Two Solution Concepts for TU Games with Cycle-Free Directed Cooperation Structures*

Anna Khmelnitskaya[1] **and Dolf Talman**[2]

[1] *SPb Institute for Economics and Mathematics Russian Academy of Sciences, 1 Tchaikovsky St., 191187, St.Petersburg, Russia,*
E-mail: `a.khmelnitskaya@math.utwente.nl`

[2] *CentER, Department of Econometrics & Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands,*
E-mail: `talman@uvt.nl`.

Abstract. For arbitrary cycle-free directed graph games tree-type values are introduced axiomatically and their explicit formula representation is provided. These values may be considered as natural extensions of the tree and sink values as has been defined correspondingly for rooted and sink forest graph games. The main property for the tree value is that every player in the game receives the worth of this player together with his successors minus what these successors receive. It implies that every coalition of players consisting of one of the players with all his successors receives precisely its worth. Additionally their efficiency and stability are studied. Simple recursive algorithms to calculate the values are also provided. The application to the water distribution problem of a river with multiple sources, a delta and possibly islands is considered.

Keywords: TU game, cooperation structure, Myerson value, efficiency, deletion link property, stability

JEL Classification Number: C71
Mathematics Subject Classification 2000: 91A12, 91A43

1. Introduction

In standard cooperative game theory it is assumed that any coalition of players may form. However, in many practical situations the collection of coalitions that can be formed is restricted by some social, economical, hierarchical, communication, or technical structure. The study of games with transferable utility and limited cooperation introduced by means of communication graphs was initiated by Myerson (Myerson, 1977). In this paper we restrict our consideration to the class of cycle-free digraph games in which the players are partially ordered and the communication via bilateral agreements between players is represented by a directed graph without directed cycles. A cycle-free digraph cooperation structure allows modeling of various flow situations when several links may merge at a node, while other links split at a node into several separate ones.

It is assumed that a directed link represents a one-way communication situation. This restricts the set of coalitions that can be formed. In the paper we consider two

* The research was supported by NWO (The Netherlands Organization for Scientific Research) grant NL-RF 047.017.017. The research was partially done during Anna Khmelnitskaya 2008 research stay at the Tilburg Center for Logic and Philosophy of Science (TiLPS, Tilburg University) whose hospitality and support are highly appreciated as well.

different scenarios possible for controlling cooperation in case of directed communication. First it is assumed that players can only control their successors and if in the underlying graph structure a player is a successor of another player and both players are members of some coalition, then also within this coalition the former player must be a successor of the last player. Another scenario assumes that players can only control their predecessors and nobody accepts that one of his predecessors becomes his equal partner if a coalition forms.

We introduce tree-types values for cycle-free digraph games axiomatically and provide their explicit formula representation. On the class of cycle-free digraph games the (root-)tree value is completely characterized by maximal-tree efficiency (MTE) and successor equivalence (SE), where a value is maximal-tree efficient if for every root of the graph, being a player without predecessors, it holds that the payoff for him and his successors is equal to the worth they can get by their own, and a value is successor equivalent if when a link towards a player is deleted this player and all his successors will get the same payoff. It implies that every player receives what he contributes when he joins his successors in the graph and that the total payoff for any player together with all his successors is equal to the worth they can get by their own. Similarly, we introduce the sink-tree value which on the class of cycle-free digraph games is completely characterized by maximal-sink efficiency (MSE) and predecessor equivalence (PE). At the sink value every player receives what he contributes when he joins his predecessors in the graph and the total payoff for this player and all his predecessors is equal their worth. It is worth to emphasize that both values should not be considered as personal payment by one player to another one (the boss to his subordinate) but as distribution of the total worth according to the proposed scheme. We also provide simple recursive computational methods for computing these values and study their efficiency and when possible their stability. The introduced tree and sink values for arbitrary cycle-free digraph games may be considered as natural extensions of the tree and sink values defined correspondingly for rooted and sink forest digraph games (cf. (Demange, 2004), (Khmelnitskaya, 2010)). Furthermore, we extend the Ambec and Sprumont line-graph river game model of sharing a river (Ambec and Sprumont, 2002) to the case of a river with multiple sources, a delta and possibly islands by applying the results obtained to this more general problem of sharing a river among different agents located at different levels along the river bed restated in terms of a cycle-free digraph game.

