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Synchronization for heterogeneous networks of weakly-non-minimum-phase, non-introspective agents without exchange of controller states

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Abstract— **This paper studies the synchronization problem for undirected, weighted networks where agents are non-introspective (i.e. they have no access to any state or output) and do not need another communication layer to exchange internal controller states. The more significant is that this paper deals with weakly-non-minimum-phase agents. We consider heterogeneous networks with linear agents. A purely decentralized linear dynamical protocol based on a low-and-high gain methodology is designed for each agent, where the only information available for each agent is a weighted linear combination of its output relative to that of its neighboring.**

I. Introduction

In the last decade, many researchers have worked on the synchronization problem of networks. The area spreads from the theoretical research to different applications, such as robot networks, sensor networks, power networks, social networks, and so on. The goal of synchronization is to secure an asymptotic agreement on a common state (*state synchronization*) or output trajectory (*output synchronization*) among agents of the network through decentralized control protocols. Part of earlier works can be seen in [8], [9],[11] for the state synchronization problem of homogeneous networks (i.e. agents are identical), and in [1], [2], [13] for the output synchronization problem of heterogeneous networks.

For heterogeneous networks, with higher-order, non-introspective agents, it becomes more challenging to achieve synchronization among agents. Grip et al solve the output synchronization problem for such a kind of network in [4] by using a distributed high-gain observer, but an extra layer of communication to exchange the internal controller states is needed as introduced in [5]. This additional layer is later dispensed in [3], where a purely distributed linear time-invariant protocol with a low-and-high gain is used. Zhang et al extend that work to more complex networks with external disturbances in [14], [15], [7].

However, all the above mentioned references which do not use an exchange of controller states require that agents are *minimum-phase*. That means all invariant zeros of agents are in the open left half complex plane. In [10], we studied *weakly-non-minimum-phase* agents for homogeneous networks, that is invariant zeros of agents can be located on the imaginary axis. The agents considered in that paper are single-input-single-output (SISO) and non-introspective.

This paper continues our studies on output synchronization. We consider networks of heterogeneous, *weakly-non-minimum-phase* linear agents with the property that the weakly-non-minimum-phase zeros are the same for each agent. The only information accessible for each agent is a linear combination of its output relative to its neighboring agents, and agents do not exchange their internal controller states.

A. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, a' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm. We denote by $\text{blkdiag}\{a_i\}$, a block-diagonal matrix with $a_1,\dots,a_n$ as the diagonal elements, and by $\text{col}\{x_i\}$, a column vector with $x_1,\dots,x_n$ stacked together, where the range of index i can be identified from the context. $A\otimes B$ indicates the Kronecker product between A and B.

Definition 1: A matrix pair (A,C) is said to contain the matrix pair (S,R) if there exists a matrix Π such that $\Pi S = A\Pi$ and $C\Pi = R$.

Remark 1: Definition 1 implies that for any initial condition $\omega(0)$ of the system $\dot\omega = S\omega$, $y_r = R\omega$, there exists an initial condition $x(0)$ of the system $\dot x = Ax$, $y = Cx$, such that $y(t) = y_r(t)$ for all $t \geq 0$ ([6]).

Definition 2: A system (A,B,C) is weakly-non-minimum-phase if all the invariant zeros are in the closed left half complex plane and the system has at least one invariant zero on the imaginary axis which is not simple.

II. Network communication

In this paper we will consider networks composed of N SISO agents, with the state and output of agent $i\in\{1,\dots,N\}$ denoted by x_i and y_i, respectively. The agents are non-introspective; hence, agent i does not have access to its own state or output. The only information available to each agent is a linear combination of its own output relative to that of the other agents:

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(y_i(t) - y_j(t)), \tag{1}$$

where $a_{ij} = a_{ji} \geq 0$ and $a_{ii} = 0$.

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The communication topology of the network can be described by an undirected graph $\mathcal{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists between agent j to i and $a_{ji} = a_{ij}$. The weight of the edge equals the magnitude of a_{ij}. A *path* from node i_1 to i_k is a squence of nodes $\{i_1, \dots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A graph is *connected* if there exists a path between every pair of nodes. A *connected subgraph* is a subset of nodes of $\mathcal{G}$ such that the subgraph is connected. For a weighted undirected graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix*. In the case where $\mathcal{G}$ is undirected and has non-negative weights, all eigenvalues of L are real and located in the closed right half complex plane; moreover at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$. In terms of the Laplacian matrix L, ζ_i can be rewritten as

$$\zeta_i(t) = \sum_{j=1}^{N} \ell_{ij} y_j(t). \tag{2}$$

III. Heterogeneous networks of linear agents

In this section we will consider heterogeneous networks where the agents are linear, SISO, non-introspective and weakly-non-minimum-phase. We will formulate the output synchronization problem and present the protocol design.

A. Problem formulation

The agents, denoted by $\tilde{\Sigma}_i$ with $i \in \{1, \dots, N\}$, have this form

$$\tilde{\Sigma}_i : \begin{cases} \dot{\tilde{x}}_i = \tilde{A}_i \tilde{x}_i + \tilde{B}_i \tilde{u}_i, \\ y_i = \tilde{C}_i \tilde{x}_i, \end{cases} \tag{3}$$

where $\tilde{x}_i \in \mathbb{R}^{\tilde{n}_i}$, $\tilde{u}_i \in \mathbb{R}$, $y_i \in \mathbb{R}$ are the state, input and output of agent i. Moreover, the order of the infinite zero for agent i is $\tilde{\rho}_i$.

Note that the dimension of each agent state is $\tilde{n}_i$, which is different for all agents, and so is the order of infinite zeros $\tilde{\rho}_i$. We make the following assumptions regarding the agent dynamics.

