THE VARIATIONAL 2D BOUSSINESQ MODEL FOR WAVE PROPAGATION OVER A SHOAL

D Adytia and **E van Groesen**, LabMath-Indonesia and University of Twente, The Netherlands

SUMMARY

The Variational Boussinesq Model (VBM) for waves [1] is based on the Hamiltonian structure of gravity surface waves. In its approximation, the fluid potential in the kinetic energy is approximated by the sum of its value at the free surface and a linear combination of vertical profiles with horizontal spatially dependent functions as coefficients. The vertical profiles are chosen a priori and determine completely the dispersive property of the model. For coastal applications, the 1D version of the model has been implemented in Finite Element with piecewise linear basis functions and has been compared with experiments from MARIN hydrodynamic laboratory for focusing wave group running above a flat bottom [2] and for irregular waves running above a sloping bottom [3]. The 2D version of the model has been derived and implemented using a pseudo-spectral method with a rectangular grid in [4,1]. A limitation of the later implementation is a lack of flexibility when the model deals with a complicated domain such as a harbour. Here, we will present an implementation of the model in 2D Finite Element which consistent with the derivation of the model via the variational formulation. To illustrate the accuracy of wave refraction and diffraction over a complex bathymetry, the experiment of Berkhoff et al, 1982 [5] is used to compare the FE results with measurements.

NOMENCLATURE

H	Hamiltonian or total energy
K	Kinetic energy
P	Potential energy
$\eta(\mathbf{x},t)$	Surface elevation(m)
$\Phi(\mathbf{x}, z,t)$	Fluid potential
$\varphi(\mathbf{x},t)$	Fluid potential at free surface
$h(\mathbf{x})$	Water depth(m)
g	Gravitational acceleration (ms^{-2})

1. INTRODUCTION

For coastal zone applications such as simulations of coastal waves in a harbour, besides the mild-slope equations, Boussinesq-type of equations have been favorite because of the ability to represent the physical phenomena of water waves such as non-linearity, dispersion and bathymetric effects. The standard Boussinesq equations which include the effect of bathymetry were first derived by Peregrine (1967) [6], who used depth averaged velocity as dependent variable. But this model is restricted to shallow water because of its rather poor dispersion characteristics in intermediate and deep water. Since then, there have been many works to improve the dispersion quality of Boussinesq-type equations, e.g. [7,8].

In order to achieve good dispersive characteristics, many of these Boussinesq-type equations include high order spatial (or mixed with time) derivatives; this makes them rather difficult for numerical implementations and for practical applications. In this paper, we will use a Boussinesq-type of equation that is derived from a variational formulation. This will lead to a system of equations with a much simpler numerical implementation since it only contains second order spatial derivatives without mixed time-space derivatives. In section 2, we derive the 2D version of this model. Section 3 describes the Finite Element implementation of the model and in Section 4 the performance of this implementation is tested for the experiment of Berhoff 1982. Section 5 provides conclusions of this paper.

2. VARIATIONAL BOUSSINESQ MODEL

2.1 VARIATIONAL FORMULATION FOR VBM

The variational principle for water waves was introduced by Luke in 1967 [9]. Related to the Hamiltonian equations in [10,11,12], the Lagrangian L in the Luke's variational principle, which depend on fluid potential in the interior $\Phi(\mathbf{x},z,t)$ and surface elevation $\eta(\mathbf{x},t)$, can be reformulated in canonical variables at the free surface. These are the surface elevation η and the fluid potential at the free surface $\varphi(\mathbf{x},t)$ as

$$L = \int \left\{ \int \varphi \partial_t \eta \, d\mathrm{x} - H(\eta, \varphi) \right\} dt \qquad (1)$$