The structure of the paper is as follows. Basic definitions and notation are introduced in Sect. 2.. Sect. 3. provides an axiomatic characterization of the tree value for a rooted-tree digraph game via component efficiency and subordinate equivalence. In Sect. 4. we discuss application to the water distribution problem of a river with multiple sources, a delta and possibly islands.

2. Preliminaries

A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1,\dots,n\}$ is a finite set of n, $n \geq 2$, players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition* and the associated real number $v(S)$ represents the *worth* of coalition S. The set of TU games with fixed player set N we denote $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when we refer to a

TU game. A game $v \in \mathcal{G}$ is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$, and $v \in \mathcal{G}$ is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$. A *value* on a subset $\mathcal{G}$ of $\mathcal{G}_N$ is a function $\xi \colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every game $v \in \mathcal{G}$ a vector $\xi(v) \in \mathbb{R}^N$; the number $\xi_i(v)$ represents the *payoff* to player i, $i \in N$, in the game v. In the sequel we use standard notation $x(S) = \sum_{i \in S} x_i$, $x_S = (x_i)_{i \in S}$ for any $x \in \mathbb{R}^N$ and $S \subseteq N$, $|A|$ for the cardinality of a given set A, and omit brackets when writing one-player coalitions such as i instead of $\{i\}$, $i \in N$.

A payoff vector $x \in \mathbb{R}^N$ in a game $v \in \mathcal{G}$ is *efficient* if it holds that $x(N) = v(N)$. We also say that a coalition $S \subseteq N$ is *efficient* in a game $v \in \mathcal{G}$ with respect to a payoff vector $x \in \mathbb{R}^N$ if $x(S) = v(S)$.

The *core* (Gillies, 1953) of a game $v \in \mathcal{G}_N$ is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S), \text{ for all } S \subseteq N\}.$$

For a game $v \in \mathcal{G}_N$, together with the core, we may consider the *weak core* defined as

$$\tilde{C}(v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N),\ x(S) \geq v(S), \text{ for all } S \subsetneq N\}.$$

A value ξ on a subset $\mathcal{G}$ of $\mathcal{G}_N$ is *stable* if for any game $v \in \mathcal{G}$ it holds that $\xi(v) \in C(v)$, and a value ξ on $\mathcal{G}$ is *weakly stable* if for any game $v \in \mathcal{G}$ it holds that $\xi(v) \in \tilde{C}(v)$.

The *cooperation structure* on the player set N is specified by a graph, directed or undirected, on N. An *undirected graph* on N consists of a set of nodes, being the elements of N, and a collection of unordered pairs of nodes $\Gamma \subseteq \Gamma^c_N = \{\{i,j\} \mid i,j \in N,\ i \neq j\}$, where Γ^c_N is the complete undirected graph without loops on N and an unordered pair $\{i,j\} \in \Gamma$ is a *link* between $i, j \in N$. A *directed graph*, or *digraph*, on N is given by a collection of directed links $\Gamma \subseteq \bar{\Gamma}^c_N = \{(i,j) \mid i,j \in N, i \neq j\}$. A subset Γ' of a graph Γ on N is a *subgraph* of Γ. For a subgraph Γ' of Γ, $S(\Gamma') \subseteq N$ is the set of nodes in Γ', i.e., $S(\Gamma') = \{i \in N \mid \exists j \in N \colon \{i,j\} \in \Gamma'\}$, if Γ is undirected, and $S(\Gamma') = \{i \in N \mid \exists j \in N \colon \{(i,j),(j,i)\} \cap \Gamma' \neq \emptyset\}$, if Γ is a digraph. For a graph Γ on N and a coalition $S \subseteq N$, the *subgraph of Γ on S* is the graph $\Gamma|_S = \{\{i,j\} \in \Gamma \mid i,j \in S\}$, if Γ is undirected, and $\Gamma|_S = \{(i,j) \in \Gamma \mid i,j \in S\}$, if Γ is directed.