Assumption 1: For each $i \in \{1, \dots, N\}$, the triple $(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$ is stabilizable and detectable. Moreover, each agent is weakly-non-minimum-phase.

Assumption 2: We assume all agents have the same invariant zeros (counting multiplicity) on the imaginary axis.

Since the agents in the network are non-identical, state synchronization is not a realistic objective. Thus, we turn to pursue regulated output synchronization among agents. In other words, our goal is to regulate the outputs of all agents asymptotically towards an a priori specified reference trajectory. The reference trajectory in this paper is generated by an autonomous exosystem

$$\begin{cases} \dot{x}_r = S x_r, & x_r(0) = x_{r0}, \\ y_r = R x_r, \end{cases} \tag{4}$$

where $x_r \in \mathbb{R}^{n_r}$, $y_r \in \mathbb{R}$. We make the following assumptions on the exosystem.

Assumption 3: We assume that

- (S, R) is observable;
- All eigenvalues of S are in the closed right-half complex plane and do not intersect with the invariant zeros of the agents.

Remark 2: It is worth noting that stable eigenvalues of S are excluded here, because stable modes vanishes asymptotically and hence play no role asymptotically.

Let $e_i = y_i - y_r$ denote the regulated output synchronization error for agent i $(i = 1, \dots, N)$. In order to achieve our goal, it is clear that a non-empty subset of agents must have knowledge of their output relative to the reference trajectory y_r generated by the reference system. We denote such a subset of agents by π. Specially, each agent has access to the quantity $\psi_i = \iota_i(y_i - y_r)$ with $\iota_i = 1$ for agent $i \in \pi$ and otherwise $\iota_i = 0$. In the following, we will refer to the node set π as the *root set*. In order to achieve regulated output synchronization for all agents, every node of the network graph $\mathcal{G}$ shouls be a member of a connected subgraph which has one node contained in the set π (when the network graph $\mathcal{G}$ is connected, the set π is completely arbitrary as long as it contains at least one agent).

Based on the Laplacian matrix L of our network graph $\mathcal{G}$, we define the expanded Laplacian matrix as

$$\bar{L} = L + \text{blkdiag}\{\iota_i\} = [\bar{\ell}_{ij}].$$

Note that $\bar{L}$ is clearly not a Laplacian matrix associated to some graph since it does not have a zero row sum. From [4, Lemma 7], all eigenvalues of $\bar{L}$ are in the open right-half complex plane.

We would like to note that, in practice, precise information of a network communication topology is usually not available for controller design and only some rough characterization of the network can be obtained. In our case, we assume only a lower bound on the smallest eigenvalue of the expanded Laplacian is given:

Definition 3: For given real number $\beta > 0$, the set $\mathbb{G}^{\pi}_{\beta,N}$ consists of all weighted and undirected graphs composed of N nodes satisfying the following property:

- The eigenvalues of the expanded Laplacian matrix $\bar{L}$, denoted by $\lambda_1, \dots, \lambda_N$, which are real, satisfy $\lambda_i > \beta$.

Remark 3: We note that the fact that all eigenvalues of the expanded Laplacian are positive guarantees that every node of our undirected network graph is a member of a connected subgraph which has one node contained in the set π.

We will define the regulated output synchronization problem for heterogenous networks of weakly-non-minimum-phase, non-introspective agents as follows.

Problem 1: Consider a multi-agent system (3), (1) and reference system (4) satisfying Assumptions 1 and 3. Moreover, all agents in the network are weakly-non-minimum-phase. Let $\beta > 0$ and let a root set π be given. The *regulated output synchronization* problem is to find, if possible, a linear time-invariant dynamic protocol such that the regulated output synchronization error satisfies

$$\lim_{t\to\infty} e_i(t) = 0, \tag{5}$$

for all $i \in \{1,\dots,N\}$, for all initial conditions $\tilde{x}_i(0)$, $x_r(0)$ and for any network graph $\mathcal{G} \in \mathbb{G}^{\pi}_{\beta,N}$.

B. Protocol design

Our protocol design is composed of two phases. We know that all agents may have different state dimensions and orders of infinite zeros; hence it is difficult to compare agents' outputs. To realize the regulation of their outputs, we will add the mode of exosystem (4) to each agent. Moreover, we want all agents to have the same order for their infinite zero. This will be achieved in Phase 1 by designing a pre-compensator for each agent. We will then consider the network of expanded agents (each original agent with associated pre-compensator. Then, we will design a distributed controller that achieves our goal: regulated output synchronization.

Phase 1: In this phase, we will generate a pre-compensator for each agent such that the interconnection of each agent (3) with its pre-compensator has three properties

- the order of the infinite zeros (i.e. the relative degree) is the same.
- the dynamics contain the dynamics of the reference system (4).
- the marginally unstable zero dynamics is the same.

Note that the second property is defined according to Definition 1. Regarding the third property note that having the same zero dynamics is a stronger property than sharing the same marginally unstable zeros (which was guaranteed by Assumption 2).

It can be shown (details are ommitted due to page limits) that there exists for each agent a linear pre-compensator of the form

$$\begin{cases} \dot{z}_i = A_{ip} z_i + B_{ip} u_i, \\ \tilde{u}_i = C_{ip} z_i, \end{cases} \tag{6}$$

for $i = 1,\dots,N$ such that interconnection of agent (3) and pre-compensator (6) is of the form:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \end{cases} \tag{7}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}$, $y_i \in \mathbb{R}$ are states, inputs and outputs of the interconnection system of agent (3) and pre-compensator (6). Moreover,

- (A_i, C_i) contains (S, R), i.e., there exists matrix Π_i such that $\Pi_i S = A_i \Pi_i$, $C_i \Pi_i = R$;
- (A_i, B_i, C_i) has relative degree ρ.

