Sequential Monte Carlo simulation for the estimation of small reachability probabilities for stochastic hybrid systems

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Abstract— **The problem of estimating the probability that a system reaches a given set within some time horizon is considered. Standard Monte Carlo methods for reachability probability estimation do not require specific assumptions on the system under consideration. However, they are computationally demanding when the probability to be estimated is small. An Interacting Particle System (IPS) approach has been developed for the estimation of small reachability probabilities for diffusion processes. IPS has then been extended so as to estimate small reachability probabilities for a certain family of stochastic hybrid processes, namely switching diffusions. This contribution further improves the hybrid IPS method by adopting an importance sampling approach that uses the interaction equations characterizing stochastic hybrid systems.**

I. INTRODUCTION

In this paper we consider the problem of estimating the probability that a system reaches a given set within some time horizon. This type of problem is of great interest for the safe design of complex safety-critical operations, such as in nuclear and chemical industries, and advanced air traffic management. The advantage of Monte Carlo (MC) methods for reachability probability estimation is that they do not require specific assumptions on the system under consideration. However, obtaining accurate estimates of rare event probabilities, say about 10^{-9}, requires to run about 10^{11} simulations, which is very time consuming.

The IPS approach developed in [1], [2] forms a sequential MC algorithm for estimating small reachability probabilities of strong Markov processes. The key idea behind the IPS approach is to express the small probability to be estimated as the product of a certain number of larger probabilities, which can be efficiently estimated by the Monte Carlo approach. This can be achieved by introducing sets of intermediate states that are visited one set after the other, in an ordered sequence, before reaching the final set of states of interest. The reachability probability of interest is then given by the product of the conditional probabilities of reaching a set of intermediate states given that the previous set of intermediate states have been reached. Each conditional probability is estimated by simulating in parallel several copies of the system, i.e. each copy is considered as a particle following the trajectory generated through the system dynamics. In [3] it was shown that the IPS approach works very well for a diffusion example, but fails when applied to a switching diffusion with large differences in mode probabilities or with rare switchings. Hybrid interacting particle system (HIPS) algorithm [3], [4] was specially developed to cope with this problems.

In this paper we continue to improve and develop efficient rare event MC simulation technique for stochastic hybrid systems the solutions of which are rarely switching diffusions with state dependent rates of switching between discrete system modes [5], [6]. We propose a new interacting particle system approach. The novelty over HIPS [3], [4] is that this time the probabilities of the discrete modes are estimated using analytical equations rather than simulating random mode switchings. Originally this approach was developed in [7] for filtering problems in hybrid system. Here we combine this with IPS for rare event probability estimation.

The new rare event probability estimation approach is based on the same idea as HIPS approach of decomposition of rare event probability into a product of "less rare" conditional probabilities and estimation of these conditional probabilities by particle systems. The difference lies in the following: only mode conditional $\mathbb{R}^n$ -valued particles are used, and the probabilities that the system is operating in one of modes are evaluated analytically. In the HIPS algorithm the time index value of the particles may differ, this is because some particles reach the level sets and stop earlier, some later, and some not at all. In our novel particle system algorithm the mode probabilities are estimated analytically, but this requires all particles to have the same time index value. In order to achieve this, we transform the original SDE, which describes the evolution of switching diffusion, into particular "equivalent" SDE, the rare event of interest is transformed correspondingly, but the probability of this "transformed" rare event is the same as of the "original" one. The novel IPS algorithm is aimed to cope with rarely switching diffusions and with large differences in mode probabilities.

The paper is organized as follows. Section II states the problem studied. The transformation of SDE and rare event is done in section III. Section IV briefly explains the theory interacting particle system based decomposition of rare event probability in case of a switching diffusion. The novel IPS algorithm is presented in section V.

II. PROBLEM CONSIDERED

Throughout this paper all stochastic processes are defined within the setting of one complete stochastic basis.

