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THE POLE ZERO CANCELLATION PROBLEM IN ADAPTIVE POLE-PLACEMENT REVISITED

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Abstract

We revisit the pole-zero cancellation problem in adaptive pole-placement. It is well known that an on-line identified model in an adaptive pole-placement algorithm need not be controllable. Since the controller design usually requires that the model is controllable, this results in an ill defined control step. By exploiting the properties of the closed loop unfalsified models set, we derive an algorithm which does not suffer this difficulty. The analysis is presented in the case that the system belongs to the model class.

1 Introduction

In the adaptive control literature dealing with the problem of adaptive stabilization of a linear system belonging to some known class of linear systems, using pole placement control ideas, one faces the so called pole-zero cancellation problem, also referred to as the stabilizability problem. A model identified on line during the adaptation process need not be controllable, resulting in an ill defined control step.

In the literature one has dealt with this problem in a number of ways. In the now classical treatment [4] see also [9, Chapter 4], one simply assumes that this problem does not occur by imposing additional a priori knowledge. In [10] it is proven that in a stochastic framework the event of hitting a non-controllable model has zero probability and could therefore be neglected. This is unsatisfactory.

It is clear that in order to achieve a completely satisfactory treatment the adaptive algorithm itself has to somehow guarantee that the control step is well posed. The problem can be circumvented through modification of the parameterization of the model class used to represent the system [1, 2]. Alternatively, the construction of the model used for control is modified. Typical modifications include the injection of external excitation as e.g. introduced in [3] or using additional time variations in the feedback loop, [13, 12]. Alternatively, identifying an overparametrized representation of the plant and the control law as expounded in [6, 9] or the modification of the identification update via a search algorithm as advocated in [5, 8, 7]. Algorithms which demonstrably possess the property that the control design step is well defined turn out to be a lot more complex than the classical pole placement algorithm. All these approaches suffer from a more or less ad hoc nature. A less ad hoc discussion about the limitations that the stabilizability problem imposes on an adaptive algorithm may be found in [11].

Here we propose to change the adaptive control algorithm by generating two sequences of estimates. The standard projection algorithm is used to generate a sequence of estimates for identification purposes only. For the controller design we generate a second sequence of estimates on the basis of the standard sequence by searching in the set of controllable (stabilizable) *unfalsified* models in a neighborhood of the standard estimate. By searching in the set of unfalsified models for a model that has 'optimal controllability', the modified estimates keep their interpretation as an estimate of the system parameters. In fact, from an identification point of view, the original and the modified estimates are equivalent since the prediction error remains the same. We demonstrate that this modification indeed avoids the pole-zero cancellation problem i.e. the set of controllable (stabilizable) unfalsified models is never empty, moreover even the limit points of the algorithm yield models that are controllable (stabilizable). The advantage of our modification over others is that the modified estimate is equivalent to the non-modified from an identification point of view in the sense that the a posteriori prediction error remains zero. This greatly simplifies the overall analysis of the adaptive algorithm.

For reasons of simplicity we restrict our analysis to the projection algorithm. However, the idea may be incorporated into any other recursive identification scheme, e.g., least squares. In this situation one replaces the set of unfalsified models with the set of models that are *prediction error bounded* by the unmodified estimate. We call a

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model prediction error bounded by another model if the prediction error of the first does not exceed that of the second.

The modification remains non trivial. In our proposal the parameters used for control purposes are on line optimized with respect to a suitable controllability/stabilizability criterion. The implementation of the optimization step is computationally expensive.

The paper is organized as follows. In the next section we briefly recall the problem statement. Section 3 deals with the set of unfalsified models. Here we derive the important property that this set always contains a controllable model, provided that the true system is controllable. Section 4 deals with interpreting the result in the context of pole placement, here our emphasis shifts towards stabilizability rather than controllability. Finally, in Section 5 we formulate and discuss the modified adaptive pole placement law.