In a graph Γ on N a sequence of different nodes $p = (i_1, \ldots, i_r)$ is a *path* in Γ from node i_1 to node i_r if $r \geq 2$ and for $h = 1, \ldots, r-1$ it holds that $\{i_h, i_{h+1}\} \in \Gamma$ when Γ is undirected and $\{(i_h, i_{h+1}), (i_{h+1}, i_h)\} \cap \Gamma \neq \emptyset$ when Γ is directed. In a digraph Γ on N a path $\boldsymbol{p} = (i_1, \ldots, i_r)$ is a *directed* path from node i_1 to node i_r if for all $h = 1, \ldots, r-1$ it holds that $(i_h, i_{h+1}) \in \Gamma$. For a digraph Γ on N and any $i, j \in N$ we denote by $\boldsymbol{P}_\Gamma(i,j)$ the set of all directed paths from i to j in Γ. Any node i of a (directed) path p we denote as an element of p, i.e., $i \in p$. Moreover, when for a directed path $\boldsymbol{p}$ in a digraph Γ we write $(i,j) \in \boldsymbol{p}$, we assume that i and j are consecutive nodes in $\boldsymbol{p}$. For any set P of (directed) paths, by $S(P) = \{i \in p \mid p \in P\}$ we denote the set of nodes determining the paths in P. A directed link $(i,j) \in \Gamma$ for which there exists a directed path $\boldsymbol{p}$ in Γ from i to j such that $\boldsymbol{p} \neq (i,j)$ is *inessential*, otherwise (i,j) is an *essential* link. A directed path $\boldsymbol{p}$ is a *proper path* if it contains only essential links.

In a graph Γ on N, undirected or directed, a sequence of nodes $(i_1, \ldots, i_{r+1})$ is a *cycle* if $r \geq 3$, $(i_1, \ldots, i_r)$ and $(i_2, \ldots, i_{r+1})$ are paths and $i_1 = i_{r+1}$. In a digraph Γ a sequence of nodes $(i_1, \ldots, i_{r+1})$ is a *directed cycle* if $r \geq 2$, $(i_1, \ldots, i_r)$ and

$(i_2, \ldots, i_{r+1})$ are directed paths, and $i_1 = i_{r+1}$. An undirected graph Γ is *cycle-free* if it contains no cycles. A digraph Γ on N is *cycle-free* if it contains no directed cycles, i.e., no node is a successor of itself. A digraph Γ on N is *strongly cycle-free* if it is cycle-free and contains no cycles. Remark that in a strongly cycle-free digraph all links are essential.