Let $\{x_t, \theta_t\}$ be a switching diffusion taking its values in

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$\mathbb{R}^n \times \mathbb{M}$ according to

$$dx_t = a(\theta_t, x_t)dt + b(\theta_t, x_t)dW_t, \tag{1}$$

$$p_{\theta_{t+\delta}|\theta_t, x_t}(\theta|\eta, x) = \lambda_{\eta\theta}(x)\delta + o(\delta),\ \eta \neq \theta, \tag{2}$$

where $\mathbb{M}$ is a finite set of modes and $(W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$ independent of $\{\theta_t\}$. We assume that switching diffusion (1,2) starts at $t = 0$ in a Borel set $\bar{D}_0 \subset \mathbb{R}^n \times \mathbb{M}$ with a known initial probability distribution $P_{x_0,\theta_0}(\cdot)$. We set $\tau_{\bar{D}} \stackrel{\triangle}{=} \inf\{t > 0 : (x_t, \theta_t) \in D \times \mathbb{M}\}$, i.e. the first passage time of $\{x_t\}$ to a closed connected Borel set D. The problem addressed in this paper is to estimate the probability $P(\tau_{\bar{D}} < T)$, i.e. the probability that $\{x_t\}$ will hit the set D on the time interval $(0, T]$, $T < \infty$. Similarly as in [2] and in [4] we will represent the probability of a rare event as a product of probabilities of intermediate "less rare" events and then use our novel particle system approach to estimate this product.

Let us define the nested sequence of sets $\bar{D} = \bar{D}_m \subset \cdots \subset \bar{D}_1$ as follows:

$$\bar{D}_k \stackrel{\triangle}{=} D_k \times \mathbb{M},\ \ k = 1, \ldots, m \tag{3}$$

where D_k is a closed Borel set of $\mathbb{R}^n$, and $\bar{D}_1$ is such that $\bar{D}_1 \cap \bar{D}_0 = \varnothing$. These nested sets aim to play an important role in decomposition of rare event probability, this will become more clear in the following sections.

The first moment in time that $\{x_t, \theta_t\}$ hits a set $\bar{D}_k$ is defined as the stopping time:

$$\tau_k \stackrel{\triangle}{=} \inf\{t \geq 0 : (x_t, \theta_t) \in \bar{D}_k\} = \inf\{t \geq 0 : x_t \in D_k\},$$

$\tau_k = \infty$ if this set is empty. The quantity of our interest is the probability of first passage time into set $\bar{D} \subset \mathbb{R}^n \times \mathbb{M}$ before time T, i.e. $P(\tau_{\bar{D}} < T) = P(\tau_m < T)$.

III. Transformation

We want to transform the process so that during simulation at each discretization step all particles will have the same time index value. This is needed in order to be able to calculate mode probabilities analytically at each simulation step. The transformation is done by properly modifying the coefficients of the SDE (1,2). It is important to note that the transformation we are going to apply depends on the sequence of nested subsets (3).

We transform the process $\{x_t, \theta_t\}$ on $[0, T]$ to a new $\mathbb{R}^{n+2} \times \mathbb{M}$-valued process $\{\tilde{x}_t, \tilde{\theta}_t\}$ on $[0, T_m]$ as the solution of the following SDE (transformation of (1,2)):

$$d\tilde{x}_t = \tilde{a}(\tilde{x}_t, \tilde{\theta}_t)dt + \tilde{b}(\tilde{x}_t, \tilde{\theta}_t)d\tilde{W}_t, \tag{4}$$

$$p_{\tilde{\theta}_{t+\delta}|\tilde{\theta}_t, \tilde{x}_t}(\theta|\eta, x) = \tilde{\lambda}_{\eta\theta}(x)\delta + o(\delta),\ \eta \neq \theta, \tag{5}$$

where $t \in [0, T_m]$, $T_k = k \cdot T$, $k = 0, 1, \ldots, m$. Roughly speaking, the stretching of $\{x_t, \theta_t\}$ on $[0, T]$ to $\{\tilde{x}_t, \tilde{\theta}_t\}$ on $[0, T_m]$ is done as follows. On the first time interval $[0, T_1] = [0, T]$ of $[0, T_m]$ $\{\tilde{x}_t, \tilde{\theta}_t\}$ equals $\{x_t, \theta_t\}$ until the moment that $\{\tilde{x}_t, \tilde{\theta}_t\}$ hits the set $\bar{D}_1$. From that moment on $\{\tilde{x}_t, \tilde{\theta}_t\}$ is frozen. On the second interval $[T_1, T_2]$ $\{\tilde{x}_t, \tilde{\theta}_t\}$ continues to evolve according to $\{x_t, \theta_t\}$ from the moment that it hit the set $\bar{D}_1$. This continues until $\{\tilde{x}_t, \tilde{\theta}_t\}$ hits the second set $\bar{D}_2$; then it is frozen. This repeats m times until $\{\tilde{x}_t, \tilde{\theta}_t\}$ hits the m-th set $\bar{D}_m$ or until $\tilde{x}_{t,1} = T$. For this the coefficients, switching rates matrix and the driving process $\{\tilde{W}_t\}$ are defined as follows:

