DISCONTINUOUS GALERKIN METHODS FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper a survey is given of the important steps in the development of discontinuous Galerkin finite element methods for hyperbolic partial differential equations. Special attention is paid to the application of the discontinuous Galerkin method to the solution of the Euler equations of gas dynamics in time-dependent flows domains and to techniques which reduce the computational complexity of the DG method.

1. Introduction

Finite element methods provide a well developed mathematical framework for the solution of elliptic partial differential equations. For an extensive survey, see for instance (Ciarlet and Lions, 1991). This has motivated the use of finite element methods also for the solution of hyperbolic partial differential equations, but this is not as straightforward as for elliptic pde's. A significant source of problems is caused by the fact that hyperbolic partial differential equations can develop non-smooth solutions, even if the initial data are very smooth. The standard Galerkin finite element discretization for hyperbolic partial differential equations results in oscillatory solutions around discontinuities and complicated stabilization and discontinuity cap-

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turing operators are necessary. These methods have resulted in Petrov-Galerkin, streamline upwind Petrov Galerkin (SUPG), and more recently Galerkin Least Squares (GLS) finite element methods. Important contributions to the development of these methods can be found in the series of papers of (Masud and Hughes, 1997; Shakib, Hughes and Johan, 1991; Hansbo and Johnson, 1995; Johnson, Szepessy, and Hansbo, 1990) and the references therein.

Discontinuous Galerkin finite element methods provide an interesting alternative to these methods and combine many of the advantages of finite volume and finite element methods. The concept of discontinuous Galerkin finite element methods has been known for quite some time, and dates back to the early 70s for hyperbolic partial differential equations, but only recently have discontinuous Galerkin methods been applied to real applications. The recent interest in discontinuous Galerkin methods is motivated by the need to develop more accurate numerical discretizations on unstructured adaptive meshes, and to use higher order accurate discretizations for acoustic problems and the direct and large eddy simulation of turbulent flows. The growing interest in discontinuous Galerkin methods was one of the motivations to have a special conference on this topic in Newport, Rhode Island, 1999.

The recent interest in discontinuous Galerkin methods is motivated by some interesting features of this discretization technique. Discontinuous Galerkin finite element methods use a local polynomial representation of the solution and test functions in each element, without requiring continuity across element faces. This results in equations for the polynomial expansion coefficients of the solution which are uncoupled from neighboring elements. An important benefit of this approach is that for higher order accuracy it is not necessary to use complicated reconstruction algorithms to obtain pointwise data from cell averaged data for the flux calculation, as is necessary for finite volume methods. Discontinuous Galerkin methods result in a very local discretization and combine well with grid adaptation using local grid refinement, and parallel computations.

Discontinuous Galerkin methods make it possible to incorporate successful upwind schemes into finite element methods, because the discontinuity in the polynomial representation at the element faces can be interpreted as a Riemann problem, which is a Cauchy initial value problem with two discontinuous initial states. The use of (approximate) Riemann solvers makes it possible to incorporate important physical information into the numerical discretization and results in a robust upwind scheme, which has become very popular for upwind finite volume techniques. An excellent survey of these methods can be found in (Toro, 1997). Another interesting

feature of discontinuous Galerkin finite element discretizations is that they result in an element-wise conservative scheme, whereas SUPG and Galerkin least squares methods only result in globally conservative schemes.

The outline of this paper is as follows. First a survey is presented of the main aspects of discontinuous Galerkin finite element methods for hyperbolic partial differential equations. An attempt will be made to give a critical assessment of discontinuous Galerkin methods, but limited space prevents discussing all aspects in great detail. Recent detailed surveys of discontinuous Galerkin methods can be found in (Cockburn, 1999; Cockburn, Karniadakis and Shu, 2000; Barth, 1998). In the second part of this paper it will be demonstrated how the discontinuous Galerkin finite element method can be used to solve the Euler equations of gas dynamics in time-dependent flow domains using translating-rotating reference frames. This technique is useful to simulate aerodynamic problems, such as aircraft maneuver and propellers. The straightforward use of a discontinuous Galerkin method results, however, in a prohibitively expensive numerical scheme and special attention will be paid to techniques which significantly reduce the computational complexity of discontinuous Galerkin methods and result in a practical algorithm for computational fluid dynamics. The discussion in this paper is mainly limited to discontinuous Galerkin discretizations in space in combination with a Runge-Kutta time integration method. A discussion of the time-discontinuous and space-time discontinuous Galerkin methods is beyond the scope of the present paper.

2. Survey of Discontinuous Galerkin Methods

Discontinuous Galerkin finite element methods have been around for a long time. Their application to elliptic second and fourth order problems already started in the early 1960's, for a brief survey see (Oden, Babuska, and Baumann, 1998). The application to hyperbolic partial differential equations started much later with the work of (Reed and Hill, 1973) for the neutron transport equation for the flux of neutrons $u(\mathbf{x}) \in R^2$:

$$\begin{aligned} \operatorname{div}(\boldsymbol{\mu} u) + \sigma u &= f, \qquad \mathbf{x} \in \Omega \subset R^2, \\ u &= 0, \qquad \text{at } \partial_{-}\Omega, \end{aligned}$$

with $\boldsymbol{\mu}$ a constant vector, $\partial_{-}\Omega$ the part of the boundary $\partial\Omega$ with $\boldsymbol{\mu} \cdot \mathbf{n} < 0$, with $\mathbf{n}$ the outward normal vector at $\partial\Omega$, and $\sigma \in R$. This equation is linear and can be solved by a marching scheme if the elements follow the characteristic lines. The first theoretical analyses of the discontinuous Galerkin method were presented by (Lesaint and Raviart, 1974) and (Johnson and Pitkaränta, 1986) for the neutron transport and linear advection equations.

The application of discontinuous Galerkin methods to non-linear scalar hyperbolic conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad (x,t) \in R \times (0,T)$$

is more complicated because the marching scheme cannot be applied in general, since the direction of propagation $(1, f'(u))$ is part of the solution. Chavent and Salzano (1982) presented the P^0P^1-DG method for the one-dimensional water flooding problem with gravity. This method uses a piecewise constant polynomial representation in time and a linear representation in space. This explicit scheme can be easily extended to more spatial dimensions, but has a severe stability restriction and is not suitable for practical calculations.