2 Problem statement

Let the true system be given by:

$$A^0(\sigma)y = B^0(\sigma)u \qquad (1)$$

Here u is the scalar input function and y is the scalar output. $(\sigma u)(k) = u(k+1)$. The system (1) is assumed to be controllable, i.e., the polynomials $A^0(\xi)$ and $B^0(\xi)$ are co-prime. The order of the system, the degree of the polynomial A^0, n, is assumed to be known, otherwise the coefficients of the polynomials $A^0(\xi)$ and $B^0(\xi)$ are constant but unknown. The objective is to control the system according to some specified criterion, e.g., pole assignment. Since the system parameters are unknown an indirect adaptive control strategy is used to achieve the objective. The polynomials $A^0(\xi)$, $B^0(\xi)$ are of the form:

$$\begin{array}{l} A^0(\xi) = \xi^n + a^0_{n-1}\xi^{n-1} + \cdots + a^0_0 \\ B^0(\xi) = b^0_{n-1}\xi^{n-1} + \cdots + b^0_0 \end{array} \qquad (2)$$

In the sequel we refer to the vector of coefficients of the pair of polynomials $(A(\xi), B(\xi))$ as (A, B). In this paper we focus the attention to the problem of how to obtain *controllable (stabilizable) estimates* of the unknown system on the basis of which we can design the controller in a *certainty equivalent* fashion. The latter is an important feature of our approach. The estimates are modified so as to improve the controllability without affecting the quality as an estimate. Other aspects of the controlled behavior, in particular the asymptotic behavior, are now well established in the literature and are not dealt with here.

3 The set of unfalsified models at time $k+1$

At time $k+1$ we have available to us $\phi(k) = (y(k), y(k-1), \ldots, y(k-n+1), u(k), u(k-1), \ldots, u(k-n+1))^T$ (the regressor vector) and $y(k+1)$ and the current best model estimate $(\hat{A}_k, \hat{B}_k)$. We want to choose the new estimate $(\hat{A}_{k+1}, \hat{B}_{k+1})$ in the set of *unfalsified models*, i.e., the set of parameter vectors consistent with the current data:

$$\begin{array}{rl} G_{k+1} := \{(A, B) \in \mathbb{R}^n \times \mathbb{R}^n \mid & \\ & y(k+1) + a_{n-1}y(k) + \cdots + a_0 y(k-n+1) \\ & b_{n-1}u(k) + \cdots + b_0 u(k-n+1)\} \end{array} \qquad (3)$$

In this paper we study the *projection algorithm*[4, 9]: $(\hat{A}_{k+1}, \hat{B}_{k+1})$ is the orthogonal projection of $(\hat{A}_k, \hat{B}_k)$ on G_{k+1}. The projection algorithm has two useful properties, namely that regardless the nature of the input, the sequence $\|(A^0, B^0) - (\hat{A}_k, \hat{B}_k)\|$ is monotonically non-increasing and $\lim_{k\to\infty} \|(\hat{A}_{k+1}, \hat{B}_{k+1}) - (\hat{A}_k, \hat{B}_k)\| = 0$. These two properties are instrumental in analyzing the asymptotic closed-loop behavior of the adaptive control scheme provided that the sequence of estimates $\{(\hat{A}_k, \hat{B}_k)\}$ remains bounded away from the set of non-controllable pairs. This assumption is not automatically satisfied. The problem is that G_{k+1} may contain parameter-values that correspond to uncontrollable models. On the other hand we may expect that the set of uncontrollable pairs will have a negligible intersection with G_{k+1}, since G_{k+1} contains at least one controllable pair, namely (A^0, B^0), and the set of uncontrollable pairs forms an algebraic set, see also [10]. This is indeed what we will prove. This property will subsequently enable us to find the *maximal controllable* pair $(\tilde{A}_{k+1}, \tilde{B}_{k+1})$ in a γ-neighborhood of $(\hat{A}_{k+1}, \hat{B}_{k+1})$ *within* G_{k+1}, to be used for controller design. Crucial for the usefulness of this modification is the property that the resulting sequence $\{(\tilde{A}_{k+1}, \tilde{B}_{k+1})\}$ stays bounded away from uncontrollable, thus avoiding the notorious stabilizability problem. So the key idea is to generate two sequences, $\{(\hat{A}_k, \hat{B}_k)\}$ and $\{(\tilde{A}_k, \tilde{B}_k)\}$, the first sequence is generated by the standard projection algorithm and is used for identification only, whereas $(\tilde{A}_k, \tilde{B}_k)$ lies somewhere in a γ neighborhood within G_{k+1} of $(\hat{A}_k, \hat{B}_k)$ and is used for controller design. Note that the boundedness of $\{(\hat{A}_k, \hat{B}_k)\}$ implies boundedness of $\{(\tilde{A}_k, \tilde{B}_k)\}$. In fact, as we will see in the next section, we use the sequence $\{(\tilde{A}_k, \tilde{B}_k)\}$ only for theoretical purposes since it is computationally too expensive to calculate it on-line. Therefore a third sequence $\{(\bar{A}_k, \bar{B}_k)\}$ that has similar properties but which is easier to compute will be introduced.

