Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

Quadratic performance of generalized first-order systems

Robert van der Geest
University of Twente
Faculty of Applied Mathematics
P.O.Box 217, 7500 AE Enschede, Netherlands
Email: vdgeest@math.utwente.nl

Anders Rantzer
Lund Institute of Technology
Department of Automatic Control
P.O.Box 118, S 22100 Lund, Sweden
Email: rantzer@control.lth.se

Abstract – In this note we formulate the Kalman-Yakubovič-Popov Lemma for generalized first-order systems, both in continuous- and discrete-time.

1 Introduction

The Kalman-Yakubovič-Popov (KYP) Lemma is a primary tool for the analysis of linear systems in state-space description. It provides a link between quadratic performance questions and the existence of a solution to a Linear Matrix Inequality (LMI). A demonstration of this connection, and some background about the KYP Lemma may be found in Willems [4].

In this note we formulate the KYP Lemma for continuous-time, *generalized first-order systems* of the form

$$G\dot{w} = Fw, \tag{1}$$

where $w \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^q)$ are the variables associated with the system, and G and F are real-valued, p by q matrices. Such a description allows for specification of a number of *algebraic constraints*, i.e., constraints of the type

$$Hw = 0, \tag{2}$$

where H is a real-valued matrix. In this respect (1) is a generalization of a state-space description, which consists of *dynamic* restrictions only. Note also that contrary to what happens in state-space theory, we do not a priori split up the variables w into inputs and outputs. An introduction to different kinds of first-order models, and some motivation for studying them may be found in Kuijper [1].

The quadratic performance criterion that we are interested in has the form

$$\int_{-\infty}^{\infty} w^T(t)Mw(t)dt \leq 0, \tag{3}$$

where M is a symmetric, real-valued, q by q matrix. It is instructive to think of the integral in (3) as the *energy* enclosed in the signal w. It turns out that a controllable system (1), without algebraic constraints, satisfies (3) if and only if there exists a symmetric solution P to the LMI

$$M + F^TPG + G^TPF \leq 0. \tag{4}$$

Analogously, it turns out that the discrete-time system

$$Gw(t+1) = Fw(t) \tag{5}$$

satisfies the performance criterion

$$\sum_{-\infty}^{\infty} w^T(t)Mw(t) \leq 0 \tag{6}$$

if and only if there exists a symmetric solution P to the discrete-time LMI

$$M + F^TPF - G^TPG \leq 0. \tag{7}$$

When the system description (1) or (5) includes algebraic constraints, the behaviour of the system is restricted to a linear subspace, and we show how the quadratic performance problem may be reduced to an equivalent problem on a subspace.

2 Quadratic performance

Before we formulate the main results, we first characterize *controllability* of a system in kernel representation (Willems [5]).

Lemma 2.1 *The system $R\left(\frac{d}{dt}\right)w = 0$ is controllable if and only if the rank of $R(\lambda)$ is constant for all $\lambda \in \mathbb{C}$.*

The KYP Lemma for continuous-time, generalized first-order systems is formulated as follows.

Theorem 2.2 *Assume that the system $G\dot{w} = Fw$ is controllable, and that the matrix G has full row-rank. Then the following two statements are equivalent:*

- *For all $w \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^q)$ such that $G\dot{w} = Fw$,*

$$\int_{-\infty}^{\infty} w^T(t)Mw(t)dt \leq 0. \tag{8}$$

- *There exists a symmetric solution P to the LMI*

$$M + F^TPG + G^TPF \leq 0. \tag{9}$$

Proof: It is possible to prove the result directly along the lines of the proof in Rantzer [2]. Here we convert the problem into state-space form instead. By Parseval's Theorem, (8) is equivalent to

$$\forall w \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^q) \text{ s.t. } G\dot{w} = Fw:$$

$$\int_{-\infty}^{\infty} \hat{w}^T(-i\omega)M\hat{w}(i\omega)d\omega \leq 0. \tag{10}$$

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By a continuity argument, (10) is equivalent to

$$\forall \omega \in \mathbb{R} : \forall v \in \mathbb{C}^q \text{ s.t. } (i\omega G - F)v = 0 : \quad v^* M v \leq 0. \quad (11)$$

Since G has full row-rank, there exist invertible matrices U and V such that $UGV = \begin{pmatrix} I & 0 \end{pmatrix}$. Define

$$UFV =: \begin{pmatrix} A & B \end{pmatrix}, \text{ and } V^{-1}v =: \begin{pmatrix} x \\ u \end{pmatrix}. \quad (12)$$

