Quickest Detection of Changes in Random Fields

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Abstract — **The quickest detection problem is formulated for processes defined on a two-dimensional lattice. Solutions are given when only the class of sequential probability ratio tests is considered.**

I. Introduction

For a process that evolves in time, there is a natural way to formulate the quickest detection problem. Quickest detection of changes then consists of finding a test statistic that minimizes the difference between the time of first occurrence of the change and the detection time. For a process that is defined on a two-dimensional lattice, several methods of processing the data, so-called *sample paths*, may exist. Therefore, not only do we have a freedom in choosing the test statistic, also the sample path may be selected to achieve quickest detection in some sense. Possible applications may be found in image processing, or the detection of faults on surfaces.

II. Quickest detection problem

Where classical quickest detection problems are specified by minimizing the delay of detection, in these multi-path situations a more suitable definition is to minimize the number of data points needed to detect a change. We define the *average site number* (ASN) as the expected number of data points processed before processing is stopped, given that a change is present. Clearly, some constraints on the detection quality have to be imposed in order for this definition to be of any use.

Here we only consider statistical tests for which the probability of false alarm is bounded by some constant. This way we may define a class of *stopping rules* for each sample path. For each sample path, we try to find the *optimal stopping rule*, i.e., the stopping rule that minimizes a certain cost function. If this cost function would be entirely determined by the number of data points needed to detect a change, the solution to this problem would be degenerate; simply stop after the first sample and choose the acceptance region such that the probability of false alarm attains its desired value. To avoid this undesirable behavior, there are two possible approaches to solve this problem.

The first one, the *Neyman-Pearson approach*, consists of adding another constraint on the miss-probability. The quickest detection problem may then be written as the minimization of the average site number, given that the error probabilities are bounded by some given constants.

The second one simply includes the probability of detection in the cost function. The cost function is chosen as

$$V = ASN + c(1-\beta)$$

where β denotes the probability of detection. The constant c, defined as the miss-cost, determines the relative importance of the probability of detection with respect to the average site number.

III. Approximate solution

The solution of the quickest detection problem may be shown to be extremely complicated. Therefore, we limit the class of stopping rules to the more tractable class of *sequential probability ratio tests* (SPRT). These tests are parameterized by two thresholds a and b. If the likelihood ratio becomes smaller than a, the null hypothesis is accepted. Alternatively, if it becomes larger than b, the alternative hypothesis is accepted. The thresholds a and b have to be chosen such that the error constraints are satisfied. In the Neyman-Pearson approach both thresholds are determined by these constraints. For the second approach, this leaves us one degree of freedom, so that we may actually find an optimal stopping rule in this class by finding the combination (a, b) for which the cost is minimal. Since this optimization procedure is in general computationally expensive, in our examples we choose to fix the lower threshold a, so that the class of stopping rules contains only one SPRT.

The case where the process is independent and identically distributed under both hypotheses on the field is examined here. The null hypothesis is assumed to be simple, and the alternative hypothesis may be composite. In case of a composite hypothesis, we assume the prior conditional distribution of the changes to be known. Assuming that the changes are parameterized by the set Θ, this gives us

$$\xi(\theta) = \mathbf{Pr}[\theta | change]$$

for all $\theta \in \Theta$.

Expressions are derived for the detection and false alarm probabilities, for given thresholds a and b. A similar expression has been derived for the average site number. Using these expressions, the optimal values of a and b may be calculated for both approaches. Consequently, we may calculate the cost for each sample path. The optimal sample path may then easily be selected as the one resulting in the minimal cost.

If we furthermore restrict the class of sample paths to the one that contains only sample paths with constant sample sizes, we may find some asymptotic results. The recursive equations obtained for the cost function may now be replaced by a Fredholm integral equation of the second type, which is easier to approximate in a numerical way. This way we may determine the asymptotically optimal sample size for the quickest detection of a change.