$$\tilde{a} : \mathbb{R}^{n+2} \times \mathbb{M} \longrightarrow \mathbb{R}^{n+2}$$

$$\tilde{b} : \mathbb{R}^{n+2} \times \mathbb{M} \longrightarrow \mathbb{R}^{(n+2)\times(n+2)}$$

and

for $x_1 \in [0, T)$, $x_2 \in [T_{k-1}, T_k)$, $x_{3:n+2} \in (D_{k-1} \setminus D_k)$, $k = 1, \ldots, m$.

$$\tilde{a}(x, \theta) = [1\ 1\ a(x_{3:n+2}, \theta)]^T$$

$$\tilde{b}(x, \theta) = \begin{bmatrix} O^{2\times 2} & O^{2\times n} \\ O^{n\times 2} & b(x_{3:n+2}, \theta) \end{bmatrix}$$

$$\tilde{\Lambda}(x) = \left(\tilde{\lambda}_{ij}(x)\right)_{i,j=1}^N = \left(\lambda_{ij}(x_{3:n+2})\right)_{i,j=1}^N$$

else

$$\tilde{a}(x, \theta) = [0\,1\,0 \ldots 0]^T$$

$$\tilde{b}(x, \theta) = \begin{bmatrix} O^{2\times 2} & O^{2\times n} \\ O^{n\times 2} & O^{n\times n} \end{bmatrix}$$

$$\tilde{\Lambda}(x) = \left(\tilde{\lambda}_{ij}(x)\right)_{i,j=1}^N = O^{N\times N}$$

$O^{n\times m}$ denotes a zero $n \times m$-matrix.

The Brownian motion driving the transformed process is defined as follows.

$$\tilde{W}_t = \begin{cases} W_{\tilde{x}_{t,1}} & \text{for } \tilde{x}_{t,1} \in [0, T), \\ & \tilde{x}_{t,2} \in [T_{k-1}, T_k), \text{ and} \\ & \tilde{x}_{t,3:n+2} \in (D_{k-1} \setminus D_k), \\ & k = 1, \ldots, m. \\ \hat{W}_t & \text{else} \end{cases}$$

where $\{\hat{W}_t\}$ is a standard Brownian motion independent of $\{W_t\}$. The first component $\tilde{x}_{t,1}$ is the time index of original process. The second component $\tilde{x}_{t,2}$ is basically the time index of transformed process. The remaining n components $\tilde{x}_{t,3:n+2}$ represent the state.

While $\tilde{x}_{t,3:n+2} \in (D_{k-1} \setminus D_k)$ and $\tilde{x}_{t,1} \in [0, T)$, $\tilde{x}_{t,2} \in [T_{k-1}, T_k)$, $k = 1, \ldots, m$, components $\{\tilde{x}_{t,3:n+2}, \tilde{\theta}_t\}$ follow the same dynamics as original process (1,2). If $\tilde{x}_{t,1} < T$, $\tilde{x}_{t,2} < T_k$ and $\tilde{x}_{t,3:n+2}$ has already reached the set D_k then evolution of components $\{\tilde{x}_{t,3:n+2}, \tilde{\theta}_t\}$ stops and they remain constant until $\tilde{x}_{t,2} = T_k$ $(k = 1, \ldots, m)$. If $\tilde{x}_{t,1} = T$ but $\tilde{x}_{t,3:n+2}$ has not yet reached the target set D then the process stops.