A significant step forward was made by (Chavent and Cockburn, 1989) which introduced a monotonicity preserving slope limiter, similar to the MUSCL scheme of (van Leer, 1974), and the use of a Godunov flux to account for the discontinuities at element faces. They proved that the scheme satisfies a maximum principle and is total variation bounded in the means (TVBM). Linear stability analysis however shows that the numerical scheme is only stable if the CFL number satisfies the condition $\text{cfl} \cong O(h^{\frac{1}{2}})$ as the element length $h \downarrow 0$.

This time step limitation can be alleviated by using the total variation diminishing Runge-Kutta schemes introduced by (Shu and Osher, 1988). This has been the topic of a series of papers by (Cockburn and Shu, 1988; Cockburn and Shu, 1989a; Cockburn, Lin and Shu, 1989b; Cockburn, Hou and Shu, 1990; Cockburn and Shu, 1991) in which they studied the $RK\Lambda\Pi P^k$ method. The $RK\Lambda\Pi P^k$ method is a combination of a TVD Runge-Kutta time integration method with a discontinuous Galerkin finite element discretization in space using polynomials of degree k. The local projection limiter $\Lambda\Pi$ is used to ensure monotonicity of the solution. It was shown by (Cockburn and Shu, 1988; Cockburn and Shu, 1991) that this method is stable and second order accurate in space and time for $k = 1$ when applied to scalar conservation laws and if the CFL number is chosen in the range $\text{cfl} \in [0, \frac{1}{3}]$. Cockburn and Shu extended the analysis of the $RK\Lambda\Pi P^k$ method to polynomials of degree $k \geq 1$ in (Cockburn and Shu, 1989a) and showed that this method is TVBM and converges to the entropy solution for the scalar conservation law. The order of accuracy of the $RK\Lambda\Pi P^k$ method is $k+1$, except at critical points. Special attention is paid to the design of a TVB limiter in order to minimize the effects of the limiter in smooth parts of the flow. The use of a TVB limiter for practical applications is, however, not straightforward since it is difficult to

estimate the coefficients in the limiter, which depend on the derivatives of the solution and the mesh size.

The extension to one-dimensional systems is discussed in (Cockburn, Lin and Shu, 1989b), where the application to the Euler equations of gas dynamics is discussed in detail. Special attention was paid to the construction of the slope limiter and the most successful and robust approach uses the characteristic decomposition, which essentially decouples the equations in a system of scalar conservation laws. The theoretical framework of the discontinuous Galerkin discretization for multi-dimensional scalar conservation laws, in combination with the TVD Runge-Kutta time integration method, was firmly established in (Cockburn, Hou and Shu, 1990). Their main results can be summarized as:

- If the quadrature rules for the flux integrals over the element faces and volumes in the discontinuous Galerkin discretization are exact for polynomials of degree $2k+1$ and $2k$, respectively, then the difference between the discrete approximation $L_h(u_h, \gamma_h)$ of $-\mathrm{div}\,\mathbf{f}(u)$ (including the boundary operator γ_h) can be estimated as:

$$\| L_h(u_h, \gamma_h) + \mathrm{div}\,\mathbf{f}(u) \|_{L^\infty(\Omega)} \leq C h^{k+1} |\mathbf{f}(u)|_{W^{k+2,\infty}(\Omega)},$$

  with $\mathbf{f}$ the flux function in the multi-dimensional scalar conservation law $\partial_t u + \partial_x \mathbf{f}(u) = 0$ and $W^{k,p}(\Omega)$ the Sobolev seminorm, see for instance (Ciarlet and Lions, 1991).
- Let the coefficients α_{il} of the Runge-Kutta time discretization be positive and such that $\sum_{l=0}^{i-1} \alpha_{il} = 1$, for $i = 1, \cdots, k+1$. Set $w_h = u_h + \triangle t L_h(u_h, \gamma_h)$, and suppose that the following maximum principle is satisfied:

$$\bar{u}_h, \gamma_h \in [a, b] \Rightarrow \bar{w}_h \in [a - Mh^2, b + Mh^2], \tag{1}$$

  where M is some nonnegative parameter and a overbar denotes the element mean value, then:

$$\bar{u}_h^n \in [a - (k+1)mMh^2, b + (k+1)mMh^2], \qquad \text{for} m = 0, \cdots, n.$$

  if $\mathrm{cfl} \in [0, \mathrm{cfl}_0 / |\max_{i,l}\{|\frac{\beta_{il}}{\alpha_{il}}\}|]$. The coefficients cfl_0, a_0 and b_0 are defined as:

$$\begin{aligned}
\mathrm{cfl}_0 &= \sup_{n=1,\cdots,N; e \in \partial K; K \in \mathcal{T}_h} \triangle t \frac{|e|}{|K|} \| \mathbf{f}' \cdot \mathbf{n} \|_{L^\infty[a_0, b_0]}, \\
a_0 &= \inf_{\mathbf{x} \in \Omega, t \in (0, t^{n+1}), \mathbf{y} \in \partial\Omega} \{u_0(\mathbf{x}), \gamma(t, \mathbf{y})\}, \\
b_0 &= \sup_{\mathbf{x} \in \Omega, t \in (0, t^{n+1}), \mathbf{y} \in \partial\Omega} \{u_0(\mathbf{x}), \gamma(t, \mathbf{y})\},
\end{aligned}$$

with $|e|$ and $|K|$ the length of the edges and the volume of element K with unit outward normal vector $\mathbf{n}$. This result shows that the combination of a TVD Runge-Kutta time integration method and a slope limiter satisfying a TVB condition results in a stable discretization with a reasonable CFL number restriction. The use of a TVD Runge-Kutta time integration method is essential and results in a very robust discretization. If one uses Runge-Kutta schemes which are not TVD then the numerical scheme will suffer from severe stability restrictions.
- For a general class of triangulations (**B**-triangulations) a $\Lambda\Pi_h$ projection was defined which satisfies the maximum principle (1).

The series of articles of (Cockburn and Shu, 1988; Cockburn and Shu, 1989a; Cockburn, Lin and Shu, 1989b; Cockburn, Hou and Shu, 1990; Cockburn and Shu, 1991) give the discontinuous Galerkin finite element method in combination with a TVD-Runge Kutta time integration method a solid mathematical background. The first applications to two-dimensional problems can be found in (Bey and Oden, 1991), which applied the method to the supersonic flow about a compression corner and a forward facing step. Special attention was paid to the local projection limiter for higher order elements, but the limiter resulted in significant smearing of discontinuities. (Lin and Chin, 1993) used a second order discontinuous Galerkin discretization for the Euler and Navier-Stokes equations and applied the method to the transonic flow in a channel with a circular bump and an oscillating NACA 0012 profile.