Here H is the Hamiltonian or the total energy, which is the sum of the kinetic energy and the potential energy that are defined as

$$K = \frac{1}{2} \int_{\mathrm{x}} \int_{-h}^{\eta} |\nabla_3 \Phi|^2 \, dz d\mathrm{x} \text{ and } P = \frac{1}{2} \int_{\mathrm{x}} g\eta \, d\mathrm{x}$$

respectively. Here, $h(\mathbf{x})$ describes the bottom. By taking variations with respect to η and φ, the Hamiltonian equation are found

$$\partial_t \eta = \delta_\varphi H \text{ and } \partial_t \varphi = -\delta_\eta H$$

where $\delta_\varphi H$ and $\delta_\eta H$ denote the variational derivatives of H with respect to η and φ respectively. The exact formulation will be obtained if we could assure that the potential velocity Φ satisfies the Laplace equation in the interior, the impermeability condition at the bottom and the condition $\Phi=\varphi$ at the free surface. The main difficulty in water wave modeling arises from the need to approximate the kinetic energy explicitly in the surface variables. Nevertheless, after finding an approximation for the kinetic energy, following the variational steps above, we obtain a consistent approximation that keeps

the consequences of the Hamiltonian form, such as energy conservation.

As described in [13,4,1,2,3], the VBM is obtained by approximating the vertical structure of the fluid velocity Φ in the expression of K with its value at the surface φ and multiple expansion terms as

$$\Phi(x,z,t) = \varphi(x,t) + \sum_m F_m(z)\psi(x,t) = \varphi + F \cdot \Psi$$

where F and Ψ are vector functions. In order to keep the canonical structure of the Hamiltonian, the condition $\Phi=\varphi$ at the free surface has to be assured. As consequence, it is required that $F_m(z=\eta)=0$. The vertical profile function $F_m(z)$ has to be chosen in advance, while $\psi_m(\mathrm{x})$ are functions that have to satisfy an optimality condition of vanishing of the kinetic energy with respect to variations in ψ_m. Such variations lead to a system of linear elliptic equations that has to be solved together with the dynamics for the variables η and φ. By substituting the approximation of Φ into K, we obtain the kinetic energy of -VBM

$$\begin{aligned} K_B &= \frac{1}{2}\int_{\mathrm{x}}\int_{-h}^{\eta} |\nabla\varphi + \nabla(F\cdot\Psi)|^2 + (\partial_z F\cdot\Psi)^2 \; dz d\mathrm{x} \\ &\approx \frac{1}{2}\int_{\mathrm{x}}\int_{-h}^{\eta} |\nabla\varphi + F\cdot\nabla\Psi|^2 + (\partial_z F\cdot\Psi)^2 \; dz d\mathrm{x} \\ &= \frac{1}{2}\int_{\mathrm{x}} \{(h+\eta)|\nabla\varphi|^2 + 2\nabla\varphi\beta\cdot\nabla\Psi + \nabla\Psi\cdot\alpha\nabla\Psi \\ &\qquad + \Psi\cdot\gamma\Psi\} d\mathrm{x} \end{aligned}$$

In the second line, we use a *weakly-nonlinear* approximation (see [13] for details) where F is assumed to be slowly varying with respect to η and h. In the last expression, matrices α and γ, and a vector β are introduced with elements given by

$$\alpha_{ij} = \int_{-h}^{\eta} F_i F_j dz;\; \gamma_{ij} = \int_{-h}^{\eta} \partial_z F_i \partial_z F_j dz;\; \beta_i = \int_{-h}^{\eta} F_i dz;$$

By substituting the expression of the kinetic energy above into the Lagrangian (1), we obtain

$$L = \int \left\{ \int \varphi \partial_t \eta \, d\mathrm{x} - H_b(\eta,\varphi,\psi_m) \right\} dt \qquad (2)$$

where H_b is the Hamiltonian for the VBM. Taking variations with respect to η, φ and ψ_m, we obtain a system of PDE

$$\begin{cases} \partial_t\eta = -\nabla\cdot[(h+\eta)\nabla\varphi] - \nabla\cdot[\beta\cdot\nabla\Psi] \\ \partial_t\varphi = -g\eta - |\nabla\varphi|^2/2 \\ -\nabla\cdot[\alpha\nabla\Psi] + \gamma\Psi = \nabla\cdot[\beta\nabla\varphi] \end{cases} \qquad (3)$$

The dispersive quality of the system above is highly determined by the choice of the vertical profile functions $F_m(z)$. In [1], it was suggested to choose a parabolic profile for rather long wave, or a cosine hyperbolic profile that is obtained from Airy's linear potential theory :

$$F_m(z;\eta,h) = \cosh(\kappa_m(z+h))/\cosh(\kappa_m(\eta+h)) - 1$$

which has exact phase and group velocities for waves with wavenumber κ_m. For periodic waves, the latter profile is the best choice, since we can choose κ_m as the wavenumber of the waves to be simulated. For broad spectra such as for focusing wave groups and irregular (coastal) waves, the optimal choice of κ_m for the Airy profiles above can be obtained by minimizing the kinetic energy functional for a given initial influx signal as described in [2,3].

