2014 European Control Conference (ECC)
June 24-27, 2014. Strasbourg, France

A Message Passing Algorithm for the Evaluation of Social Influence

Luca Vassio   Fabio Fagnani   Paolo Frasca   Asuman Ozdaglar

Abstract— **In this paper, we define a new measure of node centrality in social networks, the *Harmonic Influence Centrality*, which emerges naturally in the study of social influence over networks. Next, we introduce a distributed *message passing* algorithm to compute the Harmonic Influence Centrality of each node: its design is based on an intuitive analogy between social and electrical networks. Although our convergence analysis assumes the networks to have no cycle, the algorithm can be successfully applied on general graphs.**

I. Introduction

A key issue in the study of networks is the identification of their most important nodes: the definition of prominence is based on a suitable function of the nodes, called *centrality measure*. The appropriate notion of centrality measure of a node depends on the nature of the interactions among the nodes and on the decision and control objectives [2], [3], [4], [5], [6], [7].

In this paper, we define a new measure of centrality, which we call *Harmonic Influence Centrality* (HIC) and which emerges naturally in the context of social influence. We explain why in addition to being descriptively useful, this measure answers questions related to the optimal placement of different agents or opinions in a network with the purpose of swaying average opinion. In large real-world social networks, computation of centrality measures of all nodes is a challenging task and distributed algorithms, which do not require global information on the topology, are currently sought [8], [9]. In this paper, we present a fully distributed algorithm for computing the HIC of all nodes.

Our model of social influence builds on recent work [10], which characterizes opinion dynamics in a network consisting of two kind of nodes: *stubborn* agents, who hold a fixed opinion equal to zero or one (*i.e.*, type zero and type one stubborn agents), and *regular* agents, who hold an opinion $x_i \in [0,1]$ and update it as a weighted average of their opinion and those of their neighbors. We consider a special case of this model where a fixed subset of the agents are type zero stubborn agents and the rest are regular agents. We define the HIC of node ℓ as the asymptotic value of the average opinion when node ℓ switches from being a regular agent to a type one stubborn agent. This measure hence captures the long run influence of node ℓ on the average opinion of the social network. The HIC measure is also the precise answer to the following network decision problem: suppose you would like to have the largest influence on long run average opinion in the network and you have the capacity to change one agent from regular to type one stubborn. Which agent should be picked for this conversion? This question has a precise answer in terms of HIC: the agent with the highest HIC should be picked.

The HIC measure is intuitive, but its computation [11] in a large network would be challenging because it requires complete knowledge about the network topology and the location of the stubborn agents. We thus propose here a *distributed* algorithm whereby each agent computes its own HIC based on local information. The construction of our algorithm uses a novel analogy between social and electrical networks by relating the Laplace equation resulting from social influence dynamics to the governing equations of electrical networks. Under this analogy, the asymptotic opinion of regular agent i can be interpreted as the voltage of node i when type zero stubborn agents are kept at voltage zero and type one agents are kept at voltage one. This interpretation allows us to use tricks of electrical circuits and provide a recursive characterization of HIC in trees. Using this characterization, we develop a *message passing* [12] algorithm that computes the HIC. Our algorithm can effectively be employed in general networks, although the convergence analysis presented here is only valid for trees.

We conclude this introduction with a brief outline of the paper. In Section II we define our model of opinion dynamics with stubborn agents and our problem of interest. In Section III we review basic notions of electrical circuits and explain theirs connection with social networks: Section IV is devoted to apply this electrical analogy on tree graphs. In Section V we describe the message passing algorithm to compute the optimal solution on trees, while in Section VI we consider its extension to general graphs. Simulations and final remarks are given in Section VII.

Notation

To avoid possible ambiguities, we briefly recall some notation and a few basic notions of graph theory. The cardinality of a (finite) set E is denoted by $|E|$ and when $E \subset F$ we define its complement as $E^c = \{f \in F \mid f \notin E\}$. A square matrix P is said to be nonnegative when its entries P_{ij} are nonnegative, substochastic when it is nonnegative

An extended account of this work is available as [1]. The authors wish to thank the anonymous reviewers for their insightful comments.

L. Vassio is with Dipartimento di Ingegneria Meccanica e Aerospaziale, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10128 Torino, Italy. Tel: +39-339-2868999. luca.vassio@polito.it.

F. Fagnani is with Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10128 Torino, Italy. Tel: +39-011-0907509. fabio.fagnani@polito.it.

P. Frasca is with Department of Applied Mathematics, University of Twente, Drienerlolaan 5, 7522 NB Enschede, the Netherlands. Tel: +31-53-4893406. p.frasca@utwente.nl.