Finally, we can guarantee that (A_i, B_i, C_i) is in the Special Coordinate Bases (SCB) form, where $x_i = [x_{ia}^-; x_{ia}^0; x_{id}]$, with $x_{ia}^- \in \mathbb{R}^{n_i - r - \rho}$ representing the stable invariant zero structure, $x_{ia}^0 \in \mathbb{R}^r$ representing the marginally unstable invariant zero structure and $x_{id} \in \mathbb{R}^\rho$ the infinite zero structure, such that (7) can be written as:

$$\begin{cases} \dot{x}_{ia}^- = A_{ia}^- x_{ia}^- + L_{iad}^- y_i, \\ \dot{x}_{ia}^0 = A_a^0 x_{ia}^0 + L_{ad}^0 y_i, \\ \dot{x}_{id} = A_d x_{id} + B_d (u_i + E_{ida}^- x_{ia}^- + E_{da}^0 x_{ia}^0 + E_{idd} x_{id}), \\ y_i = C_d x_{id}, \end{cases} \tag{8}$$

for $i = 1,\dots,N$, where A_d, B_d and C_d have special structures, i.e.,

$$A_d = \begin{pmatrix} 0 & I_{\rho-1} \\ 0 & 0 \end{pmatrix}, \qquad B_d = \begin{pmatrix} 0_{\rho-1} \\ 1 \end{pmatrix}, \qquad C_d = \begin{pmatrix} 1 & 0'_{\rho-1} \end{pmatrix}. \tag{9}$$

Note that the marginally unstable zero dynamics in the above are the same for each agent in the sense that $(E_{da}^0, A_a^0, L_{ad}^0)$ are the same for each agent (and hence we do not use an index i). This extra structure will be crucial in our design and while Assumption 2 might guarantee that A_a^0 can be assumed to be identical among agents, it is our precompensator design which guarantees that we can ensure that E_{da}^0 and L_{ad}^0 are the same for each agent.

Here, we assume the dimension of this marginally stable invariant zero dynamics is r. Together with the relative degree ρ, the stable invariant zero dynamics of agent i has dimension $n_i - r - \rho$, which can clearly be different for each agent.

Phase 2: In this phase, we will design a purely decentralized controller for each interconnection system (7), which has the SCB form of (8). As mentioned at the beginning, all agents are non-introspective. So the only information utilized by the controller is ζ_i (provided by the network) and ψ_i (relative output information from the reference system only available for root set agents). The controller for agent i is designed as

$$\begin{cases} \dot{\hat{x}}_{ia}^0 = A_a^0 \hat{x}_{ia}^0 + L_{ad}^0 C_d \hat{x}_{id} + K_1(\zeta_i + \psi_i - C_d \hat{x}_{id}), \\ \dot{\hat{x}}_{id} = A_d \hat{x}_{id} + K_2(\zeta_i + \psi_i - C_d \hat{x}_{id}) \\ \qquad\quad + B_d(F_1 \hat{x}_{ia}^0 + F_2 \hat{x}_{id} + E_{da}^0 \hat{x}_{ia}^0 + E_{idd} \hat{x}_{id}), \\ u_i = F_1 \hat{x}_{ia}^0 + F_2 \hat{x}_{id} \end{cases} \tag{10}$$

with $i = 1,\dots,N$. Note that we find the estimates $\hat{x}_{ia}^0$ and $\hat{x}_{id}$ through high-gain observers. However, we need to realize that they are not estimates of x_{ia}^0 and x_{id}, but estimates of

$$\sum_{j=1}^N \bar{\ell}_{ij} x_{ia}^0 \qquad \text{and} \qquad \sum_{j=1}^N \bar{\ell}_{ij} x_{id},$$

respectively. Because the stable zeros dynamics x_{ia}^- in (8) do not affect synchronization, there is no need for an observer to estimate that part of the dynamics. Choose

$$F_1 = \varepsilon^{-\rho+1} \bar{F}_1,\ F_2 = \varepsilon^{-\rho} \bar{F}_2 S_\varepsilon,\ K_1 = \varepsilon^{-\rho} \bar{K}_1,\ K_2 = \varepsilon^{-1} S_\varepsilon^{-1} \bar{K}_2,$$

where $\varepsilon \in (0,1]$ is a high-gain parameter and $S_\varepsilon = \operatorname{blkdiag}\{1, \varepsilon, \dots, \varepsilon^{\rho-1}\}$. Moreover, $\bar{K}_2$ is selected such that $A_d - \bar{K}_2 C_d$ is asymptotically stable while $\bar{F}_2 = -B_d' P_d$, where P_d is the solution of the algebraic Riccati equation:

$$A_d' P_d + P_d A_d - \beta P_d B_d B_d' P_d + \delta I = 0, \tag{11}$$

where β is the lower bound for all eigenvalues of the expanded Laplacian matrix $\bar{L}$ and $\delta \in (0,1]$ is a low-gain parameter which needs to be chosen sufficiently small. Finally $\bar{F}_1$ and $\bar{K}_1$ will be chosen later.

In order to prove our main result, we use the following two technical lemmas, whose proof is ommitted because of page limits.

Lemma 1: There exists $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, we have that

$$\begin{pmatrix} A_d & \lambda_i B_d \bar{F}_2 \\ \bar{K}_2 C_d & A_d + B_d \bar{F}_2 - \bar{K}_2 C_d \end{pmatrix} \tag{12}$$

is asymptotically stable for all λ_i with $\mathrm{Re}(\lambda_i) \geq \beta$.

Lemma 2: Assume (A, B, C) is stabilizable and detectable and all eigenvalues of A are in the closed left-half plane with all Jordan blocks associated with imaginary axis eigenvalues have size at most 2. There exists $\bar{F}, \bar{K}$ and $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, the matrix

$$\begin{pmatrix} A & \varepsilon \lambda^{-1} B \bar{F} \\ \bar{K} C & A - \bar{K} C + \varepsilon B \bar{F} \end{pmatrix} \tag{13}$$

is asymptotically stable for any $\lambda \in \mathbb{R}$ with $\lambda > \beta$.

