L + 1 - k\}), \\
r^3(k) &= (\max\{0, L + 1 - k\}, L + 1 + \min\{0, L + 1 - k\}).
\end{aligned}$$

Let  $\lambda^d(i)$  denote the number of hops on the shortest path between the source and destination of session  $m^d(i)$ . We have  $\lambda^1(i) = \lambda^2(i) = L + 1$  and

$$\sum_{k=1}^L \lambda^3(k) + \sum_{k=L+2}^{2L+1} \lambda^3(k) = 2 \sum_{k=1}^L k = L(L + 1).$$

Therefore, the total number of hops, excluding  $m^3(L + 1)$ , is  $3L(L + 1)$ .

## 4.4 Coding

Now that we have set up the scheduling and the sessions we can directly apply the network coding scheme from [8, Section 4] to achieve unit throughput per round for each of the sessions.

The network coding from [8] has the properties that 1) no intermediate node in the network can recover the source symbols it is receiving in linear combinations (as would be required in the COPE framework [3], for instance), 2) source symbols are only retransmitted in linear combinations by nodes that are on the shortest path between source and destination, 3) destination nodes can decode the required messages.

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## References

- [1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, “Network information flow,” *IEEE Trans. Inf. Theory*, vol. 46, no. 4, pp. 1204–1216, 2000.
- [2] Y. Wu, P. A. Chou, and S.-Y. Kung, “Information exchange in wireless networks with network coding and physical-layer broadcast,” in *Proc. 39th Annual Conference on Information Sciences and Systems (CISS)*, 2005.
- [3] S. Katti, H. Rahul, W. Hu, D. Katabi, M. Médard, and J. Crowcroft, “XORs in the air: practical wireless network coding,” in *Proc. of ACM SIGCOMM*, 2006, pp. 243–254.
- [4] C. Fragouli, J. Widmer, and J.-Y. Le Boudec, “Efficient broadcasting using network coding,” *IEEE/ACM Trans. Netw.*, vol. 16, no. 2, pp. 450–463, 2008.
- [5] J. Goseling, M. Gastpar, and J. H. Weber, “Transport capacity of wireless networks: benefits from multi-access computation coding,” in *Proceedings of the 30th Symposium on Information Theory in the Benelux, Eindhoven, The Netherlands*, 2009, pp. 225–232.
- [6] J. Goseling, M. Gastpar, and J. H. Weber, “Line and lattice networks under deterministic interference models,” *IEEE Trans. Inf. Theory*, vol. 57, no. 5, pp. 3080–3099, 2011.
- [7] J. Goseling and J. H. Weber, “Energy-benefit of network coding for multiple unicast in wireless networks,” in *Proc. of 29th Symposium on Information Theory in the Benelux*, 2008, pp. 36–40.
- [8] J. Goseling, R. Matsumoto, T. Uyematsu, and J. H. Weber, “Lower bounds on the maximum energy benefit of network coding for wireless multiple unicast,” *Eurasip Journal on Wireless Communications and Networking*, 2010, special issue on network coding for wireless networks.
- [9] J. Goseling, “Network coding: Exploiting broadcast and superposition in wireless networks,” Ph.D. dissertation, Delft University of Technology, The Netherlands, 2010.
- [10] C. Fragouli and E. Soljanin, “Network coding applications,” *Foundations and Trends® in Networking*, vol. 2, no. 2, pp. 135–269, 2007.