Analysis of a state-independent change of measure for the $G|G|1$ tandem queue

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Introduction

In [4], Parekh and Walrand introduce a method to efficiently estimate the probability of a rare event in a single queue or network of queues. The event they consider is that the total number of customers in the system reaches some level N in a busy cycle. Parekh and Walrand introduce a simple change of measure, which is state-independent, in order to estimate this probability efficiently using simulation. However, they do not provide any proofs of some kind of efficiency of their method. In the remainder of this paper, the change of measure proposed by Parekh and Walrand will be referred to as the P&W change of measure.

For the single queue (with multiple servers) it has been shown by Sadowsky [5] that the P&W change of measure as proposed is asymptotically efficient under some mild conditions.

For a two node $M|M|1$ tandem queue, the state-independent change of measure has been studied thoroughly in, for example, [2] and [3]. Glasserman and Kou [3] show that using this state-independent change of measure does give an asymptotically efficient simulation for some combinations of arrival and service rates, while for some other combinations of arrival and service rates it does not give an asymptotically efficient simulation. De Boer [2] has shown that this state-independent change of measure is the only state-independent change of measure that can possibly be asymptotically efficient.

In this work we study the state-independent change of measure of the $G|G|1$ tandem queue, along the lines of Parekh and Walrand, and we provide necessary conditions for asymptotic efficiency. To the best of our knowledge, no results on asymptotic efficiency for the $G|G|1$ tandem queue had been obtained previously. Looking at the results of the $M|M|1$ tandem queue, it is expected that this state-independent change of measure is the only state-independent change of measure for the $G|G|1$ tandem queue that can possibly be asymptotically efficient. In the remaining of this paper we provide necessary conditions, but omit the proofs, for asymptotic efficiency in the $G|G|1$ tandem queue and we will give two examples.

Model and main results

We start by introducing some notation. Suppose we have d queues in tandem. Let A_k be the inter-arrival time at queue 1 between customer k and $k+1$ and let $B_k^{(j)}$ be the service time of customer k at queue j. We define K_N as the index of the first customer who reaches the overflow level N. Furthermore, queue R is the bottleneck queue when $\theta^* = \theta_R = \min_j\{\theta_j\}$. We find θ_j by solving $M_A(-\theta_j)M_{B^{(j)}}(\theta_j) = 1\ \forall j = 1, ..., d$, where $M_A(t) = \mathbb{E}\left[e^{tA}\right]$ and $M_{B^{(j)}}(t) = \mathbb{E}\left[e^{tB^{(j)}}\right]$. In [1] it is shown that the change of measure along the lines of Parekh and Walrand is

$$f_A^{\theta^*}(a) = \frac{e^{-\theta^* a}}{M_A(-\theta^*)} f_A(a), \qquad f_{B^{(R)}}^{\theta^*}(b) = \frac{e^{\theta^* b}}{M_{B^{(R)}}(\theta^*)} f_{B^{(R)}}(b),$$

where $f_X^{\theta^*}$ denotes the probability density function of distribution X with exponential tilt θ^*. Then we have the following theorem.

Theorem 1. *Suppose we have d $G|G|1$ queues in tandem and let us assume that (i) the service times of the bottleneck queue are bounded, that is, $B_k^{(R)} \leq M < \infty$ for some $M > 0$; (ii) there exist a unique bottleneck queue; and (iii) $\frac{K_N}{N}$ is uniformly integrable.*
Then the P&W change of measure is the only exponential state-independent change of measure that can possibly be asymptotically efficient.

We also have identified conditions for the two-node $G|G|1$ tandem queue under which the P&W change of measure is *not* asymptotically efficient.

Theorem 2. *Suppose queue 1 is the bottleneck queue and $B^{(2)} \sim exp(\mu_2)$. Then a necessary condition for asymptotic efficiency is*

$$\int_0^\infty e^{(2\theta^* - \mu_2)x} f_A^{\theta^*}(x) \int_0^x e^{-2\theta^* y} f_{B^{(1)}}^{\theta^*}(y)\ dydx \leq M_A(-\theta^*)^2. \tag{1}$$

Suppose queue 2 is the bottleneck queue. Then a necessary condition for asymptotic efficiency is

$$\limsup_{N\to\infty} \frac{1}{N} \log \left(\int_0^\infty e^{2\theta^* x} f^{\theta^*}_{\sum_{i=1}^{N-1} A_i}(x)[1 - F_{B^{(1)}}(x)]\ dx \right) \leq 0. \tag{2}$$

We proved Theorem 2 by considering very unlikely paths whose contribution to the likelihood ratio is so big that it does give a necessary condition. We consider some special cases of this theorem.