If A is stable (all Jordan blocks associated with imaginary axis eigenvalues have size at most 1), then there exists $\bar{F}, \bar{K}$ and $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, the matrix (13) is asymptotically stable for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\, \lambda > \beta$.

We have the main result in the following theorem:

Theorem 1: Consider a multi-agent system (3), (1), and reference system (4), where all agents in the network are weakly-non-minimum-phase. Let Assumptions 1, 2 and 3 hold. Let $\beta > 0$ and root set π be given.

If all Jordan blocks associated with imaginary axis zeros have size at most 2 then the controller described by (10) solves the regulated output synchronization problem for suitably chosen $\bar{F}_1$ and $\bar{K}_1$.

In particular, there exists a $\delta^* \in (0,1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^* \in (0,1]$ such that for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} e_i(t) = 0 \quad (i = 1, \ldots, N).$$

for all initial conditions and for any graph $\mathcal{G} \in \mathbb{G}^{\pi}_{\beta,N}$.

Remark 4: In [10], we also studied weakly minimum-phase and weakly non-minimum-phase agents however for homogeneous networks. However that paper contained some technical issues that were corrected in this paper. Beyond that, this paper extends the previous work because it addresses heterogeneous networks.

Remark 5: If, additionally, the agents are weakly mimimum-phase (Jordan blocks associated with imaginary axis zeros have size at most 1) then the controller described by (10) for suitably chosen $\bar{F}_1$ and $\bar{K}_1$ also solves the regulated output synchronization problem when the network is directed.

Proof: Let $\bar{x}_i = x_i - \Pi_i x_r$. Then we have

$$\begin{cases} \dot{\bar{x}}_i = A_i x_i + B_i u_i - \Pi_i S x_r = A_i \bar{x}_i + B_i u_i, \\ e_i = C_i x_i - R x_r = C_i \bar{x}_i, \end{cases} \tag{14}$$

which has the the same dynamics as in (7). Hence, we get the same SCB decomposition form as in (8) with $\bar{x}_{ia} = [\bar{x}_{ia}^-; \bar{x}_{ia}^0; \bar{x}_{id}]$:

$$\begin{cases} \dot{\bar{x}}_{ia}^- = A_{ia}^- \bar{x}_{ia}^- + L_{iad}^- e_i, \\ \dot{\bar{x}}_{ia}^0 = A_a^0 \bar{x}_{ia}^0 + L_{ad}^0 e_i, \\ \dot{\bar{x}}_{id} = A_d \bar{x}_{id} + B_d(u_i + E_{ida}^- \bar{x}_{ia}^- + E_{da}^0 \bar{x}_{ia}^0 + E_{idd} \bar{x}_{id}), \\ e_i = C_d \bar{x}_{id}. \end{cases} \tag{15}$$

Let

$$\xi_{ia}^- = \bar{x}_{ia}^-, \ \xi_{ia}^0 = \bar{x}_{ia}^0, \ \hat{\xi}_{ia}^0 = \hat{x}_{ia}^0,$$
$$\xi_{id} = \varepsilon^{-1} S_\varepsilon \bar{x}_{id}, \ \hat{\xi}_{id} = \varepsilon^{-1} S_\varepsilon \hat{x}_{id}.$$

Then equations (15) and (10) can be written as

$$\begin{aligned} \dot{\xi}_{ia}^- &= A_{ia}^- \xi_{ia}^- + V_{iad}^{\varepsilon -} \xi_{id}, \\ \dot{\xi}_{ia}^0 &= A_a^0 \xi_{ia}^0 + V_{ad}^{\varepsilon 0} \xi_{id}, \\ \varepsilon \dot{\xi}_{id} &= A_d \xi_{id} + B_d \bar{F}_1 \hat{\xi}_{ia}^0 + B_d \bar{F}_2 \hat{\xi}_{id} + V_{ida}^{\varepsilon -} \xi_{ia}^- \\ &\quad + V_{da}^{\varepsilon 0} \xi_{ia}^0 + V_{idd}^{\varepsilon} \xi_{id}, \end{aligned}$$

where

$$\begin{aligned} V_{iad}^{\varepsilon -} &= \varepsilon L_{iad}^- C_d, & V_{ida}^{\varepsilon -} &= \varepsilon^{\rho - 1} B_d E_{ida}^-, \\ V_{ad}^{\varepsilon 0} &= \varepsilon L_{ad}^0 C_d, & V_{da}^{\varepsilon 0} &= \varepsilon^{\rho - 1} B_d E_{da}^0, \\ V_{idd}^{\varepsilon} &= \varepsilon^{\rho} B_d E_{idd} S_\varepsilon^{-1}. \end{aligned}$$

Moreover, since the Laplacian has a zero row sum, we have

$$\zeta_i = \sum_{j=1}^{N} \ell_{ij} y_j = \sum_{j=1}^{N} \ell_{ij} e_j \quad \text{and} \quad \zeta_i + \psi_i = \sum_{j=1}^{N} \bar{\ell}_{ij} e_j.$$

Then, equation (10) can be written as

$$\begin{aligned} \dot{\hat{\xi}}_{ia}^0 &= A_a^0 \hat{\xi}_{ia}^0 + V_{ad}^{\varepsilon 0} \hat{\xi}_{id} + \textstyle\sum_{j=1}^{N} \bar{\ell}_{ij} \varepsilon K_1 C_d \xi_{id} - \varepsilon K_1 C_d \hat{\xi}_{id}, \\ \varepsilon \dot{\hat{\xi}}_{id} &= A_d \hat{\xi}_{id} + B_d \bar{F}_1 \hat{\xi}_{ia}^0 + B_d \bar{F}_2 \hat{\xi}_{id} + V_{da}^{\varepsilon 0} \hat{\xi}_{ia}^0 + V_{idd}^{\varepsilon} \hat{\xi}_{id} \\ &\quad + \textstyle\sum_{j=1}^{N} \bar{\ell}_{ij} \bar{K}_2 C_d \xi_{id} - \bar{K}_2 C_d \hat{\xi}_{id}. \end{aligned}$$