Observe that G_{k+1} is a $2n-1$ dimensional hyperplane containing the true parameter (A^0, B^0). For convenience of analysis we will consider the set of *all* such hyperplanes, therefore for every $x, y \in \mathbb{R}^n$ with $x^T x + y^T y = 1$, we define:

$$\begin{array}{l} G_{(x,y)} := \{(A, B) \in \mathbb{R}^n \times \mathbb{R}^n \mid \\ \quad x^T(A - A^0) + y^T(B - B^0) = 0\} \end{array} \qquad (4)$$

To measure the controllability of a pair (A, B) we calculate the determinant of the controllability matrix of the associated observer canonical representation of (A,B):

$$d(A,B) := \det(\mathcal{R}(\begin{bmatrix} 0 & & -a_0 \\ 1 & & \vdots \\ & \ddots & \vdots \\ 0 & 1 & -a_{n-1} \end{bmatrix}, \begin{bmatrix} b_0 \\ \vdots \\ \vdots \\ b_{n-1} \end{bmatrix})) \quad (5)$$

Where $\mathcal{R}(A,B)$ denotes the controllability matrix corresponding to the pair (A,B). The first property is that the set of controllable pairs in $G_{(x,y)}$ is open and dense in the induced topology of $G_{(x,y)}$, or otherwise stated, the set of uncontrollable pairs in $G_{(x,y)}$ is an algebraic set. We define $G^c_{(x,y)}$ as the set of controllable pairs in $G_{(x,y)}$.

Lemma 3.1 *The set $G^c_{(x,y)}$ is open and dense in $G_{(x,y)}$.*

Proof Without loss of generality we may assume that $y_{n-1} \neq 0$, then $(A,B) \in G_{(x,y)}$ if and only if $b_{n-1} = \frac{1}{y_{n-1}}[x^T(A^0 - A) + \tilde{y}^T(\tilde{B}^0 - \tilde{B})]$, where $\tilde{y}$ is obtained from y by dropping the last component, analogously for $\tilde{B}^0$ and $\tilde{B}$. This defines a complete linear affine parameterization of $G_{(x,y)}$ in terms of the first $2n-1$ components of (A,B). Define $\tilde{d} : \mathbb{R}^{2n-1} \to \mathbb{R}$ by:

$$\begin{aligned} &\tilde{d}(A,\tilde{B}) := d(A,B), \text{ with} \\ &b_{n-1} = \frac{1}{y_{n-1}}[x^T(A^0 - A) + \tilde{y}^T(\tilde{B}^0 - \tilde{B})] \end{aligned} \quad (6)$$

It follows that $\tilde{d}$ is a polynomial and that $(A,B) \in G^c_{(x,y)}$ if and only if $(A,B) \in G_{(x,y)}$ and $\tilde{d}(A,\tilde{B}) \neq 0$. Since $\tilde{d}(A^0,\tilde{B}^0) \neq 0$, we conclude that the zero set of $\tilde{d}$ is not the whole $G_{(x,y)}$ and since $\tilde{d}$ is a polynomial, it follows that $G^c_{(x,y)}$ is open and dense in $G_{(x,y)}$. ∎