The use of a discontinuous Galerkin finite element method to three-dimensional problems, in particular in computational fluid mechanics, is fairly recent. As mentioned in the introduction this is motivated by the very local behavior of the discontinuous Galerkin discretization, which makes it a good candidate for use on unstructured meshes. In a series of articles (van der Vegt, 1995a; van der Vegt and van der Ven, 1995b; van der Vegt and van der Ven, 1998a) demonstrated that the discontinuous Galerkin method can be used for practical aerodynamic calculations in combination with grid adaptation using local grid refinement. Calculations on the ONERA M6 wing and a delta wing, both in transonic flow, showed that significant improvement in the capturing of shocks and vortical structures can be obtained. Also, a face based dynamic data structure, suitable for an adaptive discontinuous Galerkin discretization, was discussed and implemented. A general discontinuous Galerkin method for arbitrary Lagrangian Eulerian hydrodynamics was developed by (Kershaw, Prasad, Shaw and Milovich, 1998). Special attention was paid to the development of a slope limiter and it was demonstrated that DG methods combine very well with object oriented programming.

Three-dimensional applications, both for the Euler and Navier-Stokes equations in combination with local grid refinement, are also demonstrated by (Baumann, 1997; Baumann and Oden, 1999), which studied the Shuttle Orbiter at a Mach number of 7.4. The most important contribution in this work is, however, the theoretical analysis of discontinuous Galerkin methods using broken spaces and the extension to the Navier-Stokes equations, see also (Oden, Babuska, and Baumann, 1998). This theoretical analysis is closely linked to the work of (Bey and Oden, 1996), which presented a very complete a posteriori error estimate for the neutron transport equation.

The application of discontinuous Galerkin finite element methods to three-dimensional problems requires, however, significant improvements in computational efficiency in order to be practical. The most significant computational expense is the use of Gauss quadrature rules for the calculation of the element face and volume integrals, because this requires for each Gauss quadrature point the calculation of the flux. Several approaches to alleviate this problem have been proposed. If only second order accuracy is required (van der Vegt and van der Ven, 1998a) demonstrated that using special element and face averages, in combination with the exact calculation of the geometric contributions, results in a practical second order accurate scheme. This method reduces the number of flux evaluations to one per element face and is analyzed in (van der Vegt and van der Ven, 1998a; van der Vegt and van der Ven, 2000; van der Ven and van der Vegt, 2000).

An alternative approach, which is not limited to second order accuracy, is followed by (Atkins and Shu, 1998). They expand the flux in terms of the basis functions used to represent the solution and integrate this representation analytically. This results in a significant reduction in the number of flux evaluations, but due to the non-linearity of the flux, and also because contributions of ρ^{-1} are required, it is not possible to obtain an exact representation for the flux. Also, the use of more advanced upwind schemes is non-trivial and Atkins and Shu therefore limit themselves to the relatively simple Lax-Friedrichs flux. The flux integrals for the Euler equations of gas dynamics can also be evaluated with the procedure used by (Lowrie, Roe, and van Leer, 1995) which uses the parameter vector $\mathbf{w} = \sqrt{\rho}(1, \mathbf{u}, H)^T$, with ρ the density, $\mathbf{u}$ the fluid velocity, and H the total enthalpy. The flux vector and conservative variables then are quadratic functions of $\mathbf{w}$. This method, however, requires a significant amount of storage for all the coefficients and is also difficult to use for advanced upwind schemes.

The application of the Runge-Kutta discontinuous Galerkin method to the Euler equations of gas dynamics in two-dimensions using a higher order accurate discretization is investigated by several people. Polynomials of degree $k = 1$ and $k = 2$ are used by (Cockburn and Shu, 1998a), which

show that the discontinuous Galerkin discretization is capable of capturing very fine details in the instability in the contact discontinuity inside a double Mach reflection and also behind a forward facing step. Discretizations up to fourth order accuracy were used by (Bassi and Rebay, 1997b), which also demonstrated that a superparametric representation of the elements at the boundary is necessary to prevent unphysical solutions. The use of higher order accurate discontinuous Galerkin methods for acoustics is investigated by (Hu, Hussaini, and Rasetarinera, 1999), which consider the dispersion and dissipation properties of the DG method using Fourier analysis.

Recently, (Lowrie and Morel, 1999) demonstrated that discontinuous Galerkin discretizations can be significantly more accurate than high resolution finite volume schemes for stiff hyperbolic systems. Discontinuous Galerkin finite element methods also combine very well with parallel computations as is demonstrated by (Biswas, Devine, and Flaherty, 1994; van der Ven and van der Vegt, 1997; van der Ven and van der Vegt, 1998).

Despite the significant progress made in the development of discontinuous Galerkin finite element methods there still are a number of important open issues. The two most important ones are the significant increase in memory use for higher order discretizations and the effects of the limiter. The use of a limiter prevents convergence to a steady state, because the limited slopes do not satisfy the discretized equations and tend to grow again, which results in limit cycle behavior. The limiter is also triggered by small disturbances and tends to be active in large parts of the flow field, causing a reduction in accuracy in smooth parts of the flow field. More efficient stabilization techniques are still a topic of active research. The memory use of DG methods rapidly increases with increasing order of the polynomials. It can be reduced by using more sophisticated basis functions, such as Legendre polynomials in one dimension, but for general elements this still is an open issue.

3. Discontinuous Galerkin Discretization of the Euler Equations in Moving Flow Domains

In the remaining part of this paper the discontinuous Galerkin finite element method will be applied to the Euler equations of gas dynamics in three-dimensional time-dependent flow domains. The Euler equations of gas dynamics are an important example of hyperbolic partial differential equations, and the use of a discontinuous Galerkin finite element method is non-trivial due to the occurrence of non-smooth solutions, such as shocks and contact discontinuities.

3.1. ALE WEAK FORMULATION OF THE EULER EQUATIONS

The calculation of the flow field in moving and deforming flow domains can be done efficiently using an arbitrary Lagrangian Eulerian (ALE) formulation. In this section we will discuss the ALE weak formulation for the Euler equations of gas dynamics using translating-rotating coordinate systems. This approach has important applications in several areas, such as aircraft maneuver and propellers. The ALE formulation, both for finite volume and finite elements methods, is analyzed in detail by (Lesoinne and Farhat, 1996). Special attention is paid to the geometric conservation law (GCL). The GCL was formulated by (Thomas and Lombard, 1979) and states that a uniform flow field should remain uniform under mesh movement and deformation. This imposes restrictions on the way grid velocities are evaluated in finite volume methods. For finite element discretizations the GCL condition can, however, be satisfied relatively easy if one calculates the element integrals with sufficient accuracy.