A. Ozaglar is with LIDS, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, 02139 Massachusetts. Tel: +1-617-3240058. asuman@mit.edu.

and $\sum_j P_{ij} \leq 1$ for every row i, and stochastic when it is nonnegative and $\sum_j P_{ij} = 1$ for every i. We denote by $\mathbf{1}$ a vector whose entries are all 1. An (undirected) graph G is a pair (I, E) where I is a finite set of *nodes* and E is a set of unordered pairs of nodes called *edges*. The neighbor set of a node $i \in I$ is defined as $N_i := \{j \in I | \{i, j\} \in E\}$ and its cardinality $d_i := |N_i|$ is said to be the *degree* of node i. A path in G is a sequence of nodes $\gamma = (j_1, \dots j_s)$ such that $\{j_t, j_{t+1}\} \in E$ for every $t = 1, \dots, s-1$. The path γ is said to connect j_1 and j_s. The path γ is said to be simple if $j_h \neq j_k$ for $h \neq k$. A graph is connected if any pair of distinct nodes can be connected by a path (which can be chosen to be simple). The length of the shortest path between two nodes i and j is said to be the *distance* between them, and is denoted as $\mathrm{dst}(i, j)$. Consequently, the *diameter* of a connected graph is defined to be $\mathrm{diam}(G) := \max_{i,j \in I}\{\mathrm{dst}(i, j)\}$. A tree is a connected graph such that for any pair of distinct nodes there is just one simple path connecting them. Finally, given a graph $G = (I, E)$ and a subset $J \subseteq I$, the subgraph induced by J is defined as $G_{|J} = (J, E_{|J})$ where $E_{|J} = \{\{i, j\} \in E \,|\, i, j \in J\}$.

II. Opinion dynamics and stubborn agent placement

Consider a connected graph $G = (I, E)$. Nodes in I will be thought as agents who can exchange information through the available edges $\{i, j\} \in E$. Each agent $i \in I$ has an opinion $x_i(t) \in \mathbb{R}$ possibly changing in time $t \in \mathbb{N}$. We assume a splitting $I = S \cup R$ with the understanding that agents in S are *stubborn* agents not changing their opinions while those in R are *regular* agents whose opinions modify in time according to the consensus dynamics

$$x_i(t+1) = \sum_{j \in I} Q_{ij} x_j(t)\,, \quad \forall i \in R$$

where $Q_{ij} \geq 0$ for all $i \in R$ and for all $j \in I$ and $\sum_j Q_{ij}$=1 for all $i \in R$. The scalar Q_{ij} represents the relative weight that agent i places on agent j's opinion. We will assume that Q only uses the available edges in G, more precisely, our standing assumption will be that

$$Q_{ij} = 0 \;\Leftrightarrow \{i, j\} \notin E \tag{1}$$

A basic example is obtained by choosing for each regular agent uniform weights along the edges incident to it, *i.e.*, $Q_{ij} = d_i^{-1}$ for all $i \in R$ and $\{i, j\} \in E$. Assembling opinions of regular and stubborn agents in vectors, denoted by $x^R(t)$ and $x^S(t)$, we can rewrite the dynamics in a more compact form as

$$\begin{aligned} x^R(t+1) &= Q^{11} x^R(t) + Q^{12} x^S(t) \\ x^S(t+1) &= x^S(t) \end{aligned}$$

where the matrices Q^{11} and Q^{12} are nonnegative matrices of appropriate dimensions.

Using the adaptivity assumption (1), it is standard to show that Q^{11} is a substochastic asymptotically stable matrix (e.g. spectral radius < 1). Henceforth, $x^R(t) \to x^R(\infty)$ for $t \to +\infty$ with the limit opinions satisfying the relation

$$x^R(\infty) = Q^{11} x^R(\infty) + Q^{12} x^S(0) \tag{2}$$

which is equivalent to

$$x^R(\infty) = (I - Q^{11})^{-1} Q^{12} x^S(0) \tag{3}$$

Notice that $[(I - Q^{11})^{-1} Q^{12}]_{hk} = [\sum_n (Q^{11})^n Q^{12}]_{hk}$ is always non negative and is nonzero if and only if there exists a path in G connecting the regular agent h to the stubborn agent k and not touching other stubborn agents. Moreover, the fact that P is stochastic easily implies that $\sum_k [(I - Q^{11})^{-1} Q^{12}]_{hk} = 1$ for all $h \in R$: asymptotic opinions of regular agents are thus convex combinations of the opinions of stubborn agents.