Define

$$\tilde{\tau}_k = \begin{cases} T_k & \text{if } \tilde{x}_{T_k,3:n+2} \in D_k \\ \infty & \text{otherwise} \end{cases} \tag{6}$$

Then, we have

$$P(\tilde{\tau}_k = T_k) = P(\tilde{x}_{T_k,3:n+2} \in D_k) \tag{7}$$

$$= P(\tilde{x}_{T_k} \in [0, T) \times [0, T_m) \times D_k) \tag{8}$$

$$= P(x_{\tau_k \wedge T} \in D_k) = P(\tau_k < T)$$

and similarly

$$P(\tilde{\tau}_k = T_k | \tilde{\tau}_{k-1} = T_{k-1}) = P(\tau_k < T | \tau_{k-1} < T). \tag{9}$$

IV. Product of conditional probabilities

Suppose the switching diffusion $\{\tilde{x}_t, \tilde{\theta}_t\}$ starts at $t=0$ in a Borel set $\{0\}\times\{0\}\times\bar{D}_0 \subset \mathbb{R}^{n+2}\times\mathbb{M}$ with a known initial probability distribution $P_{\tilde{x}_0,\tilde{\theta}_0}(\cdot)$. We assume a sequence of nested Borel sets, $\bar{D}=\bar{D}_m \subset \cdots \subset \bar{D}_1$ which are defined by (3). We want to estimate $P(\tilde{\tau}_m = T_m)$, for $T_m = T\cdot m < \infty$, what is equivalent to the probability that the "original" switching diffusion $\{x_t, \theta_t\}$ will hit the rare event set $\bar{D}$ before time T.

Let us introduce the $\{0,1\}$-valued variables $\{y_k, k=1,\ldots,m\}$ defined as follows:

$$y_k(\omega) \stackrel{\triangle}{=} \mathbf{1}_{\{\omega:\tilde{\tau}_k(\omega)=T_k\}} = \mathbf{1}_{\{\omega:\tau_k(\omega)\le T\}}. \tag{10}$$

Hence, for each k we have

$$y_k(\omega) = \mathbf{1}_{\{\omega:\tilde{\tau}_k(\omega)=T_k\}} = \prod_{i=1}^{k}\mathbf{1}_{\{\omega:\tilde{\tau}_i(\omega)=T_i\}} = \prod_{i=1}^{k} y_i(\omega). \tag{11}$$

Next we characterize $P(\tau_m < T) = P(\tilde{\tau}_m = T_m)$ in terms of the sequence $\{y_k\}$. By its definition,

$$P(\tilde{\tau}_m = T_m) = \mathbb{E}[\mathbf{1}_{\{\tilde{\tau}_m=T_m\}}]$$

Subsequent substitution of (10) and (11) yields:

$$P(\tau_m < T) = \mathbb{E}[y_m] = \mathbb{E}[\prod_{k=1}^{m} y_k]. \tag{12}$$

Since y_k assumes values from $\{0,1\}$,

$$\mathbb{E}[\prod_{k=1}^{m} y_k] = \prod_{k=1}^{m}\mathbb{E}[y_k|y_{k-1}=1,\ldots,y_1=1]$$

Substituting this into (12) yields

$$\begin{aligned} P(\tilde{\tau}_m < T) &= \prod_{k=1}^{m}\mathbb{E}[y_k|y_{k-1}=1,\ldots,y_1=1] \\ &= \prod_{k=1}^{m} P(\tilde{\tau}_k = T_k|\tilde{\tau}_{k-1}=T_{k-1},\ldots,\tilde{\tau}_1=T_1) \\ &= \prod_{k=1}^{m} P(\tilde{\tau}_k = T_k|\tilde{\tau}_{k-1}=T_{k-1}) \end{aligned} \tag{13}$$

This means that (13) characterizes the probability $P(\tau_m < T) = P(\tilde{\tau}_m = T_m)$ of the rare event as a product of conditional probabilities of intermediate "less rare" events leading to it. Thus, if we estimate the conditional probabilities

$$\gamma_k \stackrel{\triangle}{=} P(\tilde{\tau}_k = T_k|\tilde{\tau}_{k-1}=T_{k-1}) \text{ for } k=1,\ldots,m$$

and insert this in (13) we get:

$$P(\tau_m < T) = P(\tilde{\tau}_m = T_m) = \prod_{k=1}^{m}\gamma_k. \tag{14}$$

Let us denote $E=\mathbb{R}^{n+2}\times\mathbb{M}$, and let $\mathscr{E}$ be the Borel $\sigma-$algebra of E. For any $B\in\mathscr{E}$, $\pi_k(B)$ denotes the conditional probability of $\xi_k \stackrel{\triangle}{=} (x_{\tau_k\wedge T}, \theta_{\tau_k\wedge T}) \in B$ given $y_{1:k} = (1,1,\ldots,1)$. Then the estimation of the probability in subsequently hitting the nested Borel sets by $\{\xi_k\}$ is characterized through the following sequence of transformations

$$\pi_{k-1}(\cdot) \xrightarrow{\text{prediction}} p_k(\cdot) \xrightarrow{\text{conditioning}} \pi_k(\cdot),$$

where $p_k(B)$ is the condition probability of $\xi_k \in B$ given $y_{1:k-1}=(1,1,\ldots,1)$. Because $\{\xi_t\}$ is a Markov sequence the prediction satisfies:

$$p_k(B) = \int_E P_{\xi_k|\xi_{k-1}}(B|\xi)\pi_{k-1}(d\xi) \text{ for all } B\in\mathscr{E}, \tag{15}$$

and the conditioning satisfies:

$$\pi_k(B) = \frac{\int_B \mathbf{1}_{\{\xi\in([0,T)\times[0,T_k)\times\bar{D}_k)\}} p_k(d\xi)}{\int_E \mathbf{1}_{\{\xi'\in([0,T)\times[0,T_k)\times\bar{D}_k)\}} p_k(d\xi')} \text{ for all } B\in\mathscr{E}. \tag{16}$$

Then

$$\begin{aligned} \gamma_k &= P(\tau_k < T|\tau_{k-1}<T) = \mathbb{E}[y_k|y_{1:k-1}=(1,1,\ldots,1)] \\ &= \int_E \mathbf{1}_{\{\xi\in([0,T)\times[0,T_k)\times\bar{D}_k)\}} p_k(d\xi). \end{aligned}$$

Similarly as in (13), this can be written as:

$$\gamma_k = P(\tilde{\tau}_k = T_k|\tilde{\tau}_{k-1}=T_{k-1}) = \int_E \mathbf{1}_{\{\xi\in([0,T)\times[0,T_k)\times\bar{D}_k)\}} p_k(d\xi). \tag{17}$$

With this each of the m terms γ_k in (14) is characterized as a solution of a sequence of "filtering" kind of equations (15,16). However, an important difference with "filtering" equations is that (15,16) are ordinary integral equations, i.e. they have no stochastic term entering them.

V. Novel IPS Algorithm

Let $\gamma_k^{N_p}$, $p_k^{N_p}$ and $\pi_k^{N_p}$ denote the numerical approximations of γ_k, p_k and π_k respectively. By $\varphi_t(\theta)$ we will denote an approximation of mode probability $P_{\theta_t}(\theta)$ and $\varphi_0(\theta) = P_{\theta_0}(\theta)$. The particle is defined as a pair (ζ,ω), $\omega\in[0,1]$, $\zeta\in\mathbb{R}^{n+2}$. The ω component represents a weight of the particle.

Step 0. Initial setup

- Choose appropriate nested sequence of closed subsets D_j, $(j=1,\ldots,m)$, of $\mathbb{R}^n$ such that $D=D_m\subset D_{m-1}\subset\cdots\subset D_1$, and define $\bar{D}_k = D_k\times\mathbb{M}$, $k=1,\ldots,m$.
- Compute $r = \max_{\substack{i=1,\ldots,N\\ x\in\mathbb{R}^n}} \left|\sum_{\substack{j=1\\ j\neq i}}^{N}\tilde{\lambda}_{ij}(x)\right| + \varepsilon$, for some small $\varepsilon>0$.

Step 1. Initial sampling; $k=0$.

- At time $t=0$ we start with a set of N_p particles for each mode $\theta\in\mathbb{M}=\{e_1,\ldots,e_N\}$: $\{\zeta_0^{\theta,i},\omega_0^{\theta,i}\}_{i=1}^{N_p}$, $\theta\in\mathbb{M}$, here $\zeta_0^{\theta,i}=(0,0,x^{\theta,i})$ and $x^{\theta,i}$ are independently drawn from $p_{x_0|\theta_0}(\cdot|\theta)$; the initial weights $\omega_0^{\theta,i}=\frac{p_{\theta_0}(\theta)}{N_p}$, $i=1,\ldots,N_p$, $\theta\in\mathbb{M}$.
- Then $\pi_0^{N_p}(B,\theta) = \sum_{i=1}^{N_p}\omega_0^{\theta,i}\delta_{\{\zeta_0^{\theta,i}\}}(B)$, $B\in\mathscr{B}(\mathbb{R}^{n+2})$, and $\gamma_0^{N_p}=1$.