The Euler equations of gas dynamics at a point $(\mathbf{x}, t)$ fixed in space and time are defined as:

$$\frac{\partial \mathbf{U}(\mathbf{x},t)}{\partial t} + \frac{\partial \mathbf{F}_j(\mathbf{U}(\mathbf{x},t))}{\partial x_j} = 0,$$

where the vectors with conserved flow variables $\mathbf{U} : R^3 \times [t_0, T] \to R^5$, and the fluxes $\mathbf{F}_j$, $(j = 1, 2, 3)$; $\mathbf{F}_j : R^5 \to R^5$, are defined as:

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u_i \\ \rho E \end{pmatrix}; \quad \mathbf{F}_j = \begin{pmatrix} \rho u_j \\ \rho u_i u_j + p\delta_{ij} \\ u_j(\rho E + p) \end{pmatrix},$$

with $i \in \{1, 2, 3\}$, and ρ, p, E denote the density, pressure, and specific total energy, u_i the velocity component in the Cartesian coordinate direction x_i of the velocity vector $\mathbf{u} : \Omega \times [t_0, T] \to R^3$, and δ_{ij} the Kronecker delta symbol. The Euler equations are complemented with initial and boundary conditions:

$$\begin{aligned} \mathbf{U}(\mathbf{x}, t_0) &= \mathbf{U}_0(\mathbf{x}), && \mathbf{x} \in \Omega(t_0), \\ \mathbf{U}(\mathbf{x}, t) &= \mathcal{B}(\mathbf{U}, \mathbf{U}_w, t), && t \in [t_0, T],\ \mathbf{x} \in \partial\Omega(t), \end{aligned}$$

with $\mathbf{U}_w$ the boundary operator, and the equation of state for an ideal gas $p = (\gamma - 1)\rho(E - \frac{1}{2}u_i u_i)$, with γ the ratio of specific heats at constant pressure and volume.

Introduce a translating-rotating coordinate system $O_1X_1Y_1Z_1$ relative to the inertial coordinate system $O_0X_0Y_0Z_0$. The relation between between

points in both coordinate systems is:

$$\mathbf{x}(t) = \mathcal{C}(t)\ \mathbf{x}^{(1)} + \mathbf{r}(t), \tag{2}$$

with $\mathcal{C}$ the rotation matrix between the inertial and rotating reference frame, and $\mathbf{r}$ the position of the origin of the coordinate system $O_1X_1Y_1Z_1$ relative to $O_0X_0Y_0Z_0$. The superscript (1) indicates that a vector has components relative to the coordinate systems $O_1X_1Y_1Z_1$, otherwise the components are always with respect to $O_0X_0Y_0Z_0$. The Jacobian of the transformation between the two coordinate systems is equal to one. The flow domain $\Omega(t)$ becomes independent of time when expressed relative to the translating-rotating reference frame $O_1X_1Y_1Z_1$ and is denoted $\Omega^{(1)}$. The ALE weak formulation of the Euler equations can be obtained from the general ALE weak formulation discussed by (Lesoinne and Farhat, 1996) using the coordinate transformation (2):

Find a $\mathbf{U} \in [L^1(\Omega)]^5$, such that for all $\mathbf{W} \in [C^1(\Omega)]^5$, we have:

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega^{(1)}} \mathbf{W}^T(\mathbf{x}^{(1)})\mathbf{U}(\mathbf{x}^{(1)},t)d\Omega^{(1)} \\
&+\int_{\partial\Omega^{(1)}} \mathbf{W}^T(\mathbf{x}^{(1)})(n_k(\mathbf{x}^{(1)},t)\mathbf{F}_k(\mathbf{U}) - \mathbf{n}(\mathbf{x}^{(1)},t)\cdot\mathbf{s}(\mathbf{x}^{(1)},t)\ \mathbf{U}(\mathbf{x}^{(1)},t))d(\partial\Omega^{(1)}) \\
&-\int_{\Omega^{(1)}} \frac{\partial \mathbf{W}^T(\mathbf{x}^{(1)})}{\partial x_k^{(1)}}(\mathbf{F}_p(\mathbf{U})\mathcal{C}_{pk}(t) - s_k^{(1)}(\mathbf{x}^{(1)},t)\ \mathbf{U}(\mathbf{x}^{(1)},t))d\Omega^{(1)} = 0.
\end{aligned} \tag{3}$$

The components of the normal vector $\mathbf{n}$ and grid velocity $\mathbf{s} = d\mathbf{x}/dt$ in both coordinates systems are related as: $n_k^{(1)} = \mathcal{C}_{jk}n_j$ and $s_k^{(1)} = \mathcal{C}_{jk}s_j$. The use of a mixed formulation, with vectors having components in both the inertial and the translating-rotating reference frame, has as main benefit that the conservation form is maintained and no source terms, such as the Coriolis and centrifugal forces, appear.

3.2. DISCONTINUOUS GALERKIN FINITE ELEMENT DISCRETIZATION

The flow domain $\Omega^{(1)}$ is discretized using a triangulation $\mathcal{T}_h$ with elements K. As basic elements in $\mathcal{T}_h$ we use hexahedrons and each of the elements $K \in \mathcal{T}_h$ is related to the cubic master element $\hat{K} = [-1,1]^3$ by means of the isoparametric transformation F_K:

$$F_K : \hat{\mathbf{x}} = (\xi,\eta,\zeta)^T \in \hat{K} \rightarrow \mathbf{x}^{(1)} \in K : \mathbf{x}^{(1)}(\xi,\eta,\zeta) = \sum_{i=1}^{m_K} \mathbf{x}_i^{(1)}\chi_i(\hat{\mathbf{x}}), \tag{4}$$

with $m_K = 8$ for hexahedrons and χ_i the standard tri-linear basis functions. The discontinuous Galerkin discretization is now obtained using a sequence a function spaces:

- Define $P^k(\hat{K})$ as the space of polynomial functions of degree $\leq k$ on the master element $\hat{K}$: $P^k(\hat{K}) = \text{span}\{\hat{\phi}_j, j = 0, \cdots, M\}$. In this paper M is restricted to 3, so the four basis functions $\hat{\phi}_j$ are: $\hat{\phi}_j = 1, \xi, \eta, \zeta, (j = 0, \cdots, 3)$.
- Define $P^k(K)$ as the space of functions associated to functions in $P^k(\hat{K})$ through the mapping F_K: $P^k(K) = \text{span}\{\phi_j = \hat{\phi}_j \circ F_K^{-1}, j = 0, \cdots, 3\}$.
- Define $\mathbf{V}_h^1(K) = \{\mathbf{P}(K) = (p_1, \cdots, p_5)^T \mid p_i \in P^1(K)\}$.