In this paper we will focus on the situation when $S = S^0 \cup \{\ell\}$ and $R = I \backslash S$ assuming that $x_i(0) = 0$ for all $i \in S^0$ while $x_\ell(0) = 1$, *i.e.*, there are two types of stubborn agents: one type consisting of those in set S^0 that have opinion 0 and the other type consisting of the single agent ℓ that has opinion 1. We investigate how to choose ℓ in $I \setminus S^0$ in such a way to maximize the influence of opinion 1 on the limit opinions. More precisely, let us denote as $x_i^{R,\ell}(\infty)$ the asymptotic opinion of the regular agent $i \in R$ under the above stubborn configuration, and define the objective function.

$$H(\ell) := \sum_{i \in R} x_i^{R,\ell}(\infty) \tag{4}$$

In order to use the electrical circuit analogy, we need to make an extra "reciprocity" assumption on the weights Q_{ij} assuming that they can be represented through a symmetric matrix $C \in \mathbb{R}^{I \times I}$ (called conductance matrix) with non negative elements and $C_{ij} > 0$ iff $\{i, j\} \in E$ by posing

$$Q_{ij} = \frac{C_{ij}}{\sum_j C_{ij}}\,, \quad i \in I\,,\ j \in I \tag{5}$$

The value $C_{ij} = C_{ji}$ can be interpreted as a measure of the "strength" of the relation between i and j. For two regular nodes connected by an edge, the interpretation is a sort of reciprocity in the way the nodes trust each other. Notice that C_{ij} when $i \in S$ is not used in defining the weights, but is anyhow completely determined by the symmetry assumption. Finally, the terms Q_{ij} when $i, j \in S$ do not play any role in the sequel and for simplicity we can assume they are all equal to 0. By the definition (5) and from matrix C we are actually defining a square matrix $Q \in \mathbb{R}^{I \times I}$. Compactly, if we consider the diagonal matrix $D_{C\mathbf{1}} \in \mathbb{R}^{I \times I}$ defined by $(D_{C\mathbf{1}})_{ii} = (C\mathbf{1})_i$, where $\mathbf{1}$ is all ones vector with appropriate dimension, the extension is obtained by putting $Q = D_{C\mathbf{1}}^{-1} C$. The matrix Q is said to be a *time-reversible* stochastic matrix in the probability jargon. The special case of uniform weights considered before fits in this framework, by simply choosing $C = A_G$, where A_G is the adjacency matrix of the graph. In this case all edges have equal strengths and the resulting time-reversible stochastic matrix Q is known as the simple random walk (SRW) on G.

III. THE ELECTRICAL NETWORK ANALOGY

In this section we briefly recall the basic notions of electrical circuits and we illustrate the relation with our problem. A connected graph $G = (I, E)$ together with a conductance matrix $C \in \mathbb{R}^{I \times I}$ can be interpreted as an electrical circuit where an edge $\{i, j\}$ has electrical conductance $C_{ij} = C_{ji}$ (and thus resistance $R_{ij} = C_{ij}^{-1}$). The pair (G, C) will be called an *electrical network* from now on.

An *incidence matrix* on G is any matrix $B \in \{0, +1, -1\}^{E \times I}$ such that $B\mathbf{1} = 0$ and $B_{ei} \neq 0$ iff $i \in e$. It is immediate to see that given $e = \{i, j\}$, the e-th row of B has all zeroes except B_{ei} and B_{ej}: necessarily one of them will be $+1$ and the other one -1 and this will be interpreted as choosing a direction in e from the node corresponding to $+1$ to the one corresponding to -1. Define $D_C \in \mathbb{R}^{E \times E}$ to be the diagonal matrix such that $(D_C)_{ee} = C_{ij} = C_{ji}$ if $e = \{i, j\}$. A standard computation shows that $B^* D_C B = D_{C\mathbf{1}} - C$.

On the electrical network (G, C) we now introduce current flows. Consider a vector $\eta \in \mathbb{R}^I$ such that $\eta^* \mathbf{1} = 0$: we interpret η_i as the input current injected at node i (if negative being an outgoing current). Given C and η, we can define the voltage $W \in \mathbb{R}^I$ and the current flow $\Phi \in \mathbb{R}^E$ in such a way that the usual Kirchoff and Ohm's law are satisfied on the network. Compactly, they can be expressed as

$$\begin{cases} B^* \Phi = \eta \\ D_C B W = \Phi \end{cases}$$

Notice that Φ_e is the current flowing on edge e and sign is positive iff flow is along the conventional direction individuated by B on edge e. Coupling the two equations we obtain $(D_{C\mathbf{1}} - C)W = \eta$ which can be rewritten as