Corollary 3. *Suppose queue 1 is the bottleneck queue, $B^{(1)} \sim exp(\mu_1)$ and $B^{(2)} \sim exp(\mu_2)$. Then a necessary condition for asymptotic efficiency is*

$$\frac{\mu_1 - \theta^*}{\theta^* + \mu_1} \left[M_A(\theta^* - \mu_2) - M_A(-(\mu_1 + \mu_2)) \right] \quad \leq \quad M_A(-\theta^*)^3.$$

Suppose queue 2 is the bottleneck queue and $B^{(1)} \sim exp(\mu_1)$. Then a necessary condition for asymptotic efficiency is

$$M_A(\theta^* - \mu_1) \leq M_A(-\theta^*).$$

Suppose we have an $M|M|1$ tandem queue with $A \sim exp(\lambda)$, $B^{(1)} \sim exp(\mu_1)$ and $B^{(2)} \sim exp(\mu_2)$. When queue 1 is the bottleneck queue, a necessary condition for asymptotic efficiency is

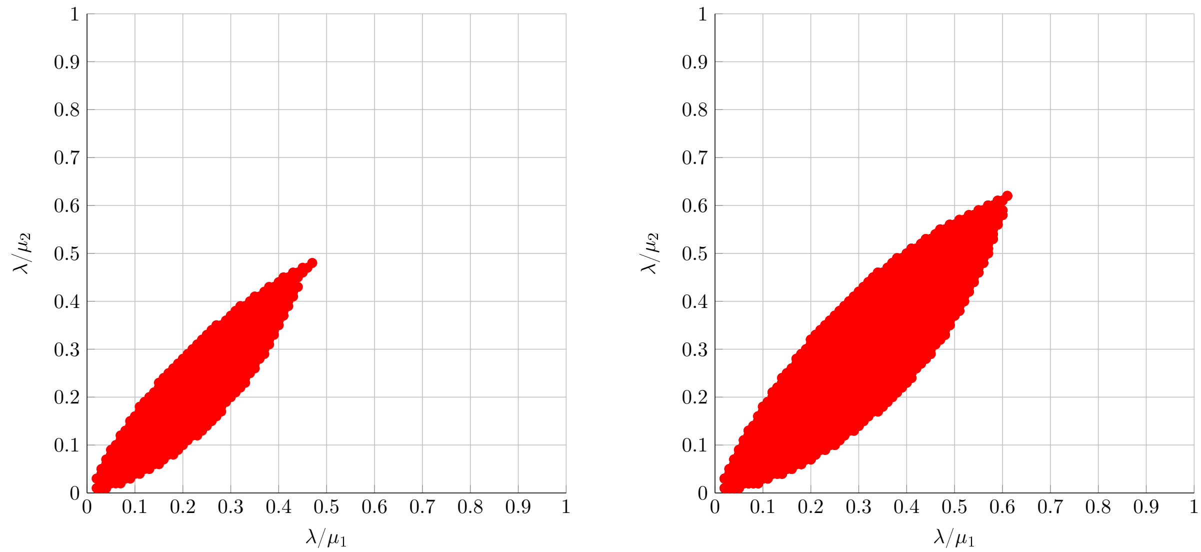
$$\frac{\mu_1}{2\mu_1 - \lambda} \left[\frac{1}{2\lambda + \mu_2 - \mu_1} - \frac{1}{\lambda + \mu_1 + \mu_2} \right] \leq \frac{\lambda}{\mu_1^2},$$

and when queue 2 is the bottleneck queue a necessary condition for asymptotic efficiency is

$$2\mu_2 \leq 2\lambda + \mu_1.$$

Examples

In Figure 1 we give two examples of tandem queues to show that the necessary conditions are not always satisfied.



(a) A $M|M|1$ tandem queue with $A \sim exp(\lambda)$, $B^{(1)} \sim exp(\mu_1)$ and $B^{(2)} \sim exp(\mu_2)$.

(b) A tandem queue with $A \sim U[0, 2]$, $B^{(1)} \sim exp(\mu_1)$ and $B^{(2)} \sim exp(\mu_2)$. Here $\lambda = \frac{1}{\mathbb{E}[A]} = 1$.

Figure 1: Two examples where the colored area shows for which parameter values the necessary conditions are *not* satisfied.

Future work

In this paper we gave conditions for an asymptotically efficient change of measure for a $G|G|1$ tandem queue. As there are combinations of parameters where the P&W change of measure can not be asymptotically efficient and where another exponential twist also does not work, a highly challenging next step is to find a state-dependent change of measure that is asymptotically efficient.

References

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