2.2 FINITE ELEMENT IMPLEMENTATION

The VBM is obtained from a variational formulation by minimizing the Lagrangian of VBM (2) with respect to η, φ and ψ_m. For this reason, it is quite natural to implement the VBM by using the Finite Element Method (FEM). Besides this, we can use piecewise linear local basis functions since the highest derivatives in (2) are of first order. In this implementation we discretize the solutions in space by using FEM which leads to a system of ordinary differential equations (ODEs) that is solved using an explicit time integrator such as a Runge-Kutta method.

We start the spatial discretization of the solutions η, φ and ψ_m into $\overline{\eta}$, $\overline{\varphi}$ and $\overline{\psi}$ by using standard 2D triangular basis functions $\mathrm{T}(\mathbf{x})$, then substitute them into the Lagrangian of VBM (2). This leads to a Lagrangian with vector state variables

$$\overline{L} = \int \left\{ \mathrm{M}\overline{\varphi}\cdot\partial_t\overline{\eta} - \overline{H}_b(\overline{\eta},\overline{\varphi},\overline{\psi}_m) \right\} dt \qquad (4)$$

with

$$\begin{aligned} \overline{H}_b &= \frac{1}{2} g\mathrm{M}\overline{\eta}\cdot\overline{\eta} + \frac{1}{2}\mathrm{D}\overline{\varphi}\cdot\overline{\varphi} + \overline{H}_3(\overline{\eta},\overline{\varphi}) + \mathrm{B}\cdot\overline{\psi} \\ &\quad + \frac{1}{2}\mathrm{A}\overline{\psi}\cdot\overline{\psi} + \frac{1}{2}\mathrm{G}\overline{\psi}\cdot\overline{\psi} \end{aligned}$$

Here M is the so-called *mass matrix* with elements $\mathrm{M}_{\mathrm{i,j}}=\int \mathrm{T}_\mathrm{i}(\mathbf{x})\mathrm{T}_\mathrm{j}(\mathbf{x})d\mathbf{x}$, D and A are so-called *stiffness-matrices* with elements

$$\mathrm{D}_{\mathrm{i,j}} = \int h(\mathrm{x})\nabla\mathrm{T}_i(\mathrm{x})\cdot\nabla\mathrm{T}_i(\mathrm{x})d\mathrm{x}$$

$$\mathrm{A}_{\mathrm{i,j}} = \int \alpha(\mathrm{x})\nabla\mathrm{T}_i(\mathrm{x})\cdot\nabla\mathrm{T}_i(\mathrm{x})d\mathrm{x}$$

respectively. G is the matrix with elements $\mathrm{G}_{\mathrm{i,j}}=\int\gamma(\mathbf{x})\mathrm{T}_\mathrm{i}(\mathbf{x})\mathrm{T}_\mathrm{j}(\mathbf{x})d\mathbf{x}$, $\overline{\mathrm{B}}$ is the column vector $\overline{\mathrm{B}} = [B(\beta_1)\overline{\varphi},\cdots,B(\beta_N)\overline{\varphi}]$ with

$$B(\beta_i) = \int \beta_i(\mathrm{x})\nabla\mathrm{T}_i(\mathrm{x})\cdot\nabla\mathrm{T}_i(\mathrm{x})d\mathrm{x},$$

and $\overline{H}_3(\overline{\eta},\overline{\varphi})$ is the nonlinear cubic term :

$$\overline{H}_3(\overline{\eta},\overline{\varphi}) = \frac{1}{2}\sum_{i,j,k}\eta_i\varphi_j\varphi_k C_{ijk}$$