$$L(Q)W = D_{C\mathbf{1}}^{-1} \eta \qquad (6)$$

where $L(Q) := I - Q$ is the so called Laplacian of Q. Since the graph is connected, $L(Q)$ has rank $|I| - 1$ and $L(Q)\mathbf{1} = 0$. This shows that (6) determines W up to translations. Notice that $(L(Q)W)_i = 0$ for every $i \in I$ such that $\eta_i = 0$. For this reason, in analogy with the Laplacian equation in continuous spaces, W is said to be *harmonic* on $\{i \in I \,|\, \eta_i = 0\}$. Clearly, given a subset $S \subseteq I$ and a $W \in \mathbb{R}^I$ which is harmonic on S^c, we can always interpret W as a voltage with input currents given by $\eta = D_{C\mathbf{1}} L(Q) W$ which will necessarily be supported on S. W is actually the only voltage harmonic on S^c and with assigned values on S.

It is often possible to replace an electrical network by a simplified one without changing certain quantities of interest. An useful operation is *gluing:* if we merge vertices having the same voltage into a single one, while keeping all existing edges, voltages and currents are unchanged, because current never flows between vertices with the same voltage. Another useful operation is replacing a portion of the electrical network connecting two nodes h, k by an *equivalent resistance*, a single resistance denoted as R_{hk}^{eff} which keeps the difference of voltage $W(h) - W(k)$ unchanged. Two basic cases consist in deleting degree two nodes by adding resistances (series law) and replacing multiple edges between two nodes with a single one having conductance equal to the sum of the various conductances (parallel law). These techniques will be heavily used in deriving our algorithm.

Social networks as electrical networks

We are now ready to state the relationship between social and electrical networks. Consider a connected graph $G = (I, E)$, a subset of stubborn agents $S \subseteq I$, and a stochastic time-reversible matrix Q having conductance matrix C. Notice that relation (2) can also be written as

$$L(Q) \begin{pmatrix} x^{R,\ell}(\infty) \\ x^S(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \end{pmatrix} \qquad (7)$$

for some suitable vector $\theta \in \mathbb{R}^S$ (where θ represents the initial opinions of the stubborn agents). Comparing with (6), it follows that $x^{R,\ell}(\infty)$ can be interpreted as the voltage at the regular agents when stubborn agents have fixed voltage $x^S(0)$ or, equivalently, when input currents $D_{C\mathbf{1}}\theta$ are applied to the stubborn agents. Because of equation (7), the vector $x^{R,\ell}(\infty)$ is called the *harmonic* extension of $x_S(0)$ and the function H defined in (4) the *Harmonic Influence Centrality (HIC)*.

Thanks to the electrical analogy we can compute the asymptotic opinion of the agents by computing the voltages in the graph seen as an electrical network. From now on, we will stick to this equivalence and we will exclusively talk about an electrical network (G, C) with a subset $S^0 \subseteq I$ of nodes at voltage 0. For any $\ell \in I \setminus S_0$, $W^{(\ell)}$ denotes the voltage on I such that $W^{(\ell)}(i) = 0$ for every $i \in S^0$ and $W^{(\ell)}(\ell) = 1$. Using this notation and the association between limiting opinions and electric voltages provided in Eqs. (6) and (7), we can express the Harmonic Influence Centrality of node ℓ as

$$H(\ell) = \sum_{i \in I, i \neq \ell} W^{(\ell)}(i).$$

The following result, which is an immediate consequence of (7), provides a formula that is useful in computing the voltages.

Lemma 1 (Voltage scaling): Consider the electrical network (G, C) and a subset $S^0 \subseteq I$. Let $W^{(\ell)}$ be the voltage which is 0 on S^0 and 1 on ℓ. Let W be another voltage such that $W(s) = w_0$ if $s \in S^0$ and $W(\ell) = w_1$. Then, for every node $i \in I$ it holds

$$W(i) = w_0 + (w_1 - w_0) W^{(\ell)}(i) \qquad (8)$$

IV. THE ELECTRICAL ANALOGY ON TREES

The case when the graph G is a *tree* is very important to us. In the foregoing, we assume to have fixed a tree $T = (I, E)$, a conductance matrix C and the subset $S^0 \subseteq I$ of 0 voltage nodes. We next define subsets of nodes of the given tree, which enable us to isolate the effects of the upstream and downstream neighbors of a node in computing its harmonic centrality and its voltage. Given a pair of distinct nodes $i, j \in I$, we let $I^{<ij}$ denote the subset of nodes that

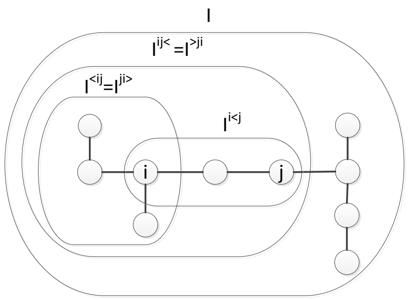

Fig. 1. An example of tree presenting the notation of the subsets of I.