Then, we define

$$\xi_a^- = \mathrm{col}\{\xi_{ia}^-\}, \ \xi_a^0 = \mathrm{col}\{\xi_{ia}^0\}, \ \hat{\xi}_a^0 = \mathrm{col}\{\hat{\xi}_{ia}^0\},$$
$$\xi_d = \mathrm{col}\{\xi_{id}\}, \ \hat{\xi}_d = \mathrm{col}\{\hat{\xi}_{id}\}.$$

The dynamics of the whole network system looks like

$$\begin{aligned} \dot{\xi}_a^- &= A_a^- \xi_a^- + V_{ad}^{\varepsilon -} \xi_d, \\ \dot{\xi}_a^0 &= (I_N \otimes A_a^0) \xi_a^0 + (I_N \otimes V_{ad}^{\varepsilon 0}) \xi_d, \\ \dot{\hat{\xi}}_a^0 &= (I_N \otimes A_a^0) \hat{\xi}_a^0 + (I_N \otimes V_{ad}^0) \hat{\xi}_d + \varepsilon (\bar{L} \otimes K_1 C_d) \xi_d \\ &\quad - \varepsilon (I_N \otimes K_1 C_d) \hat{\xi}_d, \\ \varepsilon \dot{\xi}_d &= (I_N \otimes A_d) \xi_d + (I_N \otimes B_d \bar{F}_1) \hat{\xi}_a^0 + (I_N \otimes B_d \bar{F}_2) \hat{\xi}_d \\ &\quad + (I_N \otimes V_{da}^{\varepsilon 0}) \xi_a^0 + V_{dd}^{\varepsilon} \xi_d + V_{da}^{\varepsilon -} \xi_a^-, \\ \varepsilon \dot{\hat{\xi}}_d &= (I_N \otimes A_d) \hat{\xi}_d + (I_N \otimes B_d \bar{F}_1) \hat{\xi}_a^0 + (I_N \otimes B_d \bar{F}_2) \hat{\xi}_d \\ &\quad + (I_N \otimes V_{da}^{\varepsilon 0}) \hat{\xi}_a^0 + V_{dd}^{\varepsilon} \hat{\xi}_d + (\bar{L} \otimes \bar{K}_2 C_d) \xi_d \\ &\quad - (I_N \otimes \bar{K}_2 C_d) \hat{\xi}_d, \end{aligned}$$

where

$$\begin{aligned} A_a^- &= \mathrm{blkdiag}\{A_{ia}^-\}, & V_{da}^{\varepsilon -} &= \varepsilon^{\rho - 1} \mathrm{blkdiag}\{B_d E_{ida}^-\}, \\ V_{ad}^{\varepsilon -} &= \varepsilon\, \mathrm{blkdiag}\{L_{iad}^- C_d\}, & V_{dd}^{\varepsilon} &= \varepsilon^{\rho}\, \mathrm{blkdiag}\{B_d E_{idd} S_\varepsilon^{-1}\}, \end{aligned}$$

Define $\bar{L} = UJU^{-1}$, where J is the Jordan form of $\bar{L}$. Clearly, the eigenvalues of $\bar{L}$, denoted by λ_i ($i = 1, \ldots, N$), are exactly the diagonal elements of J. Now we define