It is important to realize that the polynomial expansions in each element are purely local without any connection to neighboring elements. This is the main difference of the discontinuous Galerkin finite element method with standard node based Galerkin methods. The approximate flow field $\mathbf{U}_h$ can be defined using the basis functions ϕ_j, but for the definition of the slope limiter and the multigrid convergence acceleration it is beneficial to split $\mathbf{U}_h$ into an element mean $\bar{\mathbf{U}}_h$ and a fluctuation $\tilde{\mathbf{U}}_h$. This can be accomplished by splitting $\mathbf{V}_h^1(K)$ into two spaces: $\bar{\mathbf{V}}_h^0(K)$ and $\tilde{\mathbf{V}}_h^1(K)$, such that:

$$\mathbf{V}_h^1(K) = \bar{\mathbf{V}}_h^0(K) \oplus \tilde{\mathbf{V}}_h^1(K),$$

with :

$$\bar{\mathbf{V}}_h^0(K) = \{\mathbf{P}(K) = (p_1, \cdots, p_5)^T | p_i \in P^0(K)\},$$

$$\tilde{\mathbf{V}}_h^1(K) = \{\mathbf{P}(K) = (p_1, \cdots, p_5)^T | \int_K p_i dK = 0, p_i \in P^1(K)\}.$$

The element mean flow field $\bar{\mathbf{U}}_h \in \bar{\mathbf{V}}_h^0(K) \otimes C^1[0,T]$ can now be defined as:

$$\bar{\mathbf{U}}_h(t) = \frac{1}{|K|} \int_K \mathbf{U}(\mathbf{x}^{(1)}, t) dK,$$

and the flow field fluctuations $\tilde{\mathbf{U}}_h \in \tilde{\mathbf{V}}_h^1(K) \otimes C^1[0,T]$ as:

$$\tilde{\mathbf{U}}_h(\mathbf{x}^{(1)}, t) = \mathbf{U}_h(\mathbf{x}^{(1)}, t) - \bar{\mathbf{U}}_h(t) = \sum_{m=1}^{3} \hat{\mathbf{U}}_m(K, t) \psi_m(\mathbf{x}^{(1)}),$$

with the basisfunctions $\psi_m \in P^1(K)$ defined as:

$$\psi_m(\mathbf{x}^{(1)}) = \phi_m(\mathbf{x}^{(1)}) - \frac{1}{|K|} \int_K \phi_m(\mathbf{x}^{(1)}) dK, \quad m = 1, 2, 3.$$

The element mean and fluctuating flow fields are orthogonal with respect to the L_2 inner product, and this relation can be used to simplify the discretized equations. The test functions $\mathbf{W}_h$ are also split into an element mean and a fluctuating part, $\bar{\mathbf{W}}_h \in \bar{\mathbf{V}}_h^0(K)$ and $\tilde{\mathbf{W}}_h \in \tilde{\mathbf{V}}_h^1(K)$.

The discontinuous Galerkin finite element discretization for the Euler equations in a translating-rotating reference frame is obtained if we introduce the polynomial representations for $\bar{\mathbf{U}}_h$ and $\tilde{\mathbf{U}}_h$, and the equivalent expressions for the test functions $\bar{\mathbf{W}}_h$ and $\tilde{\mathbf{W}}_h$, into the weak formulation (3):

$$|K| \frac{d\bar{U}_i(K)}{dt} = -\int_{\partial K} (n_k F_{ik}(\mathbf{U}_h) - n_k s_k U_i) d(\partial K) \equiv -\bar{R}_i(K) \tag{5}$$

$$M_{nm} \frac{d\tilde{U}_{mi}(K)}{dt} = -\int_{\partial K} \phi_n (n_k F_{ik}(\mathbf{U}_h) - n_k s_k U_i) d(\partial K) + \int_K \frac{\partial \phi_n}{\partial x_k^{(1)}} \left(F_{ip}(\mathbf{U}_h) \mathcal{C}_{pk}(t) - s_k^{(1)} U_i \right) dK + \frac{1}{|K|} M_{n0} \bar{R}_i(K), \tag{6}$$

with the matrix $M \in R^{3\times 3}$ defined as:

$$M_{nm}(K) = \int_K \phi_n \phi_m dK.$$

The equation for the element mean flow field $\bar{\mathbf{U}}_h$ (5) is identical to the equations for a finite volume discretization and is only weakly coupled with the equation for the flow field fluctuations (6). The mass matrix of the moving elements in the translating-rotating flow domain needs to be calculated only once since the grid is not deforming. An analytic expression for the mass matrix can be found in (van der Vegt and van der Ven, 1998a). For use in the time integration method and the multigrid convergence acceleration scheme it is beneficial to express (5) and (6) symbolically as:

$$\mathcal{M} \frac{d}{dt} \hat{\mathbf{U}}_m = \mathcal{R}_m(\mathbf{U}_h), \quad m = 0, \cdots, 3 \tag{7}$$

with $\mathcal{M} \in R^{4\times 4}$ defined as:

$$\mathcal{M} = \begin{bmatrix} |K| & 0 \\ 0 & M \end{bmatrix}.$$

4. Flux Calculation

The discontinuous Galerkin discretization results in expressions for the flow field $\mathbf{U}_h$ and test functions $\mathbf{W}_h$ which are discontinuous at element faces. This discontinuity can be interpreted as a Riemann problem from gas dynamics and can be used to give a suitable definition for the flux at the

element boundary ∂K. This is accomplished by replacing the flux function with a numerical flux. Any of the well-known (approximate) Riemann solvers, such as those from Godunov, Roe, Lax-Friedrichs or Osher, can be used in the definition of the numerical flux. For an overview of (approximate) Riemann solvers for gas dynamics, see (Toro, 1997). This procedure introduces upwinding into the discontinuous Galerkin formulation and does not require the design of elaborate stabilization and discontinuity capturing operators.