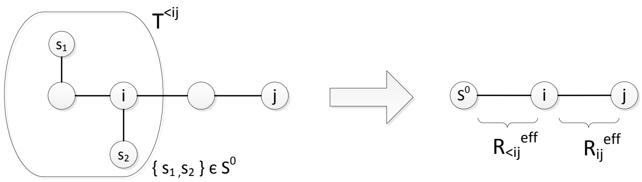

Fig. 2. A subtree equivalently represented as a line graph.

form the subtree rooted at node i that does not contain node j. Formally,

$$I^{<ij} := \{h \in I \mid \text{the simple path from } h \text{ to } j \text{ goes through } i\}.$$

We also define $I^{ij>} := I^{<ji}$, $I^{ij<} := (I^{ij>})^c \cup \{j\}$, $I^{>ij} := I^{ji<}$, and $I^{i<j} := I^{ij<} \cap I^{>ij}$. Figure 1 illustrates these definitions.

The induced subtrees is denoted using the same apex $T^{<ij}$ and so on; similarly the HIC of the nodes on each of the trees above is denoted as $H^{<ij}(\cdot)$ and so on. Finally, we use the notation $R^{\text{eff}}_{<ij}$ to denote the effective resistance inside $T^{<ij}$ between $S^0 \cap I^{<ij}$ (considered as a unique collapsed node) and node i. We will conventionally interpret this resistance as infinite in case $S^0 \cap I^{<ij} = \emptyset$. Analogously, we define $R^{\text{eff}}_{ij>} := R^{\text{eff}}_{<ji}$.

Given a pair of distinct nodes i, $j \in I$, consider the two voltages $W^{(i)}$ and $W^{(j)}$. If we restrict them to $T^{<ij}$, we may interpret them as two voltages on $T^{<ij}$ which are 0 on S^0 and take values in node i, respectively, $W^{(i)}(i) = 1$ and $W^{(j)}(i)$. It follows by applying Lemma 1 that

$$W^{(j)}(\ell) = W^{(j)}(i)W^{(i)}(\ell) \quad \forall \ell \in I^{<ij} \tag{9}$$

Moreover, $W^{(j)}(i)$ can be computed through effective resistances replacing the circuit determined by the subtree $T^{<ij}$, by an equivalent circuit represented by a line graph with three nodes S^0, i, and j as in Figure 2. We recall that collapsing all nodes of S^0 in a single node is possible since they all have the same voltage. Moreover, by definition, the edge $\{S^0, i\}$ has resistance $R^{\text{eff}}_{<ij}$, while $\{i, j\}$ has resistance R^{eff}_{ij}. Therefore, since the current flowing along the two edges is the same, Ohm's law yields

$$\frac{W^{(j)}(j)}{R^{\text{eff}}_{<ij} + R^{\text{eff}}_{ij}} = \frac{W^{(j)}(i)}{R^{\text{eff}}_{<ij}}$$

yielding

$$W^{(j)}(i) = \frac{R^{\text{eff}}_{<ij}}{R^{\text{eff}}_{<ij} + R^{\text{eff}}_{ij}} \tag{10}$$

(equal to 1 in case $S^0 \cap I^{<ij} = \emptyset$). From relations (9) and (10), later on we will derive iterative equations for the computation of voltages on a tree.

The absence of cycles has the important consequence that nodes in S^0 break the computation of the HIC into separate non-interacting components. Indeed, the induced subgraph $T_{|I \setminus S_0}$ is a forest composed of subtrees $\{T_h = (J_h, E_h)\}_{h \in \{1,\dots,n\}}$. For every $h \in \{1, \dots, n\}$, define the set S^0_h as the set of type 0 stubborn nodes that are adjacent to nodes in J_h in the graph G, that is,

$$S^0_h := \{i \in S^0 | \exists j \in J_h : \{i, j\} \in E\}.$$

Then, define the tree $\widehat{T}_h = G|_{S^0_h \cup I_h}$, which is therefore the tree T_h augmented with its type 0 stubborn neighbors in the original graph G. An example of this procedure is shown in Figure 3. It is then immediate to see that it is sufficient to compute the HIC on the subtrees $\widehat{T}_h$: consequently, we will assume from now on that *stubborn agents are located in the leaves*, without any loss of generality.