$$
\begin{aligned}
&v_a^- = \xi_a^-, && v_d = (JU^{-1} \otimes I_\rho)\xi_d,\\
&v_a^0 = (JU^{-1} \otimes I_r)\xi_a^0, && \tilde{v}_d = v_d - (U^{-1} \otimes I_\rho)\hat{\xi}_d.\\
&\tilde{v}_a^0 = v_a^0 - (JU^{-1} \otimes I_r)\hat{\xi}_a^0, &&
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\dot{v}_a^- &= A_a^- v_a^- + W_{ad}^{\varepsilon-} v_d,\\
\dot{v}_a^0 &= (I_N \otimes A_a^0) v_a^0 + W_{ad}^{\varepsilon 0} v_d,\\
\dot{\tilde{v}}_a^0 &= (I_N \otimes A_a^0)\tilde{v}_a^0 + W_{ad}^{\varepsilon 0} v_d - \hat{W}_{ad}^{\varepsilon 0}(v_d - \tilde{v}_d)\\
&\qquad - \varepsilon (J \otimes K_1 C_d)\tilde{v}_d,\\
\varepsilon \dot{v}_d &= (I_N \otimes A_d) v_d + (J \otimes B_d \bar{F}_2)(v_d - \tilde{v}_d)\\
&\quad + (I_N \otimes B_d \bar{F}_1)(v_a^0 - \tilde{v}_a^0) + W_{da}^{\varepsilon-} v_a^- + W_{da}^{\varepsilon 0} v_a^0 + W_{dd}^{\varepsilon} v_d,\\
\varepsilon \dot{\tilde{v}}_d &= (I_N \otimes (A_d - \bar{K}_2 C_d))\tilde{v}_d + (J \otimes B_d \bar{F}_2)(v_d - \tilde{v}_d)\\
&\quad + (I_N \otimes B_d \bar{F}_1)(v_a^0 - \tilde{v}_a^0) + W_{da}^{\varepsilon-} v_a^- + W_{da}^{\varepsilon 0} v_a^0\\
&\quad - \hat{W}_{da}^{\varepsilon 0}(v_a^0 - \tilde{v}_a^0) + W_{dd}^{\varepsilon} v_d - \hat{W}_{dd}^{\varepsilon}(v_d - \tilde{v}_d)\\
&\quad - (J^{-1} \otimes B_d \bar{F}_1)(v_a^0 - \tilde{v}_a^0) - (I_N \otimes B_d \bar{F}_2)(v_d - \tilde{v}_d),
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
W_{ad}^{\varepsilon-} &= V_{ad}^{\varepsilon-}(UJ^{-1} \otimes I_\rho),\\
W_{ad}^{\varepsilon 0} &= (JU^{-1} \otimes I_r)(I_N \otimes V_{ad}^{\varepsilon 0})(UJ^{-1} \otimes I_\rho)\\
&= \varepsilon (I_N \otimes L_{ad}^0 C_d),\\
\hat{W}_{ad}^{\varepsilon 0} &= (JU^{-1} \otimes I_r)(I_N \otimes V_{ad}^{\varepsilon 0})(U \otimes I_\rho) = \varepsilon (J \otimes L_{ad}^0 C_d),\\
W_{da}^{\varepsilon-} &= (JU^{-1} \otimes I_\rho) V_{da}^{\varepsilon-} = \varepsilon^{\rho-1}(JU^{-1} \otimes B_d)\operatorname{diag}(E_{ida}^-),\\
W_{da}^{\varepsilon 0} &= (JU^{-1} \otimes I_\rho)(I_N \otimes V_{da}^{\varepsilon 0})(UJ^{-1} \otimes I_r)\\
&= \varepsilon^{\rho-1}(I_N \otimes B_d E_{da}^0),\\
\hat{W}_{da}^{\varepsilon 0} &= (U^{-1} \otimes I_\rho)(I_N \otimes V_{da}^{\varepsilon 0})(UJ^{-1} \otimes I_r)\\
&= \varepsilon^{\rho-1}(J^{-1} \otimes B_d E_{da}^0),\\
W_{dd}^{\varepsilon} &= (JU^{-1} \otimes I_\rho) V_{dd}^{\varepsilon}(UJ^{-1} \otimes I_\rho)\\
&= \varepsilon^{\rho}(JU^{-1} \otimes B_d)\operatorname{diag}(E_{idd})(UJ^{-1} \otimes S_\varepsilon^{-1}),\\
\hat{W}_{dd}^{\varepsilon} &= (U^{-1} \otimes I_\rho) V_{dd}^{\varepsilon}(U \otimes I_\rho)\\
&= \varepsilon^{\rho}(U^{-1} \otimes B_d)\operatorname{diag}(E_{idd})(U \otimes S_\varepsilon^{-1}),
\end{aligned}
$$

In order to prove stability of this system we are going to use singular perturbations. We first note that we can ignore the stable dynamics for v_a^- since this part of the dynamics clearly does not affect the stability of the overall systems. Note that the stability of the fast dynamics is determined by the stability of the following matrix:

$$
\begin{pmatrix}
I_N \otimes A_d + J \otimes B_d F_2 & -J \otimes B_d \bar{F}_2\\
(J - I_N) \otimes B_d \bar{F}_2 & I_N \otimes (A_d - \bar{K}_2 C_d) - (J - I_N) \otimes B_d \bar{F}_2
\end{pmatrix}
+ \begin{pmatrix}
W_{dd}^{\varepsilon} & 0\\
\hat{W}_{dd}^{\varepsilon} - W_{dd}^{\varepsilon} & W_{dd}^{\varepsilon}
\end{pmatrix}
\tag{17}
$$

Since W_{dd}^{ε} and $\hat{W}_{dd}^{\varepsilon}$ are both of order ε, the stability of the first matrix is determined by the stability of the matrix:

$$
\begin{aligned}
&\begin{pmatrix}
A_d + \lambda_i B_d \bar{F}_2 & -\lambda_i B_d \bar{F}_2\\
(\lambda_i - 1) B_d \bar{F}_2 & A_d - \bar{K}_2 C_d - (\lambda_i - 1) B_d \bar{F}_2
\end{pmatrix}\\
&\quad = \begin{pmatrix} I & 0\\ I & -I \end{pmatrix}
\begin{pmatrix} A_d & \lambda_i B_d \bar{F}_2\\ \bar{K}_2 \bar{C}_d & A_d + B_d \bar{F}_2 - \bar{K}_2 C_d \end{pmatrix}
\begin{pmatrix} I & 0\\ I & -I \end{pmatrix}
\end{aligned}
$$

for $i = 1, \ldots, \lambda_N$. The stability of this matrix follows from Lemma 1 for suitably chosen δ. The stability of the matrix in (17) then follows for ε small enough.

Since the fast dynamics is asymptotically stable, it remains to show that the slow dynamics is asymptotically stable. Recall that we ignore the asymptotically stable dynamics for v_a^-. To obtain the slow dynamics we set $\dot{v}_d$ and $\dot{\tilde{v}}_d$ to zero and replace the differential equations for v_d and $\tilde{v}_d$ by the following algebraic equations:

$$
\begin{aligned}
0 &= (I_N \otimes A_d) v_d + W_{da}^{\varepsilon 0} v_a^0 + (J \otimes B_d \bar{F}_2)(v_d - \tilde{v}_d)\\
&\qquad + (I_N \otimes B_d \bar{F}_1)(v_a^0 - \tilde{v}_a^0) + W_{dd}^{\varepsilon} v_d,\\
0 &= (I_N \otimes (A_d - \bar{K}_2 C_d))\tilde{v}_d + W_{da}^{\varepsilon 0} v_a^0 - \hat{W}_{da}^{\varepsilon 0}(v_a^0 - \tilde{v}_a^0)\\
&\quad + ((J - I_N) \otimes B_d \bar{F}_2)(v_d - \tilde{v}_d)\\
&\quad + ((I_N - J^{-1}) \otimes B_d \bar{F}_1)(v_a^0 - \tilde{v}_a^0) + W_{dd}^{\varepsilon} v_d - \hat{W}_{dd}^{\varepsilon}(v_d - \tilde{v}_d)
\end{aligned}
\tag{18}
$$

Let $v_d = \begin{pmatrix} v_{1d}, v_{2d}, \ldots, v_{Nd} \end{pmatrix}'$ and let v_{idj} ($j = 1, \ldots, \rho$) denote the j^{th} element of v_{id}. We decompose $\tilde{v}_d$ in the same way.