with $C_{ijk} = \int T_i(\mathrm{x})\nabla T(\mathrm{x})\cdot\nabla T(\mathrm{x})d\mathrm{x}$. Now, from the Lagrangian (4), we obtain the Hamiltonian equations as the following matrix system

$$\begin{aligned} \mathrm{M}\partial_t\overline{\eta} &= \partial_{\overline{\varphi}}\overline{H}_b = \mathrm{D}\overline{\varphi} + \overline{b}(\overline{\eta},\overline{\varphi}) + \overline{\mathrm{B}}(\beta)\cdot\overline{\psi} \\ \mathrm{M}\partial_t\overline{\varphi} &= -\partial_{\overline{\eta}}\overline{H}_b = -g\mathrm{M}\overline{\varphi} - \overline{v}(\overline{\varphi}) \end{aligned} \quad (5)$$

and an additional matrix system of linear elliptic equations from $\partial_{\overline{\psi}}\overline{H}_b = 0$

$$[\mathrm{A}+\mathrm{G}]\overline{\psi} = -\mathrm{B} \quad (6)$$

where $\overline{b}(\overline{\eta},\overline{\varphi})$ and $\overline{v}(\overline{\varphi})$ are column vectors with elements

$$b_j = \partial_{\varphi_j}\overline{H}_3 = \sum_{i,k}\eta_i\varphi_k C_{ijk}$$

$$v_j = \partial_{\eta_j}\overline{H}_3 = \frac{1}{2}\sum_{i,k}\varphi_i\varphi_k C_{ijk}$$

The algorithm to obtain numerical solutions for the system (5) and (6) at every time step can be obtained as follows. For given initial conditions η_0 and φ_0, we calculate ψ_0 by solving the elliptic system in (6). Then by using the new ψ_1, we solve the dynamics system in (5) using a time integrator, a Runge-Kutta method (or ODE-solver in MATLAB) to obtain new η_1 and φ_1. These steps can be repeated until the desired end-time. In the calculation of the elliptic system in (6), we use an iterative method, i.e a preconditioned conjugate gradient method with a preconditioner obtained from an incomplete Cholesky factorization. This iterative method only needs 1-4 steps since we have an appropriate initial guess for ψ from the previous time step. Besides that, the matrices in (5) and (6) are very sparse since we use the local basis functions for FEM, so the calculation is quite efficient.

3. WAVE PROPAGATION OVER A SHOAL

To show the performance of the 2D implementation of the model above, we perform a simulation of a monochromatic wave propagating above an elliptic shoal on a slope as in the experiment of Berkhoff et al in 1982 [5]. This case is often used to demonstrate the stability, accuracy and efficiency of a model and its numerical implementation, since the waves are affected by shoaling, refraction, diffraction and non-linearity [14].

The laboratory setup for this experiment and the measurement sections are shown in Figure 1. An elliptic shoal is placed above 1:50 sloping bottom and turned at an angle 20° with the x-axis. The thickness of the shoal is

$$d = -0.3 + 0.5\sqrt{1-(x'/5)^2-(y'/3.75)^2}\,.$$

Monochromatic waves with period 1*s* and amplitude 2.32*cm* are generated at the north boundary (y=10) and propagate in the south direction.

For the computation, the west and east boundary conditions (at x=-10 and x=10) are set to be a fully reflective wall, while at the south an absorbing boundary condition is implemented using a sponge layer of 5*m* wide. To influx a wave into the domain at the north boundary (y=10), we use internal wave generation with a spatial delta function and modified influx signal with the group velocity of the model, similar with the method described in [15]. We use an unstructured triangular grid with grid spacing approximately 0.1*m*. For the vertical potential profiles in the model, since we simulate monochromatic waves, we use one cosine hyperbolic profile with the value for κ the wavenumber of the wave with period 1*s* at the depth that is described in Figure 1. The calculation was done for the linear version of the model over a time interval of 50 wave periods without any stability problem. The wave amplitude of the simulation is obtained by averaging the maximal amplitudes of the last 10 wave periods of the simulation (i.e. from t=40*s* to t=50*s*). In Figure 2, the amplitude contours are given of the measurement (upper part) and of the simulation (middle part). In the lower part the average of the maximal wave amplitude from t=40*s* to t=50*s* is shown.