For a directed link $(i,j) \in \Gamma$, i is the *origin* and j is the *terminus*, i is a *superior* of j and j is a *subordinate* or *follower* of i. If a directed link (i,j) is essential, then j is a *proper subordinate* of i and i is a *proper superior* of j. All nodes having the same superior in Γ are called *brothers*. Besides, for $i, j \in N$, j is a *(proper) successor* of i and i is a *(proper) predecessor* of j if there is a directed (proper) path from i to j. For $i \in N$, we denote by $P_\Gamma(i)$ the set of all predecessors of i in Γ, by $O_\Gamma(i)$ the set of all superiors of i in Γ, by $O^*_\Gamma(i)$ the set of all proper superiors of i, by $F_\Gamma(i)$ the set of all subordinates of i in Γ, by $F^*_\Gamma(i)$ the set of all proper subordinates of i, by $S_\Gamma(i)$ the set of all successors of i in Γ, and by $B_\Gamma(i)$ the set of all brothers of i in Γ. Moreover, for $i \in N$, we define $\bar{P}_\Gamma(i) = P_\Gamma(i) \cup i$, $\bar{S}_\Gamma(i) = S_\Gamma(i) \cup i$, and $\bar{B}_\Gamma(i) = B_\Gamma(i) \cup i$. A coalition $S \subseteq N$ is a *full successors set in* Γ, if $S = \bar{S}_\Gamma(i)$ for some $i \in N$, and is a *full predecessors set in* Γ, if $S = \bar{P}_\Gamma(i)$ for some $i \in N$. A node $i \in N$ having no superior in Γ, i.e., $O_\Gamma(i) = \emptyset$, is a *root* in Γ. A node $i \in N$ having no subordinate in Γ, i.e., $F_\Gamma(i) = \emptyset$, is a *leaf* in Γ. For any $S \subseteq N$ denote by $R_\Gamma(S)$ the set of all roots in $\Gamma|_S$ and by $L_\Gamma(S)$ the set of all leaves in $\Gamma|_S$. For simplicity of notation, for a digraph Γ on N and $i \in N$, by Γ^i we denote the subgraph $\Gamma|_{\bar{S}_\Gamma(i)}$ and by Γ_i the subgraph $\Gamma|_{\bar{P}_\Gamma(i)}$. Given a digraph Γ on N and $i \in N$, the *in-degree* of i is defined as $d_\Gamma(i) = |O^*_\Gamma(i)|$ and the *out-degree* of i as $\tilde{d}_\Gamma(i) = |F^*_\Gamma(i)|$, while for any $i \in N$ and $j \in S_\Gamma(i)$ the *in-degree of* j *with respect to* i is equal to $d^i(j) = |O^*_{\Gamma^i}(j)|$ and for any $j \in P_\Gamma(i)$ the *out-degree of* j *with respect to* i is equal to $d_i(j) = |F^*_{\Gamma_i}(j)|$. Given a digraph Γ on N, $i \in N$ and $j \in P_\Gamma(i)$, a node $h \in S(\boldsymbol{P}_\Gamma(i,j))$ such that $d^i(h) \cdot d_j(h) > 1$ is called a *proper intersection point* in $S(\boldsymbol{P}_\Gamma(i,j))$.

Given a graph Γ on N, two nodes i and j in N are *connected* if there exists a path from node i to node j. Graph Γ on N is *connected* if any two nodes in N are connected. Given a graph Γ on N, a coalition $S \subseteq N$ is *connected* if the subgraph $\Gamma|_S$ is connected. For a graph Γ on N and coalition $S \subseteq N$, $C^\Gamma(S)$ is the set of all connected subcoalitions of S, S/Γ is the set of maximally connected subcoalitions of S, called the *components of* S, and $(S/\Gamma)_i$ is the component of S containing player $i \in S$.

A directed graph Γ on N is a *(rooted) tree* if it has precisely one root, denoted $r(\Gamma)$, and there is a unique directed path in Γ from this node to any other node in N. In a tree the root plays the role of the source of the stream presented via this graph. A directed graph Γ on N is a *sink tree* if the directed graph composed by the same set of links as Γ but with the opposite orientation is a rooted tree; in this case a root of a tree changes its meaning to an absorbing sink. A directed graph Γ is a *(rooted/sink) forest* if it is composed by a number of disjoint (rooted/sink) trees. A *line-graph* is a forest that contains links only between subsequent nodes. Both a rooted tree and a sink tree, and in particular a line-graph, are strongly cycle-free. For a directed graph Γ, a subgraph T is a *subtree* of Γ if T is a tree on $S(T)$. A subtree T of a digraph Γ is a *full subtree* if it contains together with its root $r(T)$ all successors of $r(T)$, in other words, $S(T) = \bar{S}_\Gamma(r(T))$. A full subtree T of Γ is a *maximal subtree* if its root is a root of Γ.

In what follows it is assumed that the cooperation structure on the player set N is specified by a cycle-free directed graph, not necessarily being strongly cycle-free. A pair $\langle v, \Gamma\rangle$ of a game $v \in \mathcal{G}_N$ and a cycle-free directed communication graph Γ on N constitutes a *game with cycle-free limited cooperation* or *cycle-free digraph structure* and is also called a *directed cycle-free graph game* or just a *cycle-free digraph game*. The set of all cycle-free digraph games on a fixed player set N is denoted $\mathcal{G}_N^\Gamma$. A *value* on a subset $\mathcal{G}$ of $\mathcal{G}_N^\Gamma$ is a function $\xi\colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}$ a vector of payoffs $\xi(v, \Gamma) \in \mathbb{R}^N$. For any graph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$, a payoff vector $x \in \mathbb{R}^N$ is *component efficient* if for every component $C \in N/\Gamma$ it holds that $x(C) = v(C)$.