V. Message passing on trees

In this section, we design a message passing algorithm (MPA), which computes the HIC of every node of a tree in a distributed way. We begin by outlining the structure of a general message passing algorithm on a tree. Preliminarily, define any node i in the graph as the *root*. In the first phase, messages are passed inwards: starting at the leaves, each node passes a message along the unique edge towards the root node. The tree structure guarantees that it is possible to obtain messages from all other neighbor nodes before passing the message on. This process continues until the root i has obtained messages from all its neighbors. The second phase involves passing the messages back out: starting at the root, messages are passed in the reverse direction. The algorithm is completed when all leaves have received their messages.

Next, we show how this approach can be effective in our problem. Take a generic root node $i \in I \setminus S^0$ and, for every $j \in N_i$, notice the following iterative structure of the subtree rooted in i and not containing j:

$$I^{<ij} = \bigcup_{k \in N_i \setminus \{j\}} I^{<ki} \cup \{i\}$$

This, together with relation (9), yields

$$\begin{aligned} H^{<ij}(i) &= \sum_{k \in N_i \setminus \{j\}} \sum_{\ell \in T^{<ki}} W^{(i)}(\ell) + 1 \\ &= \sum_{k \in N_i \setminus \{j\}} W^{(i)}(k) H^{<ki}(k) + 1. \end{aligned} \tag{11}$$

Forthermore, (10) yields

$$W^{(i)}(k) = \frac{R^{\text{eff}}_{<ki}}{R^{\text{eff}}_{<ki} + R_{ik}} \tag{12}$$

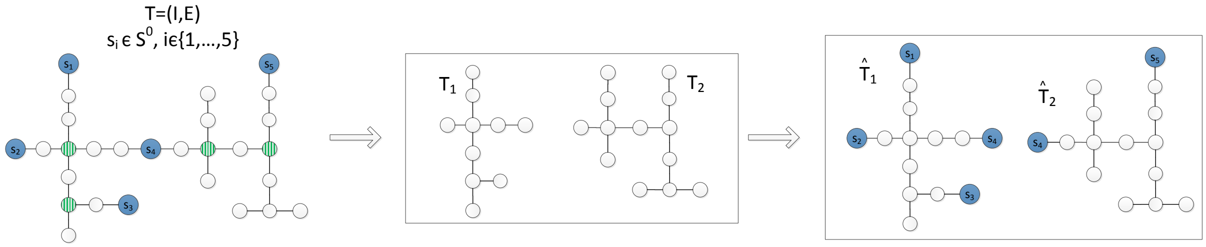

Fig. 3. A tree with type 0 stubborn nodes in blue, together with its decomposition according to Proposition **??**.

where we conventionally assume that $R^{\text{eff}}_{<ki} = \infty$ and $W^{(i)}(k) = 1$ if $S^0 \cap I^{<ki} = \emptyset$. On the other hand, also effective resistances $R^{\text{eff}}_{<ki}$ admit an iterative representation. Indeed, replace $T^{<ij}$ with the equivalent circuit consisting of nodes: S^0, i, j, and all the nodes $k \in N_i \setminus \{j\}$. Between S^0 and i there are $|N_i| - 1$ parallel (length 2) paths each passing through a different $k \in N_i \setminus \{j\}$ and having resistance $R^{\text{eff}}_{<ki} + R_{ik}$. Therefore, using the parallel law for resistances we obtain

$$R^{\text{eff}}_{<ij} = \left(\sum_{k \in N_i \setminus \{j\}} \frac{1}{R^{\text{eff}}_{<ki} + R_{ik}} \right)^{-1} \tag{13}$$

The three relations (11), (12), and (13) determine an iterative algorithm to compute $H(i)$ at every node i starting from leaves and propagating to the rest of the graph. More precisely, define $H^{i\to j} := H^{<ij}(i)$, and $W^{i\to j} := W^{(i)}(j)$ to be thought as messages sent by node i to node j along the edge $\{i, j\}$. From (11), (12), and (13), we easily obtain the following iterative relations

$$\begin{aligned} H^{i\to j} &= \sum_{k \in N_i \setminus \{j\}} W^{k\to i} H^{k\to i} + 1 \\ W^{i\to j} &= \left(1 + R_{ij} \sum_{k \in N_i \setminus \{j\}} \frac{1 - W^{k\to i}}{R_{ik}} \right)^{-1} \end{aligned} \tag{14}$$

Notice that a node i can only send messages to a neighbor j, once he has received messages $H^{k\to i}$ and $W^{k\to i}$ from all its neighbors k but j. Iteration can start at leaves (having just one neighbor) with the following initialization step

$$\begin{aligned} H^{i\to j} &= \mathbf{1}_{i \notin S^0} \\ W^{i\to j} &= \mathbf{1}_{i \notin S^0} \end{aligned} \tag{15}$$

where we denote by $\mathbf{1}_{i \notin S^0}$ a vector indexed in I which has entry 1 if $i \notin S^0$ and entry 0 elsewhere. Notice that each regular agent i can finally compute $H(i)$ by the formula

$$H(i) = \sum_{k \in N_i} W^{k\to i} H^{k\to i} + 1$$

Hence, the algorithm converges in a number of steps not larger than the diameter of the tree. Furthermore, it can easily be shown that the needed number of operations for the whole network is $O(\sum_{i \in I \setminus S^0} d_i^2)$.