Iteration k; $k = 1,\dots,m$ over step 2 (prediction), step 3 (assessment) and step 4 (resampling)

Step 2. Prediction: $\pi_{k-1}^{N_p} \longrightarrow p_k^{N_p}$;

For $j = 1,\dots,J$, with time discretization step $h = \frac{T}{J}$ and $\hat{t}_j := T_{k-1} + h \cdot j$

Substep 2.a (Interaction based resampling):

- For all $\theta, \eta \in \mathbb{M}$, $i = 1,\dots,N_p$ evaluate the transition probabilities

$$p_{\tilde{\theta}^h_{\hat{t}_j}|\tilde{X}^h_{\hat{t}_{j-1}},\tilde{\theta}^h_{\hat{t}_{j-1}}}(\theta|\zeta^{\eta,i}_{\hat{t}_{j-1}},\eta) \approx [I_{\eta\theta} + (P^*_{\eta\theta}(\zeta^{\eta,i}_{\hat{t}_{j-1}}))rh]e^{-rh},$$

where $P^*(\zeta^{\eta,i}_{\hat{t}_{j-1}}) = \frac{1}{r}\tilde{\Lambda}(\zeta^{\eta,i}_{\hat{t}_{j-1}}) + I$.

- Evaluate probabilities of modes:

$$p_{\tilde{\theta}^h_{\hat{t}_j}}(\theta) \approx \varphi_{\hat{t}_j}(\theta) = \sum_{\eta\in\mathbb{M}}\sum_{i=1}^{N_p} p_{\tilde{\theta}^h_{\hat{t}_j}|\tilde{X}^h_{\hat{t}_{j-1}},\tilde{\theta}^h_{\hat{t}_{j-1}}}(\theta|\zeta^{\eta,i}_{\hat{t}_{j-1}},\eta)\omega^{\eta,i}_{\hat{t}_{j-1}}$$

- If all modes have zero probabilities, i.e. $\varphi_{\hat{t}_j}(\theta) = 0$ for all $\theta \in \mathbb{M}$ then the algorithm is stopped and we set $P(\tau_m < T) \approx 0$.
- For each $\theta \in \mathbb{M}$ draw N_p random vectors $\tilde{\zeta}^{\theta,i}_{\hat{t}_{j-1}}$, $i = 1,\dots,N_p$, from the particle spanned measure:

$$p_{\tilde{X}^h_{\hat{t}_{j-1}},\tilde{\theta}^h_{\hat{t}_j}}(B,\theta) \approx$$
$$\approx \sum_{\eta\in\mathbb{M}}\sum_{i=1}^{N_p} p_{\tilde{\theta}^h_{\hat{t}_j}|\tilde{X}^h_{\hat{t}_{j-1}},\tilde{\theta}^h_{\hat{t}_{j-1}}}(\theta|\zeta^{\eta,i}_{\hat{t}_{j-1}},\eta)\omega^{\eta,i}_{\hat{t}_{j-1}}\,\delta_{\{\zeta^{\eta,i}_{\hat{t}_{j-1}}\}}(B).$$

- This yields for each $\theta \in \mathbb{M}$ the following set of particles $\{\tilde{\zeta}^{\theta,i}_{\hat{t}_{j-1}}, \omega^{\theta,i}_{\hat{t}_j}\}_{i=1}^{N_p}$ with $\omega^{\theta,i}_{\hat{t}_j} = \varphi_{\hat{t}_j}(\theta)/N_p$.