3. Main results

In this section we introduce two values for the class of cycle-free digraph games, not being necessarily strongly cycle-free.

For a directed link in an arbitrary digraph there are two different interpretations possible. One interpretation is that a link is directed to indicate which player has initiated the communication, but at the same time it represents a fully developed communication link. In such a case, following Myerson (Myerson, 1977), it is assumed that cooperation is possible among any set of connected players, i.e., the coalitions in which players are able to cooperate, the *productive coalitions*, are all the connected coalitions. In this case the focus is on component efficient values. Another interpretation of a directed link assumes that a directed link represents the only one-way communication situation. In that case not every connected coalition might be productive. In this paper we abide by the second interpretation of a directed link and consider two different options for creation of the productive coalitions.

3.1. Tree connectedness

In a cycle-free digraph Γ there is at least one node having no superior. A node without superior, i.e., any root in the graph, can be seen as a *topman* of the communication structure given by Γ. There are different scenarios possible for controlling cooperation in case of directed communication. First we assume that in any coalition each player can be controlled only by his predecessors and that nobody accepts that one of his subordinates becomes his equal partner if a coalition forms. This entails the assumption that the only productive coalitions are the so-called *tree connected*, or simply *t-connected*, coalitions, being the connected coalitions $S \in C^\Gamma(N)$ that also meet the condition that for every root $i \in R_\Gamma(S)$ it holds that $i \notin S_\Gamma(j)$ for any other root $j \in R_\Gamma(S)$. It is not difficult to see that the latter condition guarantees that every t-connected coalition inherits the subordination of players prescribed by Γ in N. Obviously, every component $C \in N/\Gamma$ is t-connected. Moreover, any full successors set in Γ is t-connected. A t-connected coalition is *full t-connected*, if it together with its roots contains all successors of these roots. Observe that a full t-connected coalition is the union of several full successors sets.

In what follows for a cycle-free digraph Γ on N and a coalition $S \subseteq N$, let $C_t^\Gamma(S)$ denote the set of all t-connected subsets of S, $[S/\Gamma]^t$ the set of maximally t-connected subsets of S, called the *t-connected components of* S, and $[S/\Gamma]_i^t$ the t-connected component of S containing player $i \in S$.

Since the communication is assumed to be one-way, we require for efficiency of a value that the t-connected coalition consisting of one of the roots of the graph together with all his successors realizes its worth. This gives the first property a value must satisfy, what we call maximal-tree efficiency.

A value ξ on $\mathcal{G}_N^\Gamma$ is *maximal-tree efficient* (MTE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j\in\bar{S}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in R_\Gamma(N).$$

MTE generalizes the usual definition of efficiency for a tree. In a digraph with only one topman, the maximal-tree efficiency just says that the total payoff should be equal to the worth of the grand coalition. Still, MTE is not the productive component efficiency condition. Different from the Myerson case with undirected communication graph (Myerson, 1977) we assume that not every productive component is able to realize its exact capacity but only those with a tree structure. For example if one worker works in two different divisions, the two managers of these firms and the worker create a productive coalition. Yet, it is impossible to guarantee the efficiency of this coalition because there is no communication link between the managers of the two divisions.

The next property, what we call successor equivalence, says that if a link is deleted, each successor of the terminus of this link still receives the same payoff.

A value ξ on $\mathcal{G}_N^\Gamma$ is *successor equivalent* (SE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that for all $(i,j)\in\Gamma$

$$\xi_k(v,\Gamma\backslash(i,j)) = \xi_k(v,\Gamma), \qquad \text{for all } k \in \bar{S}_\Gamma(j).$$

SE means that the payoff to any member in the full successors set of a player does not change if any of the superiors of that player breaks his link to that player. It implies that for each successors set the payoff distribution is completely determined by the players of this set.

Along with MTE we consider a stronger efficiency property, what we call full-tree efficiency, that requires that every full successors set realizes its worth.