Figure 3 shows the comparison between computed and measured normalized wave amplitude for the eight sections shown in Figure 1. Figure 2 shows that the model can follow the diffraction pattern, including the refraction effect in the wave focusing by the shoal. Figure 3 shows that the agreement between the measurement and the simulation is good.

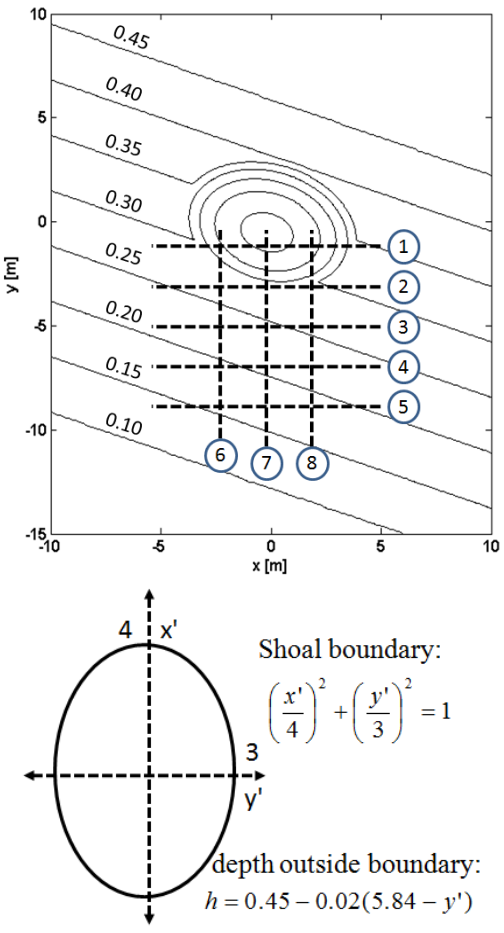

Figure 1: Bottom configuration for the experiment of Berkhoff et al 1982 [5]. Dashed lines (labeled from 1 to 8) indicate sections where measurement data are available.