We find that

$$
A_d' A_d = \begin{pmatrix} 0 & 0\\ 0 & I \end{pmatrix}, \quad A_d' B_d = 0.
$$

Then, by multiplying (18) by $(I_N \otimes A_d')$ on the left we get

$$
\begin{pmatrix} v_{id2}\\ \vdots\\ v_{id\rho} \end{pmatrix} = 0, \quad \bar{K}_{21}\tilde{v}_{id1} = \begin{pmatrix} \tilde{v}_{id2}\\ \vdots\\ \tilde{v}_{id\rho} \end{pmatrix}, \quad (i = 1, \ldots, N)
$$

or

$$
v_{id} = C_d' v_{id1}, \quad \tilde{v}_{id} = \tilde{K}_2 \tilde{v}_{id1}
$$

where

$$
\bar{K}_2 = \begin{pmatrix} \bar{K}_{21}\\ \bar{K}_{22} \end{pmatrix}, \qquad \tilde{K}_2 = \begin{pmatrix} 1\\ \bar{K}_{21} \end{pmatrix},
$$

and $\bar{K}_{22}$ is a scalar. Similarly, we decompose $\bar{F}_2 = \begin{pmatrix} \bar{F}_{21} & \bar{F}_{22} \end{pmatrix}$ and $\bar{F}_{21}$ is a scalar. Multiplying (18) by $I_N \otimes B_d'$ on the left and using the structure of v_{id} and $\tilde{v}_{id}$ found above we get:

$$
\begin{aligned}
0 &= (J \otimes \bar{F}_{21}) v_{d1} - (J \otimes \bar{F}_2 \tilde{K}_2)\tilde{v}_{d1} + (I_N \otimes \bar{F}_1)(v_a^0 - \tilde{v}_a^0)\\
&\quad + \varepsilon^{\rho-1}(I_N \otimes E_{da}^0) v_a^0\\
&\quad + \varepsilon^{\rho}(JU^{-1} \otimes I)\operatorname{diag}(E_{idd})(UJ^{-1} \otimes C_d') v_{d1},
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= -(I_N \otimes \bar{K}_{22})\tilde{v}_{d1} + \varepsilon^{\rho-1}((I_N - J^{-1}) \otimes E_{da}^0) v_a^0\\
&\quad + \varepsilon^{\rho-1}(J^{-1} \otimes E_{da}^0)\tilde{v}_a^0\\
&\quad + \varepsilon^{\rho}(JU^{-1} \otimes I)\operatorname{diag}(E_{idd})(UJ^{-1} \otimes C_d') v_{d1}\\
&\quad - \varepsilon^{\rho}(U^{-1} \otimes I)\operatorname{diag}(E_{idd})(U \otimes C_d') v_{d1}\\
&\quad + \varepsilon^{\rho}(U^{-1} \otimes I)\operatorname{diag}(E_{idd})(U \otimes S_\varepsilon^{-1}\tilde{K}_2)\tilde{v}_{d1}\\
&\quad + ((J - I_N) \otimes \bar{F}_{21}) v_{d1} - ((J - I_N) \otimes \bar{F}_2 \tilde{K}_2)\tilde{v}_{d1}\\
&\quad + ((I_N - J^{-1}) \otimes \bar{F}_1)(v_a^0 - \tilde{v}_a^0)
\end{aligned}
$$

where $v_{d1} = \operatorname{vec}\{v_{id1}\}$ and $\tilde{v}_{d1} = \operatorname{vec}\{\tilde{v}_{id1}\}$. Subtracting $(I_N - J^{-1}) \otimes I$ times the first equation from the second equation yields:

$$\begin{aligned}0 = &-(I_N \otimes \bar{K}_{22})\tilde{v}_{d1} + \varepsilon^{\rho-1}(J^{-1} \otimes E_{da}^0)\tilde{v}_a^0 \\ &+ \varepsilon^{\rho}(U^{-1} \otimes I)\,\mathrm{diag}(E_{idd})(U(J^{-1} - I_N) \otimes C_d')v_{d1} \\ &+ \varepsilon^{\rho}(U^{-1} \otimes I)\,\mathrm{diag}(E_{idd})(U \otimes S_\varepsilon^{-1}\tilde{K}_2)\tilde{v}_{d1}\end{aligned}$$

We obtain:

$$(X_1 + X_2 + X_3)\begin{pmatrix} v_{d1} \\ \tilde{v}_{d1}\end{pmatrix} = Y\begin{pmatrix} v_a^0 \\ \tilde{v}_a^0\end{pmatrix}$$

where

$$\begin{aligned}X_1 &= \begin{pmatrix}(J \otimes \bar{F}_{21}) & -(J \otimes \bar{F}_2\tilde{K}_2) \\ 0 & -(I_N \otimes \bar{K}_{22})\end{pmatrix} \\ X_2 &= \varepsilon^{\rho}\,\mathrm{blkdiag}\Big((JU^{-1} \otimes I)\,\mathrm{diag}(E_{idd})(UJ^{-1} \otimes C_d'), \\ &\quad (U^{-1} \otimes I)\,\mathrm{diag}(E_{idd})(U \otimes S_\varepsilon^{-1}\tilde{K}_2)\Big) \\ X_3 &= \varepsilon^{\rho}\begin{pmatrix} 0 & 0 \\ (U^{-1} \otimes I)\,\mathrm{diag}(E_{idd})(U(J^{-1} - I_N) \otimes C_d') & 0\end{pmatrix} \\ Y &= \begin{pmatrix}(I_N \otimes \bar{F}_1) + \varepsilon^{\rho-1}(I_N \otimes E_{da}^0) & -(I_N \otimes \bar{F}_1) \\ 0 & \varepsilon^{\rho-1}(J^{-1} \otimes E_{da}^0)\end{pmatrix}\end{aligned}$$