A centralized algorithm to compute the HIC in any connected network was proposed in [11]: its computational complexity is $O((|I| - |S^0|)^3)$. Since $\sum_{i \in I \setminus S^0} d_i^2 \leq |I \setminus S^0| \, d^2_{\max}$, on graphs with bounded degrees the MPA has a much smaller complexity $O(|I| - |S^0|)$.

VI. Message passing on general graphs

The MPA presented above is limited to trees. Message passing algorithms are commonly designed on trees, but also implemented with some modification over general graphs. In many cases, the application is just empirical, without a proof of convergence. We will see in this section how to apply the MPA to every graph, with suitable modifications in order to manage the new issues. Namely, we design an "iterative" version of the message passing algorithm of Section V, which can run on every network, regardless of the presence of cycles.

We let the nodes send their messages at every time step, so that we denote them as $W^{i\to j}(t)$ and $H^{i\to j}(t)$ for all $t \geq 0$. The dynamics of messages are

$$H^{i\to j}(t+1) = \sum_{k \in N_i \setminus \{j\}} W^{k\to i}(t) H^{k\to i}(t) + 1 \tag{16a}$$

$$W^{i\to j}(t+1) = \left(1 + R_{ij} \sum_{k \in N_i \setminus \{j\}} \frac{1 - W^{k\to i}(t)}{R_{ik}} \right)^{-1} \tag{16b}$$

if $i \notin S^0$ and

$$H^{i\to j}(t+1) = 0 \tag{17a}$$

$$W^{i\to j}(t+1) = 0 \tag{17b}$$

otherwise. The initialization is

$$H^{i\to j}(0) = \mathbf{1}_{i \notin S^0} \qquad W^{i\to j}(0) = \mathbf{1}_{i \notin S^0} \tag{18}$$

By these definitions of messages we have defined the message passing algorithm for general graphs. We should choose a termination criterion: for instance, the algorithm may stop after a number of steps which is chosen *a priori*. At every time t, each agent i can compute an approximate $H(i)^{(t)}$ by the formula

$$H(i)^{(t)} = \sum_{k \in N_i} W^{k\to i}(t) H^{k\to i}(t) + 1$$

In order to illustrate how this algorithm is affected by the presence of cycles, we can use the so called *computation trees*, which we construct in the following way. Given a graph G, we focus on a 'root' node, and for all $t \in \mathbb{N}$ we let the nodes at distance t from the root (the level t of the tree) be the nodes whose messages reach the root after t iterations of the message-passing algorithm. Note that if the graph G

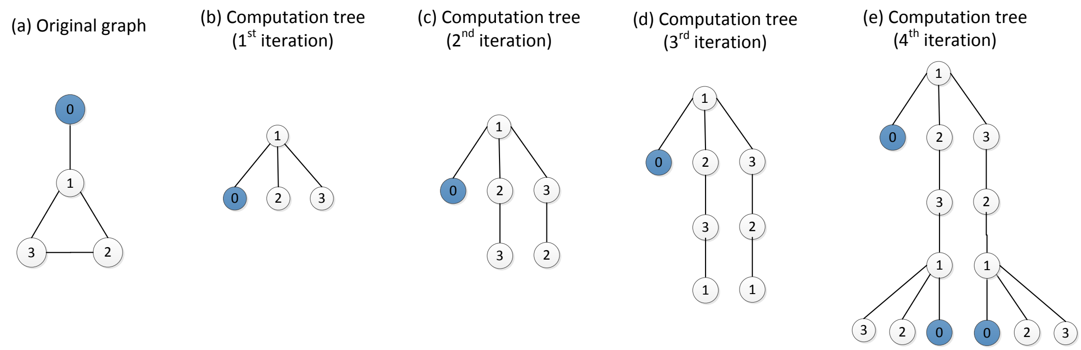

Fig. 4. A graph (a) and computation trees of increasing depth (b)-(c)-(d)-(e) of a MPA from root 1.

is a tree, the computation tree is just equal to G; otherwise, it has a number of nodes which diverges when t goes to infinity. As an example, Figure 4 shows the first 4 levels of a computation tree. In our MPA, each node i is computing its own Harmonic Influence Centrality *in the computation tree instead than on the original graph*. As the number of levels of the computation tree diverges, the computation procedure may not converge, and –if converging– may not converge to the harmonic influence in the original graph. Sufficient conditions for convergence are given in [1].