In this paper the Osher scheme is used because it is a very accurate upwind scheme with good shock capturing capabilities. More details can be found in (Osher and Chakravarthy, 1983). The main difference in calculating the Osher flux for moving elements in comparison with non-moving elements is that the eigenvalues used in determining the path integrals in phase space must be corrected for the grid velocity. More details can be found in (van der Vegt and van der Ven, 2000). The Osher flux for moving element faces can be split into the standard Osher flux for non-moving grids $\mathbf{H}$ and a part directly related to the grid velocity:

$$\mathbf{H}^c(\mathbf{U}_h^{\text{int}(K)}, \mathbf{U}_h^{\text{ext}(K)}) = \mathbf{H}(\mathbf{U}_h^{\text{int}(K)}, \mathbf{U}_h^{\text{ext}(K)}) - \mathbf{n} \cdot \mathbf{s}\, \mathbf{G}(\mathbf{U}_h^{\text{int}(K)}, \mathbf{U}_h^{\text{ext}(K)}),$$

where the flux vector $\mathbf{G}$ is obtained by replacing the normal flux vector $n_k \mathbf{F}_k(\mathbf{U})$ in the Osher flux with $\mathbf{U}_h$.

If only second order accuracy is required then the flux integration does not require Gauss quadrature rules and can be done using special face and element flux averages, after which the geometric contributions can be calculated analytically or numerically. This method was proposed and analyzed by (van der Vegt and van der Ven, 1998a; van der Vegt and van der Ven, 2000) and is briefly described below. The main benefit of this method is that only one flux calculation for each element face is necessary and relatively simple and exact relations for the geometric contributions are obtained, which automatically satisfy the GCL.

The integral of the numerical flux $\mathbf{H}^c$ over the element faces can be approximated using the following approximation to the flux integrals at the element boundary ∂K:

$$\int_{\partial K} \phi_n \mathbf{H}^c dK \cong \frac{1}{2}\left(\mathbf{F}_j(\bar{\mathbf{U}}_h^{\text{int}(K)}) + \mathbf{F}_j(\bar{\mathbf{U}}_h^{\text{ext}(K)})\right) \int_{\partial K} \phi_n n_j d(\partial K) -$$

$$\frac{1}{2}\left(\sum_\alpha \int_{\Gamma_\alpha} |\partial \hat{\mathbf{F}}| d\Gamma\right) \int_{\partial K} \phi_n d(\partial K) -$$

$$\mathbf{G}(\bar{\mathbf{U}}_h^{\text{int}(K)}, \bar{\mathbf{U}}_h^{\text{ext}(K)}) \int_{\partial K} \phi_n \mathbf{n} \cdot \mathbf{s}\, d(\partial K).$$

The integral along Γ_α uses a path in phase space between $\bar{\mathbf{U}}_h^{\mathrm{int}(K)}$ and $\bar{\mathbf{U}}_h^{\mathrm{ext}(K)}$. For details, see (Osher and Chakravarthy, 1983). The flow states $\bar{\mathbf{U}}_h^{\mathrm{int}(K)}$ and $\bar{\mathbf{U}}_h^{\mathrm{ext}(K)}$ in the element face are defined as:

$$\bar{\mathbf{U}}_h^{\mathrm{int}(K)} = \frac{1}{|K|}\Big(\bar{\mathbf{U}}_h(K) + \sum_{m=1}^{3} \hat{\mathbf{U}}_{m,K} \int_{\partial K} \psi_{m,K} d(\partial K)\Big)$$

$$\bar{\mathbf{U}}_h^{\mathrm{ext}(K)} = \frac{1}{|K|}\Big(\bar{\mathbf{U}}_h(K') + \sum_{m=1}^{3} \hat{\mathbf{U}}_{m,K'} \int_{\partial K} \psi_{m,K'} d(\partial K)\Big),$$

with K' the index of the element connected to element K at the ∂K. The suffices K and K' of $\psi_m(\mathbf{x}^{(1)})$ refer to the limit of $\psi_m(\mathbf{x}^{(1)})$ taken from the interior and exterior of element K at the boundary ∂K, respectively.

Analytic expressions for the element face moments $\int_{\partial K} \phi_n n_j dK$ are given in (van der Vegt and van der Ven, 1998a). The flux contribution at element faces which is related to the body motion is evaluated using the representation for the velocity of a point in the moving reference frame:

$$\mathbf{s}(t) = \frac{d\mathbf{x}}{dt} = \mathbf{v}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_b = \mathbf{v}^{(1)}(t) + \boldsymbol{\omega}^{(1)}(t) \times \mathbf{r}_b^{(1)},$$

with $\mathbf{v}$ the velocity vector of the origin of the moving reference frame, $\boldsymbol{\omega}$ the angular velocity vector of the moving reference frame, and $\mathbf{r}_b = \mathbf{x} - \mathbf{x}_b$ a vector pointing from the center of rotation $\mathbf{x}_b$ in coordinate system $O_1X_1Y_1Z_1$ to a point $\mathbf{x}$ in this reference system. Introducing the representation for the grid velocity $\mathbf{s}$ into the approximation of the element face flux integrals we obtain:

$$\int_{\partial K} \phi_n \mathbf{n}\cdot\mathbf{s}\, d(\partial K) = \Big(\mathbf{v}_0^{(1)} \cdot \int_{\partial K} \phi_n \mathbf{n}^{(1)} d(\partial K) + \boldsymbol{\omega}^{(1)} \cdot \int_{\partial K} (\mathbf{r}_b^{(1)} \times \mathbf{n}^{(1)}) \phi_n d(\partial K)\Big).$$

The volume flux integrals in (6) are calculated analogously:

$$\int_K \frac{\partial \phi_n}{\partial \mathbf{x}^{(1)}} \cdot \mathbf{s}^{(1)} \mathbf{U}_h dK \cong \bar{\mathbf{U}}_h\Big(\mathbf{v}_0^{(1)} \cdot \int_K \frac{\partial \phi_n}{\partial \mathbf{x}^{(1)}} dK + \boldsymbol{\omega}^{(1)} \cdot \int_K \mathbf{r}_b^{(1)} \times \frac{\partial \phi_n}{\partial \mathbf{x}^{(1)}} dK\Big).$$

The advantage of this splitting is that we can calculate the contribution of the grid velocity to the fluxes exactly and therefore automatically satisfy the geometric conservation law. The geometric integrals can be calculated analytically or can be found in (van der Vegt, 1998b).