Substep 2.b:

- Determine the new set of particles (the weights are not changed) $\{\zeta^{\theta,i}_{\hat{t}_j}, \omega^{\theta,i}_{\hat{t}_j}\}_{i=1}^{N_p}$, $\theta \in \mathbb{M}$ by evaluating for each particle a new value according to Euler discretization scheme:

$$\zeta^{\theta,i}_{\hat{t}_j} = \tilde{\zeta}^{\theta,i}_{\hat{t}_{j-1}} + \tilde{a}(\tilde{\zeta}^{\theta,i}_{\hat{t}_{j-1}},\theta)\cdot h + \tilde{b}(\tilde{\zeta}^{\theta,i}_{\hat{t}_{j-1}},\theta)(\tilde{W}^i_{\hat{t}_j} - \tilde{W}^i_{\hat{t}_{j-1}})$$

- Set $j := j+1$, if $j < J$ then go to Substep 2.a, else we have:

$p_k^{N_p}$, the empirical distribution associated with the cloud of particles $\{\bar{\zeta}^{\theta,i}_k, \bar{\omega}^{\theta,i}_k\}_{i=1}^{N_p}$, with $\bar{\omega}^{\theta,i}_k = \omega^{\theta,i}_{\hat{t}_J}$ and $\bar{\zeta}^{\theta,i}_k = \zeta^{\theta,i}_{\hat{t}_J}$ obtained after series of substeps 2.a and 2.b ($j = 1,\dots,J$):

$$p_k^{N_p}(B,\theta) = \sum_{i=1}^{N_p} \bar{\omega}^{\theta,i}_k \delta_{\{\bar{\zeta}^{\theta,i}_k\}}(B).$$

Step 3. Assess particle arrival:

- We set $\hat{\omega}^{\theta,i}_k = \bar{\omega}^{\theta,i}_k$ if $\bar{\zeta}^{\theta,i}_k \in ([0,T)\times[0,T_m)\times\bar{D}_k)$, and otherwise $\hat{\omega}^{\theta,i}_k = 0$. This yields the set of particles $\{\bar{\zeta}^{\theta,i}_k, \hat{\omega}^{\theta,i}_k\}_{i=1}^{N_p}$, $\theta \in \mathbb{M}$.
- Approximation of π_k:

$$\pi_k(B,\theta) \approx \pi_k^{N_p}(B,\theta) = \frac{1}{\gamma_k^{N_p}}\sum_{i=1}^{N_p} \hat{\omega}^{\theta,i}_k \delta_{\{\bar{\zeta}^{\theta,i}_k\}}(B).$$

- Approximation of γ_k: $\gamma_k \approx \gamma_k^{N_p} = \sum_{\theta\in\mathbb{M}}\sum_{i=1}^{N_p} \hat{\omega}^{\theta,i}_k$.
- If $\gamma_k^{N_p} = 0$ then the algorithm is stopped and we set $P(\tau_m < T) \approx 0$.

Step 4. Resampling:

- For each mode $\theta \in \mathbb{M}$ resample with replacement N_p values of $\zeta^{\theta,i}_k$ according to the empirical measure $\pi_k^{N_p}(\cdot,\theta)$ and assign new weights to particles:

$$\omega^{\theta,i}_k = \frac{1}{N_p\gamma_k^{N_p}}\sum_{j=1}^{N_p} \hat{\omega}^{\theta,j}_k.$$

Hence at time $t = T_k$ we have a set of N_p particles per mode θ: $\{\zeta^{\theta,i}_k, \omega^{\theta,i}_k\}_{i=1}^{N_p}$, $\theta \in \mathbb{M}$.

- $k := k+1$. If $k < m$ then repeat step 2, 3 and 4.
- Otherwise, algorithm stops with $P(\tau_m < T) \approx \prod_{k=1}^m \gamma_k^{N_p}$.

This completes our novel IPS particle system algorithm for estimating the probability of reaching a subset D by a switching diffusion. In a follow up study we plan to compare for an illustrative example this novel IPS estimation algorithm with the Hybrid IPS estimation algorithm of [4]. The main difference between this new IPS and HIPS consist in the way these two approaches deal with the problem of rare switches in stochastic hybrid systems. Our novel IPS approach uses only mode conditional $\mathbb{R}^n$ -valued particles, and the probabilities of switches and mode probabilities (i.e. probabilities on $\mathbb{M}$) are evaluated analytically at each discretization step. In HIPS approach particles are $\mathbb{R}^n \times \mathbb{M}$ valued, and, in order to capture the interaction between system operation modes, during simulation it uses importance switching procedure.

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