VII. Simulations and final remarks

We have performed extensive simulations of our algorithm on well-known families of random graphs such as Erdős-Rényi and Watts-Strogatz, obtaining very encouraging results. First, the algorithm is convergent in every test. Second, in many cases the computed values of HIC are very close to the correct values, which we can obtain by the benchmark algorithm in [11].

In this note, we limit ourselves to a small example for the sake of illustration: a more extended set of simulations is reported in [1]. We simulate an Erdős-Rényi random graph [13] with $n = 15$ and edge probability $p = 0.2$ and we set $S^0 = \{1, 2, 3\}$. In spite of the presence of several cycles, we see in see Figure 5 that the algorithm finds the maximum of the HIC correctly: in fact, the three nodes with highest HIC are identified and the HIC profile is well approximated. The same diagram compares the HIC with some heuristics that can be computed in a distributed way: the degree centrality (*i.e.,* the number of neighbors) and the eigenvector centrality. We clearly see that these measures are inadequate to our problem. Indeed, both the degree centrality and the eigenvector centrality evaluate the influence of a node within a network, but they do not consider the different role of stubborn nodes, treating them as normal nodes.

Further research will be devoted to an extensive experimental analysis of the proposed algorithms on real-world networks and to extending the analysis of the algorithm beyond the scope of the current assumptions to include general networks with cycles and directed edges.

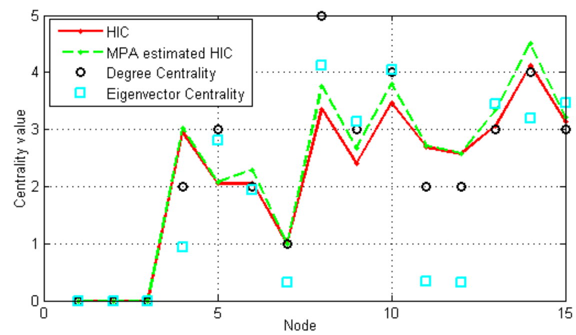

Fig. 5. Values of H computed by the MPA, compared with the actual values and to the degree centrality and the (rescaled) eigenvector centrality, in a graph with 15 nodes.

References

[1] L. Vassio, F. Fagnani, P. Frasca, and A. Ozdaglar, “Message passing optimization of harmonic influence centrality,” *IEEE Transactions on Control of Network Systems*, 2014, to appear. [Online]. Available: http://arxiv.org/abs/1310.0005
[2] P. Bonacich, “Power and centrality: A family of measures,” *The American Journal of Sociology*, vol. 92, no. 5, pp. 1170–1182, 1987.
[3] K. Stephenson and M. Zelen, “Rethinking centrality: Methods and examples,” *Social Networks*, vol. 11, no. 1, pp. 1–37, 1989.
[4] N. E. Friedkin, “Theoretical foundations for centrality measures,” *American Journal of Sociology*, vol. 96, no. 6, pp. 1478–1504, 1991.
[5] D. Kempe, J. Kleinberg, and É. Tardos, “Maximizing the spread of influence through a social network,” in *ACM International Conference on Knowledge Discovery and Data Mining*, 2003, pp. 137–146.
[6] M. Newman, “A measure of betweenness centrality based on random walks,” *Social Networks*, vol. 27, no. 1, pp. 39–54, 2005.
[7] D. Easley and J. Kleinberg, *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*. Cambridge University Press, 2010.
[8] H. Ishii and R. Tempo, “Distributed randomized algorithms for the PageRank computation,” *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 1987–2002, 2010.
[9] W. Wang and C. Tang, “Distributed computation of node and edge betweenness on tree graphs,” in *IEEE Conference on Decision and Control*, Florence, Italy, Dec. 2013, pp. 43–48.
[10] D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar, “Opinion fluctuations and disagreement in social networks,” *Mathematics of Operations Research*, vol. 38, no. 1, pp. 1–27, 2013.
[11] E. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi, and A. Scaglione, “Binary opinion dynamics with stubborn agents,” *ACM Transactions on Economy and Computation*, vol. 1, no. 4, pp. 1–30, 2013.
[12] M. Mézard and A. Montanari, *Information, Physics, and Computation*. Oxford, 2009.
[13] P. Erdős and A. Rényi, “On random graphs,” *Publ. Math. Debrecen*, vol. 6, pp. 290–297, 1959.