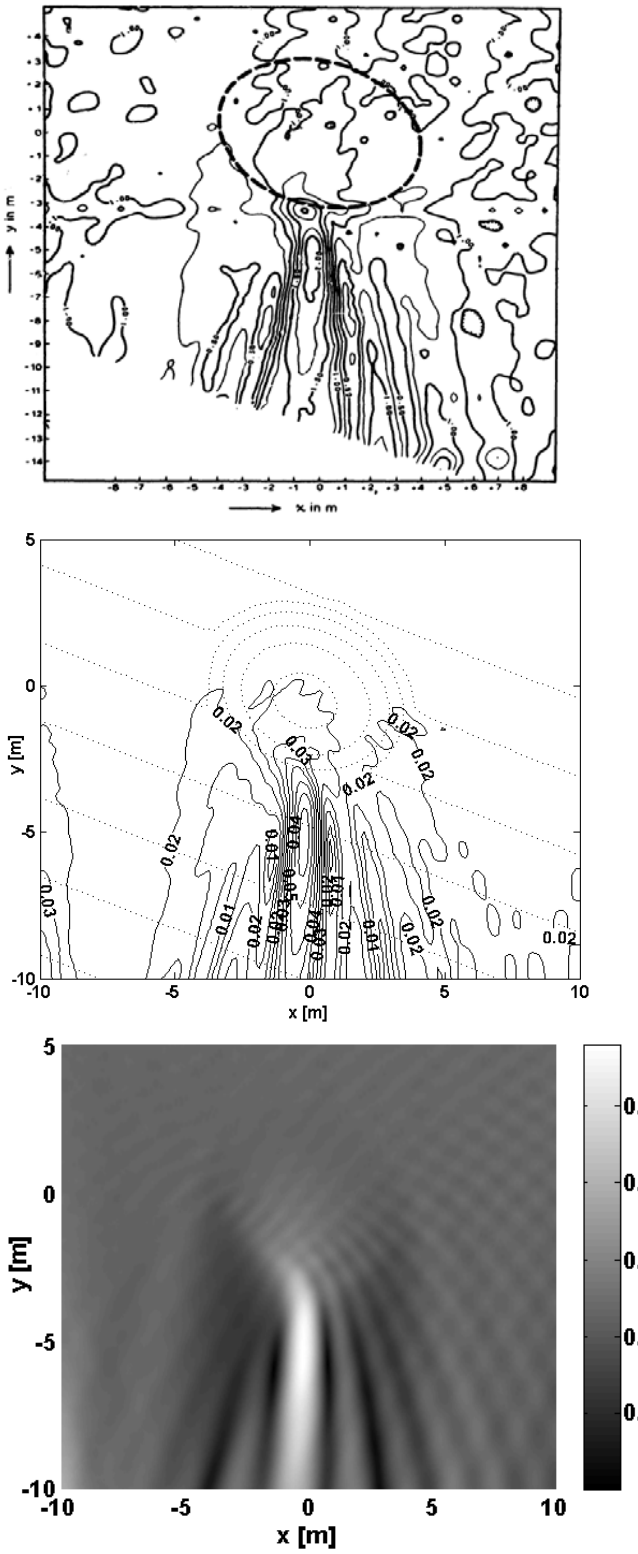

Figure 2: Measured (upper plot) and computed (middle plot) amplitude contours (solid lines). In the lower plot the average of the computed maximal wave amplitude is shown.

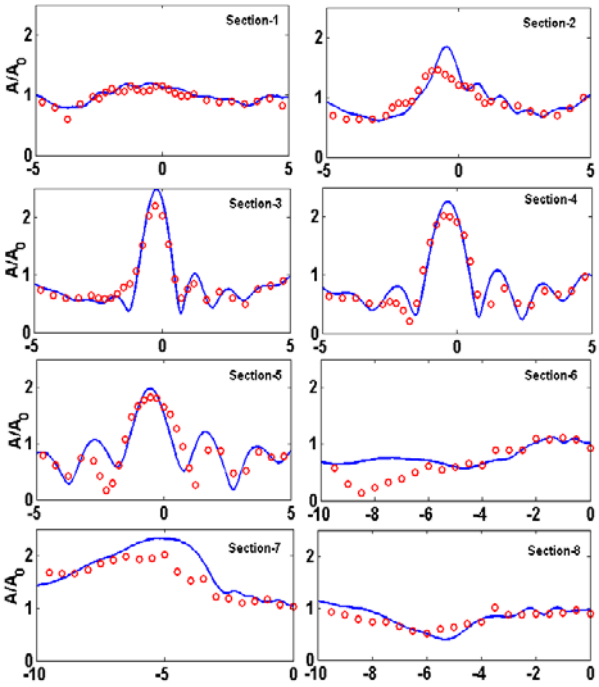

Figure 3: Comparion of the measured (o o) and computed (-) normalized wave amplitude in eight sections.

4. CONCLUSIONS

Via the variational formulation the VBM was derived by approximating the kinetic energy using specific vertical structures in the fluid potential. The resulting system consists of two dynamic equations and an additional linear elliptic equation. In contrast with other Boussinesq-type equations, the system has no spatial derivatives of higher than second order, which allows for simple numerical implementation. In this paper, the model has been implemented in a 2D Finite Element (FE) method in a way that is consistent with the variational formulation. This implementation has been tested to simulate the experiment of Berkhoff et al 1982 [5] which represents the importance of diffraction, refraction and shoaling; non-linear effects have been discarded in the linear simulation presented here. Even without the nonlinear effects, the comparison between the model and measurements shows a good agreement.

5. ACKNOWLEDGEMENTS

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7. AUTHORS' BIOGRAPHIES

Didit Adytia is PhD candidate in Applied Mathematics at University of Twente and Laboratorium Matematika Indonesia (LabMath-Indonesia). His work concentrates on wave simulations with Boussinesq-type of models for applications in the coastal zone such as harbour simulations.

E. van Groesen holds the position of full time Professor at Departement of Applied Mathematics in University of Twente and is Scientific Director of Laboratorium Matematika Indonesia (LabMath-Indonesia).