A value ξ on $\mathcal{G}_N^\Gamma$ is *full-tree efficient* (FTE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j\in\bar{S}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in N. \tag{1}$$

Proposition 1. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ MTE and SE together imply FTE.*

Proof. Let ξ be a value on $\mathcal{G}_N^\Gamma$ that meets MTE and SE, and let a cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ be arbitrarily chosen. For every $i\in N$ the subgraph Γ^i is a maximal tree in the subgraph $\Gamma\backslash\bigcup_{j\in O_\Gamma(i)}\{(j,i)\}$. Hence, due to MTE,

$$\sum_{j\in\bar{S}_\Gamma(i)} \xi_j(v,\Gamma\backslash \bigcup_{k\in O_\Gamma(i)}\{(k,i)\}) \overset{\text{MTE}}{=} v(\bar{S}_\Gamma(i)).$$

By successive application of SE,

$$\xi_j(v, \Gamma \setminus \bigcup_{k \in O_\Gamma(i)} \{(k,i)\}) \overset{\text{SE}}{=} \xi_j(v, \Gamma), \qquad \text{for all } j \in \bar{S}_\Gamma(i).$$

Whence,

$$\sum_{j \in \bar{S}_\Gamma(i)} \xi_j(v, \Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in N,$$

i.e., the value ξ meets FTE. ∎

It turns out that MTE and SE uniquely define a value on the class of cycle-free digraph games.

Theorem 1. *On the class of cycle-free digraph games* $\mathcal{G}_N^\Gamma$ *there is a unique value* t *that satisfies MTE and SE. For every cycle-free digraph game* $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$*, the value* $t(v, \Gamma)$ *satisfies the following conditions:*

(i) *it obeys the recursive equality*

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} t_j(v, \Gamma), \qquad \textit{for all } i \in N; \tag{2}$$

(ii) *it admits the explicit representation in the form*

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(j)), \qquad \textit{for all } i \in N, \tag{3}$$

where for all $i \in N$, $j \in S_\Gamma(i)$

$$\kappa_i(j) = \sum_{r=0}^{n-2} (-1)^r \kappa_i^r(j), \tag{4}$$

and $\kappa_i^r(j)$ *is the number of tuples* $(i_0, \ldots, i_{r+1})$ *such that* $i_0 = i$, $i_{r+1} = j$, $i_h \in S_\Gamma(i_{h-1})$, $h = 1, \ldots, r+1$.

Proof. Due to Proposition 1 the value t on $\mathcal{G}_N^\Gamma$ that satisfies MTE and SE meets FTE as well, wherefrom the recursive equality (2) follows straightforwardly. Next, we show that the representation in the form (2) is equivalent to the representation in the form (3). According to (2) it holds for the value t that every player receives what this player together with his successors can get on their own, their worth, minus what all his successors will receive by themselves. Since the same property holds for these successors as well, it is not difficult to see that (3) follows directly from (2) by successive substitution. Indeed,

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} t_j(v, \Gamma) \overset{(2)}{=}$$

$$v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} v(\bar{S}_\Gamma(j)) + \sum_{j \in S_\Gamma(i)} \sum_{k \in S_\Gamma(j)} t_k(v, \Gamma) \overset{(2)}{=}$$

$$v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} v(\bar{S}_\Gamma(j)) + \sum_{j \in S_\Gamma(i)} \sum_{k \in S_\Gamma(j)} v(\bar{S}_\Gamma(k)) - \sum_{j \in S_\Gamma(i)} \sum_{k \in S_\Gamma(j)} \sum_{h \in S_\Gamma(k)} t_h(v, \Gamma) \overset{(2)}{=}$$

$$\ldots = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \sum_{r=0}^{n-2} (-1)^r \kappa_i^r(j) v(\bar{S}_\Gamma(j)) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(j)).$$

From (3), we obtain immediately that the value t meets SE, because in any digraph Γ for all $(i,j)\in\Gamma$ and for every $k\in\bar{S}_\Gamma(j)$ the full subtrees Γ^k and $(\Gamma\backslash(i,j))^k$ coincide. This completes the proof, since MTE follows from FTE automatically. ∎

Corollary 1. According to (2) the value t assigns to every player the worth of his full successors set minus the total payoff to his successors. Wherefrom we obtain a simple recursive algorithm for computing the value t going upstream from the leaves of the given digraph.