What we would like to do next is the following. Given a pair $(A,B) \in G_{(x,y)}$, we define a closed γ-neighborhood of (A,B) in $G_{(x,y)}$, where γ is a fixed positive constant. Within this neighborhood we want to maximize $|d|$, the measure of controllability. Therefore we define the function $m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by:

$$m(x,y,A,B) := \max_{(C,D)\in G_{(x,y)}\cap B_\gamma(A,B)} |d(C,D)| \quad (7)$$

where $B_\gamma(A,B) = \{(C,D) \mid \|(A,B) - (C,D)\| \leq \gamma\}$. The following lemma states that by maximizing m over $B_\gamma(A,B)$ we will always find a controllable pair.

Lemma 3.2 *For all (x,y,A,B), we have that $m(x,y,A,B) > 0$.*

Proof This follows immediately from Lemma 3.1. ∎

Referring to our original problem, we do not only wish to obtain controllable pairs $(\tilde{A}_k,\tilde{B}_k)$, we also want to stay bounded away from non-controllable. In general the estimates $(\hat{A}_k,\hat{B}_k)$ belong to different $G_{(x_k,y_k)}$'s, so that it is not yet clear whether replacing $(\hat{A}_k,\hat{B}_k)$ by maximizers $(\tilde{A}_k,\tilde{B}_k)$ of m, will lead to a sequence for which $\liminf_{k\to 0} d(\tilde{A}_k,\tilde{B}_k) > 0$. Yet this is true. The way to see this is to show that m is a continuous positive valued function. As the sequence of estimates is bounded and because all $2n-1$ dimensional hyperplanes through (A^0,B^0) can be compactly parametrized with $x^Tx + y^Ty = 1$, it follows that $(x_k,y_k,\hat{A}_k,\hat{B}_k)$ remains in a compact set. It then follows that $m(x_k,y_k,\hat{A}_k,\hat{B}_k)$ remains bounded away from zero.

Lemma 3.3 *The function m defined by (7) is continuous.*

Proof The proof is elementary, therefore we leave it to the reader. ∎

Corollary 3.4 *There exists $\epsilon > 0$ such that for all k:*

$|d(\tilde{A}_k,\tilde{B}_k)| \geq \epsilon$

Proof Let $G_k = G_{(x_k,y_k)}$. Since $\{(\hat{A}_k,\hat{B}_k)\}$ is a bounded sequence, there exists a compact set $K \subset \mathbb{R}^n \times \mathbb{R}^n$ such that for all k $(\hat{A}_k,\hat{B}_k) \in K$. As a consequence we have that for all k:

$$m(x_k,y_k,\hat{A}_k,\hat{B}_k) \geq \min_{x^Tx+y^T=1,(A,B)\in K} m(x,y,A,B) > 0 \quad (8)$$

The second inequality in (8) follows from the continuity of m (Lemma 3.3) the positiveness of m (Lemma 3.2) and the fact that we minimize over a compact set. ∎

4 The maximization problem

In the previous section we have demonstrated that the construction of a sequence of estimates on the basis of which the controller design may be carried out, is in principle possible. In this section we indicate how to apply the ideas to adaptive pole placement. The remaining difficulty is of course to determine the 'maximal controllable' pair (A,B) in a neighborhood of the standard estimate intersected with G_k. Due to its highly non-linear nature, maximizing the determinant of the associated controllability matrix over a disk seems not to be a feasible idea. Instead we propose to search for a pair (A,B) such that the compensator $(R(\xi),S(\xi))$ that assigns the correct poles has minimal norm. Although we cannot prove this rigorously, we expect that the computations that are involved are easier and that in fact we will be able to come up with an analytical expression. To explain what we have in mind, let $P(\xi) = \xi^{2n-1} + p_{2n-2}\xi^{2n-2} + \cdots + p_0$ be the desired closed-loop characteristic polynomial. For every controllable pair $(A(\xi),B(\xi))$ there exists exactly one pair of polynomials $(R(\xi),S(\xi)$ of the form:

$$\begin{aligned} R(\xi) &= \xi^{n-1} + r_{n-2}\xi^{n-2} + \cdots + r_0 \\ S(\xi) &= s_{n-1}\xi^{n-1} + s_{n-2}\xi^{n-2} + \cdots + s_0, \end{aligned} \quad (9)$$

such that

$$A(\xi)R(\xi) - B(\xi)S(\xi) = P(\xi) \tag{10}$$

To make the dependence on $(A(\xi), B(\xi))$ explicit, we denote the polynomials $(R(\xi), S(\xi))$ such that (10) is satisfied by $R(A,B)(\xi)$ and $S(A,B)(\xi)$ respectively. We choose $(\bar{A}_k, \bar{B}_k)$ as:

$$\begin{aligned}(\bar{A}_k, \bar{B}_k) \in \arg\min\{ &\|(R(A,B), S(A,B)\| \mid \\ &(A,B) \in B_\gamma(A,B) \cap G_k\}\end{aligned} \tag{11}$$

The norm of $(R(A,B), S(A,B))$ is defined as ∞ if (10) has no solution for that particular pair (A,B). Notice that we are indeed minimizing over a non-empty set since the γ neighborhood in G_k contains controllable pairs. The definition of the sequence $(\bar{A}_k, \bar{B}_k)$, may not be complete because it is not all clear whether the minimization in (11) yields a unique answer. The precise implementation of the sequence $(\bar{A}_k, \bar{B}_k)$ is still under investigation. At this point in time we assume that uniqueness is guaranteed by some additional selection criterion. This is not essential for the sequel. Searching for a pair for which the pole placement compensator has minimal norm is of course not the same as searching for a pair that is maximally controllable. (In particular, in the event that the polynomials $A(\xi), B(\xi)$ and $P(\xi)$ have a common root, a non controllable model may yield a minimum norm controller.) However, both optimization methods have the same effect on the analysis of the adaptive scheme and are highly correlated. The following theorem shows that the sequence of controllers based on $(\bar{A}_k, \bar{B}_k)$ is bounded.

Theorem 4.1 *The sequence $(R(\bar{A}_k, \bar{B}_k), S(\bar{A}_k, \bar{B}_k))$ defined by (10,11) is bounded.*

Proof From Corollary 3.4 we know that $|d(\tilde{A}_k, \tilde{B}_k)| \geq \epsilon$ for some $\epsilon > 0$. Since moreover $(\tilde{A}_k, \tilde{B}_k)$ remains in a compact set, we conclude that the sequence of compensators based on $(\tilde{A}_k, \tilde{B}_k)$, $(R(\tilde{A}_k, \tilde{B}_k), S(\tilde{A}_k, \tilde{B}_k))$ is bounded. By the very definition of $(\bar{A}_k, \bar{B}_k)$ the statement follows. ∎

5 Application to adaptive pole assignment

Using the results of the previous sections we can now define an adaptive pole assignment algorithm without having to impose any assumptions about the controllability of the estimates.

1. Generate the sequence of estimates $(\hat{A}_k, \hat{B}_k)$ by the projection algorithm.
2. In parallel generate the sequence of modified estimates $(\bar{A}_k, \bar{B}_k)$ according to (11).
3. Calculate the controller parameters on the basis of $(\bar{A}_k, \bar{B}_k)$.

The analysis of the closed-loop behavior can be done by just mimicking the analysis in [14, 9]. To save space we do not provide the details here. The main property of the adaptively controlled system is that asymptotically the system behaves as if the poles are assigned while the controller parameters remain bounded. The proof of this statement goes along the same lines as that of its counterpart in [9, Theorems 4.6.5 and 4.7.5]. The main reason why we can indeed mimic the steps provided there, is that our modification, contrary to other methods, generates estimates that, from an identification point of view, are equivalent to the unmodified estimates.

6 Conclusion

A new method for the avoidance of pole-zero cancellation problems in adaptive pole placement has been introduced. The solution is in principle effective and feasible. Further work needs to be completed in the characterization of the computational cost of the proposed optimization based method.

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