5. Slope Limiter

The discontinuous Galerkin finite element method requires a slope limiting algorithm to obtain monotone solutions. A local projection limiter, which

guarantees monotonicity for multi-dimensional scalar equations, was derived by (Cockburn, Hou and Shu, 1990). Several alternative multi-dimensional limiters have also been proposed which attempt to minimize the loss in accuracy caused by the limiting process, but they are difficult to apply to hexahedral elements. In this paper the Barth and Jespersen slope limiter with the modifications proposed by (Venkatakrishnan, 1995), is used, because it results in a robust formulation and is easy to apply to hexahedral elements. The limiting operator $\Pi_h(\mathbf{U}_h) \in [0,1]$ is applied directly to the flow field fluctuations $\tilde{\mathbf{U}}_h$, because the element mean flow field $\bar{\mathbf{U}}_h$ remains unchanged by the limiter: $\Pi_h(\bar{\mathbf{U}}_h) = \bar{\mathbf{U}}_h$.

6. Implicit Time Integration

Calculations of unsteady flows frequently suffer from a large disparity between the physically relevant time scales and the time step limitations imposed by the stability constraints of explicit time integration methods. These limitations can be alleviated for the discontinuous Galerkin method presented in this paper by using a three-point backward implicit time integration method to integrate the semi-discrete equations (7) in time. The resulting set of non-linear equations is solved by augmenting these equations with a pseudo-time derivative of the flow field expansion coefficients $\partial\tilde{\mathbf{U}}_m/\partial\tau$ and marching the solution to a steady state in pseudo-time:

$$\frac{\partial\hat{\mathbf{U}}_m}{\partial\tau} = \mathcal{L}_m(\hat{\mathbf{U}}_m, \hat{\mathbf{U}}_m^n, \hat{\mathbf{U}}_m^{n-1}), \quad m = 0, \cdots, 3, \tag{8}$$

with:

$$\mathcal{L}_m(\hat{\mathbf{U}}_m, \hat{\mathbf{U}}_m^n, \hat{\mathbf{U}}_m^{n-1}) = \mathcal{R}_m(\mathbf{U}_h) - \mathcal{M}\left(\tfrac{3}{2}\hat{\mathbf{U}}_m - 2\hat{\mathbf{U}}_m^n + \tfrac{1}{2}\hat{\mathbf{U}}_m^{n-1}\right)/\triangle t,$$

and $\triangle t$ the global time step. At steady state the new solution then is equal to $\hat{\mathbf{U}}_m^{n+1}$. This technique was first proposed by (Jameson, 1991), and made unconditionally stable by (Melson, Sanetrik and Atkins, 1993) for the Jameson scheme.

The equations in pseudo-time τ are integrated using the third order accurate TVD Runge-Kutta time integration method from (Shu and Osher, 1988):

1. Set $\hat{\mathbf{U}}_m^{(0)} = \hat{\mathbf{U}}_m^n$
2. For $i = 1, 2, 3$ compute the intermediate Runge-Kutta stages:

$$\hat{\mathbf{U}}_m^{(i)} = \Pi_h\left(\sum_{l=0}^{i-1}\left(\alpha_{il}\mathbf{U}_m^{(l)} + \beta_{il}\triangle\tau\mathcal{L}_m(\hat{\mathbf{U}}_m^{(l)}, \hat{\mathbf{U}}_m^n, \hat{\mathbf{U}}_m^{n-1})\right)\right)$$

3. $\hat{\mathbf{U}}_m^{n+1} = \hat{\mathbf{U}}_m^3$

The coefficients in the TVD Runge-Kutta scheme are equal to: $\alpha_{10} = 1, \alpha_{20} = \frac{3}{4}, \alpha_{21} = \frac{1}{4}, \alpha_{30} = \frac{1}{3}, \alpha_{32} = \frac{2}{3}, \beta_{10} = 1, \beta_{21} = \frac{1}{4}, \beta_{32} = \frac{2}{3}$ and zero otherwise. This Runge-Kutta scheme is stable for CFL numbers less than one, but all calculations discussed in this paper have been done using a CFL number of 0.7. The convergence to steady state is accelerated using the FAS multigrid scheme proposed by (Brandt, 1977). The restriction operator uses volume weighted averages and the prolongation operator pure injection. The FAS algorithm is only applied to the element mean flow field. The application of the FAS scheme to the equations for the flow field fluctuations has not been successful, because the corrections in the flow field fluctuation expansion coefficients caused by the limiter disturb the multigrid process. Also, the restriction and prolongation operators are considerably more complicated for the flow field fluctuations for hexahedral elements.

7. Applications

The discontinuous Galerkin finite element discretization is tested by simulating the unsteady flow field about a generic wing, called simple strake wing (SIS). The wing consists of two parts, an outer part and a strake connected to it and represents a generic model for a fighter aircraft. The mesh used for the calculations consists of 189184 grid points with hexahedral elements. The wing is oscillating in pitch with a mean angle of attack $\alpha = 6.157°$ and amplitude $\triangle\alpha = 2.141°$. The oscillating frequency is $\omega = 0.241$ and the free stream Mach number $M_\infty = 0.899$. The simulations were started by first calculating the steady flow field at an angle of attack $\alpha = 8.298°$, see Fig. 1. The flow field has a shock close to the trailing edge and a lambda shock from the junction of the strake with the leading edge of the wing to the wing tip. Both the strake and outer wing generate a vortex system, which merge in the far wake. This vortex system produces significant additional lift. The hysteresis curves of both the lift coefficient C_L and the drag coefficient C_D are plotted in Fig. 2. In these figures also the results of experiments and the effect of different time steps (20 and 40 time steps per period), mesh size (one time coarsened=C and fine=M) in the simulation are plotted.