Noting that X_1 is independent of ε and invertible while X_2 is at least of order ε while X_3 is of order ε^ρ we find that:

$$\begin{aligned}(X_1+X_2+X_3)^{-1} = &\begin{pmatrix}(J^{-1} \otimes \bar{F}_{21}^{-1}) & -(I_N \otimes \bar{F}_{21}^{-1}\bar{F}_2\tilde{K}_2\bar{K}_{22}^{-1}) \\ 0 & -(I_N \otimes \bar{K}_{22}^{-1})\end{pmatrix} \\ &+ \begin{pmatrix}\varepsilon\tilde{X}_{11}(\varepsilon) & \varepsilon\tilde{X}_{12}(\varepsilon) \\ \varepsilon^{\rho}\tilde{X}_{21}(\varepsilon) & \varepsilon\tilde{X}_{22}(\varepsilon)\end{pmatrix}\end{aligned}$$

with $\tilde{X}_{11}(\varepsilon)$, $\tilde{X}_{12}(\varepsilon)$, $\tilde{X}_{21}(\varepsilon)$ and $\tilde{X}_{22}(\varepsilon)$ bounded functions of ε. Here we exploit that X_1 and X_2 are upper triangular and hence $(X_1 + X_2)^{-1}$ has an upper triangular structure. Finally, we noted that $(X_1 + X_2 + X_3)^{-1}$ is an ε^ρ perturbation of an upper triangular structure. Their approximate solutions are

$$\begin{aligned}v_{d1} &= -(J^{-1} \times \bar{F}_1\bar{F}_{21}^{-1})(v_a^0 - \tilde{v}_a^0) + O(\varepsilon) \\ \tilde{v}_{d1} &= \varepsilon^{\rho-1}(J^{-1} \otimes \bar{K}_{22}^{-1}E_{da}^0)\tilde{v}_a^0 + O(\varepsilon^{\rho})\end{aligned} \tag{19}$$

Furthermore, (16) implies that

$$\begin{aligned}\dot{v}_a^0 &= (I_N \otimes A_a^0)v_a^0 + \varepsilon(I_N \otimes L_{ad}^0)v_{d1} \\ \dot{\tilde{v}}_a^0 &= (I_N \otimes A_a^0)\tilde{v}_a^0 + \varepsilon((I_N - J) \otimes L_{ad}^0)v_{d1} \\ &\quad + \left[\varepsilon(J \otimes L_{ad}^0) - \varepsilon^{-\rho+1}(J \otimes \bar{K}_1)\right]\tilde{v}_{d1}\end{aligned} \tag{20}$$

Using

$$\tilde{K}_1 = \frac{\bar{K}_1}{\bar{K}_{22}}, \quad \text{and} \quad \tilde{F}_1 = -\frac{\bar{F}_1}{\bar{F}_{21}}.$$

and ignoring higher order terms we obtain:

$$\begin{aligned}\dot{v}_a^0 &= \left[(I_N \otimes A_a^0) + \varepsilon(J^{-1} \otimes L_{ad}^0\tilde{F}_1)\right]v_a^0 - \varepsilon(J^{-1} \otimes L_{ad}^0\tilde{F}_1)\tilde{v}_a^0, \\ \dot{\tilde{v}}_a^0 &= -\varepsilon\left[(I_N - J^{-1}) \otimes L_{ad}^0\tilde{F}_1\right]v_a^0 + \Big[(I_N \otimes (A_a^0 - \tilde{K}_1E_{da}^0)) \\ &\quad +\varepsilon((I_N - J^{-1}) \otimes L_{ad}^0\tilde{F}_1)\Big]v_a^0.\end{aligned}$$

Define

$$\check{v}_a^0 = \begin{pmatrix} I & 0 \\ I & -I\end{pmatrix}\begin{pmatrix} v_a^0 \\ \tilde{v}_a^0\end{pmatrix}$$

and we get:

$$\dot{\check{v}}_a^0 = \begin{pmatrix} I_N \otimes A_a^0 & \varepsilon(J^{-1} \otimes L_{ad}^0\tilde{F}_1) \\ I_N \otimes \tilde{K}_1E_{da}^0 & I_N \otimes (A_a^0 - \tilde{K}_1E_{da}^0 + \varepsilon L_{ad}^0\tilde{F}_1)\end{pmatrix}\check{v}_a^0$$

which clearly is stable if:

$$\begin{pmatrix} A_a^0 & \varepsilon\lambda_i^{-1}L_{ad}^0\tilde{F}_1 \\ \tilde{K}_1E_{da}^0 & A_a^0 - \tilde{K}_1E_{da}^0 + \varepsilon L_{ad}^0\tilde{F}_1\end{pmatrix}$$

is asymptotically stable for all $i = 1, \ldots, N$. Since λ_i^{-1} are bounded, we can use Lemma 2 to design $\tilde{F}_1$ and $\tilde{K}_1$ such that this matrix is asymptotically stable.

Since both the slow and fast dynamics are asymptotically stable for ε small enough, singular perturbations guarantees that the closed-loop system is asymptotically stable for small enough ε. ■

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