Then (A, B) is a controllable pair, and (11) is equivalent to

$$\forall \omega \in \mathbb{R} : \forall \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{C}^q \text{ s.t. } i\omega x = Ax + Bu : \quad \begin{pmatrix} x \\ u \end{pmatrix}^* V^T M V \begin{pmatrix} x \\ u \end{pmatrix} \leq 0. \quad (13)$$

The KYP Lemma for continuous-time systems in state-space form may be found in Yakubovich [7]. By this Lemma, (13) is equivalent to

$$\exists Q = Q^T s.t. \quad V^T M V + \begin{pmatrix} A^T Q + QA & QB \\ B^T Q & 0 \end{pmatrix} \leq 0. \quad (14)$$

Take $P = U^T Q U$. Then (14) is equivalent to (9). □

Remark: Note the similarity between Theorem 2.2 and the so-called Projection Lemma, see e.g. Scherer [3].

The discrete-time counterpart of Theorem 2.2 is the following.

Theorem 2.3 *Assume that the system $Gw(t+1) = Fw(t)$ is controllable, and that the matrix G has full row-rank. Then the following two statements are equivalent:*

- *For all $w \in \ell_2(\mathbb{R}, \mathbb{R}^q)$ such that $Gw(t+1) = Fw(t)$,*

$$\sum_{-\infty}^{\infty} w^T(t) M w(t) \leq 0. \quad (15)$$

- *There exists a symmetric solution P to the LMI*

$$M + F^T P F - G^T P G \leq 0. \quad (16)$$

Proof: The proof is analogous to that of Theorem 2.2, using the KYP Lemma for discrete-time systems in state-space form. □

3 Algebraic constraints

The condition that G has full row-rank is equivalent to excluding algebraic constraints on the system (1). If the system description *does* include algebraic constraints, the behaviour of (1) is restricted to live on a linear subspace of $\mathbb{R}^q$, and Theorem 2.2 should be adjusted accordingly.

Theorem 3.1 *Say that the description $G\dot{w} = Fw$ includes algebraic constraints restricting the behaviour to* image$(W) \subseteq \mathbb{R}^q$. *Assume that the system is controllable on* image(W). *Then the following two statements are equivalent:*

- *For all $w \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^q)$ such that $G\dot{w} = Fw$,*

$$\int_{-\infty}^{\infty} w^T(t) M w(t) dt \leq 0. \quad (17)$$

- *There exists a symmetric solution P to the LMI*

$$W^T(M + F^T P G + G^T P F)W \leq 0. \quad (18)$$

3.1 Reduction procedure

The following procedure may be used to detect any algebraic constraints in the description $G\dot{w} = Fw$. Assume that the matrix G does not have full row-rank. Then there exists an invertible matrix U such that

$$UG = \begin{pmatrix} \tilde{G} \\ 0 \end{pmatrix}, \text{ and } UF = \begin{pmatrix} \tilde{F} \\ H \end{pmatrix}, \quad (19)$$

where $\tilde{G}$ has full row-rank. The behaviour of the system is equivalently described as

$$\tilde{G}\dot{w} = \tilde{F}w, \text{ and } Hw = 0. \quad (20)$$

The algebraic constraints $Hw = 0$ may be rewritten in image representation as $w = \text{image}\left(H^\perp\right)^T$.

Remark: In Theorem 2.2, the quadratic function

$$f(w) := w^T G^T P G w \quad (21)$$

is a storage function in the sense of dissipative systems theory. It is possible to prove the equivalence in Theorem 2.2 by finding this storage function using the results in a forthcoming paper by Willems and Trentelman[6].

References

[1] M. Kuijper, *First-order representations of linear systems*, Birkhäuser, 1994.

[2] A. Rantzer, 'On the Kalman-Yakubovich-Popov lemma,' *Systems & Control Letters*, Vol.28, 1996, pp.7-10.

[3] C.W. Scherer, 'Mixed H_2/H_∞ control,' in: A. Isidori (ed.), *Trends in control: a European perspective*, Springer-Verlag, 1995.

[4] J.C. Willems, 'Least squares stationary optimal control and the algebraic Riccati equation,' *IEEE T-AC*, Vol.16, No.6, Dec.1971, pp.621-634.

[5] J.C. Willems, 'Paradigms and puzzles in the theory of dynamical systems,' *IEEE T-AC*, Vol.36, 1991, pp.259-294.

[6] J.C. Willems and H.L. Trentelman, 'On quadratic differential forms,' preprint.

[7] V.A. Yakubovich, 'A frequency theorem in control theory,' translated from *Sibirskii Matematicheskii Zhurnal*, Vol.14, No.2, 1973, pp.384-419.