Figure 2 shows that the hysteresis effects are small for this specific condition. The comparison of the induced drag force with the experiments is good, whereas the lift force in the calculations is slightly overpredicted, as can be expected from inviscid flow simulations. Fig. 3 shows the zeroth harmonic and the real and imaginary part of the first harmonic of the pressure coefficient at the spanwise location $y/b = 0.5$, with b the wing

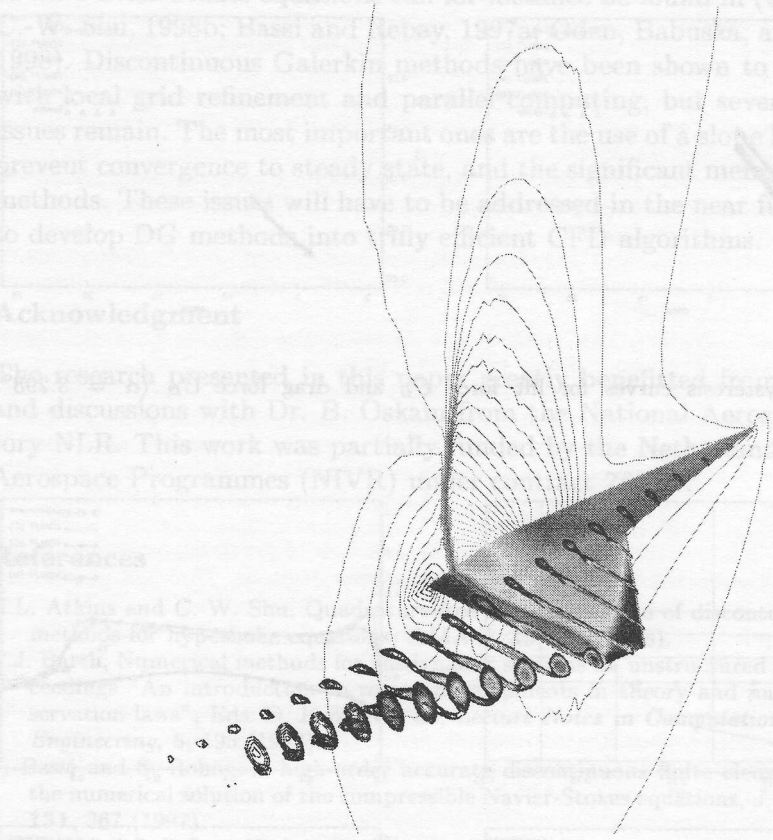

Figure 1. Steady pressure field on Simple Strake wing and total pressure loss in the wake ($\alpha = 8.298°$, $M_\infty = 0.899$).

span, for both the experiments and the numerical results for different time steps $\triangle t$ and mesh size (one time coarsened=C and fine=M). Apart from the position of the shock the agreement is generally good. These simulations show that discontinuous Galerkin finite element discretization can be used for practical aerodynamic calculations, which until now have been done only in a few cases, see (van der Vegt, 1995a; van der Vegt and van der Ven, 1995b; van der Vegt and van der Ven, 1998a; Baumann, 1997; Baumann and Oden, 1999).

8. Conclusions

Discontinuous Galerkin methods combine many of the nice features of finite volume and finite element methods and can be developed into algorithms

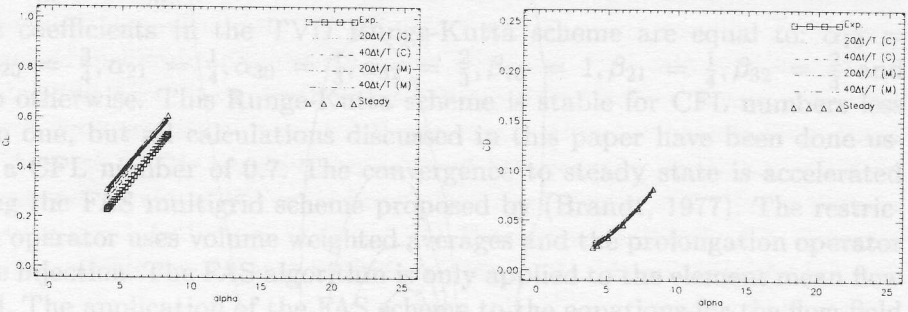

Figure 2. Hysteresis curves for lift force C_L and drag force C_D ($\alpha = 8.298°$, $M_\infty = 0.899$).

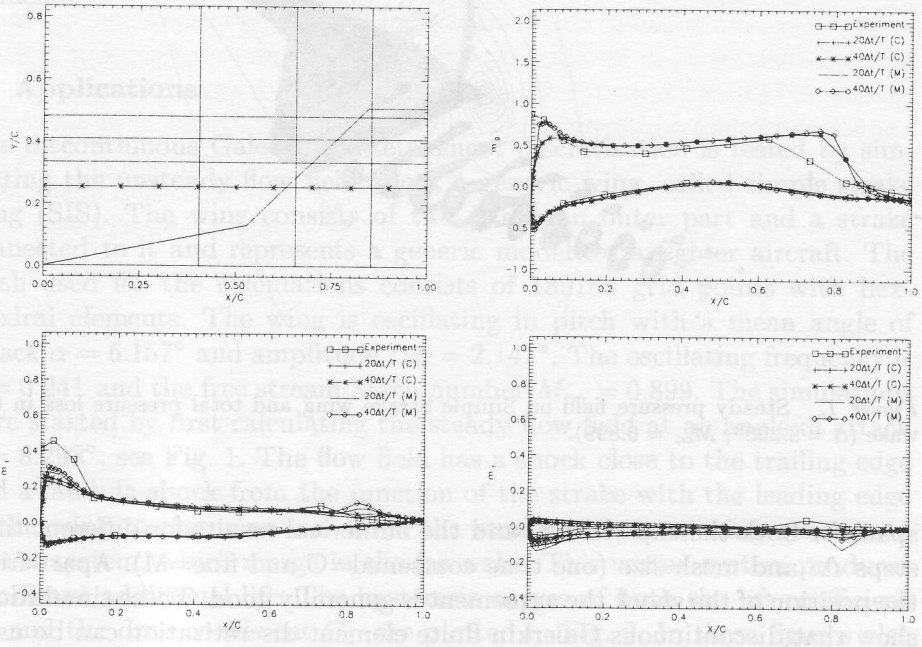

Figure 3. Position of cross-section and zeroth harmonic and real and imaginary part of first harmonic of the pressure coefficient C_p, ($\alpha = 6.157°$, $\Delta\alpha = 2.141°$, $\omega = 0.241$, $M_\infty = 0.899$).

suitable for complicated CFD calculations. In this paper only hyperbolic partial differential equations have been discussed, but presently there is also a significant development of DG methods for elliptic and parabolic partial differential equations. Extensions of the DG method to the solution

of the Navier-Stokes equations can for instance be found in (Cockburn and C.-W. Shu, 1998b; Bassi and Rebay, 1997a; Oden, Babuska, and Baumann, 1998). Discontinuous Galerkin methods have been shown to combine well with local grid refinement and parallel computing, but several important issues remain. The most important ones are the use of a slope limiter, which prevent convergence to steady state, and the significant memory use of DG methods. These issues will have to be addressed in the near future in order to develop DG methods into truly efficient CFD algorithms.

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