Observe that the computation of the coefficients $\kappa_i(j)$, $i\in N$, $j\in S_\Gamma(i)$, in the explicit formula representation (3) requires, in general, the enumeration of quite a lot of possibilities. We show below that in many cases the coefficients $\kappa_i(j)$ can be easily computed and the value t can be presented in a computationally more transparent and simpler form. Before formulating the next theorem we introduce some additional notation.

For any digraph Γ on N and $i\in N$ the set $S_\Gamma(i)$ of all successors of i can be partitioned into three disjoint subsets $F^*_\Gamma(i)$, $S^1_\Gamma(i)$, and $S^2_\Gamma(i)$, i.e.,

$$S_\Gamma(i)=F^*_\Gamma(i)\cup S^1_\Gamma(i)\cup S^2_\Gamma(i),$$

where both sets $S^1_\Gamma(i)$ and $S^2_\Gamma(i)$ are composed by successors of i that are not proper subordinates of i. $S^1_\Gamma(i)$ consists of any of them for which all paths from i to that node j can be partitioned into a number of separate groups, might be only one group, such that all paths in the same group have at least one common node different from i and j and paths from different groups do not intersect between i and j, namely,

$$\begin{aligned}S^1_\Gamma(i)=\Big\{j\in S_\Gamma(i)\backslash F^*_\Gamma(i)\mid \boldsymbol{P}_\Gamma(i,j)=\bigcup_{h=1}^{q}\boldsymbol{P}_h,\boldsymbol{P}_h\cap\boldsymbol{P}_l=\emptyset,h\neq l:&\\ \forall h=1,...,q,\exists k_h\in S(\boldsymbol{P}_h)\backslash\{i,j\}:&\\ k_h\in\boldsymbol{p},\forall\boldsymbol{p}\in\boldsymbol{P}_h\text{ and }\boldsymbol{p}_h\cap\boldsymbol{p}_l=\{i,j\},\forall\boldsymbol{p}_h\in\boldsymbol{P}_h,\forall\boldsymbol{p}_l\in\boldsymbol{P}_l,h\neq l\Big\}&\end{aligned}$$

and

$$S^2_\Gamma(i)=S_\Gamma(i)\setminus\big(F^*_\Gamma(i)\cup S^1_\Gamma(i)\big).$$

Remark that all $j\in S_\Gamma(i)\setminus F^*_\Gamma(i)$ with $d^i(j)=1$ belong to $S^1_\Gamma(i)$ since the unique proper superior of j belongs to all paths $\boldsymbol{p}\in\boldsymbol{P}_\Gamma(i,j)$; in particular, it holds that $j\in S^1_\Gamma(i)$, when there is only one path from i to j, i.e., when $|\boldsymbol{P}_\Gamma(i,j)|=1$. From here besides it follows that for all $j\in S^2_\Gamma(i)$, $d^i(j)>1$. For every $j\in S^1_\Gamma(i)$ we define the *proper in-degree* $\tilde{d}^i(j)$ of j with respect to i as the number of groups $\boldsymbol{P}_h$, $h=1,...,q$, in the partition of $\boldsymbol{P}_\Gamma(i,j)$.

Next, observe that for a given digraph Γ on N, for any $i\in N$ and $j\in S_\Gamma(i)$, all nodes forming a tuple $(i_0,...,i_{r+1})$ in which $i_0=i$, $i_{r+1}=j$, $i_h\in S_\Gamma(i_{h-1})$, $h=1,...,r+1$, belong to the same directed path $\boldsymbol{p}\in\boldsymbol{P}_\Gamma(i,j)$. Wherefrom it easily follows that for all $i\in N$ and $j\in S_\Gamma(i)$, $\kappa_i(j)$ given by (4) in fact is defined via tuples of nodes from the set of nodes $S(\boldsymbol{P}_\Gamma(i,j))$ that determine the set of directed paths $\boldsymbol{P}_\Gamma(i,j)$. Similar to the definition of $\kappa_i(j)$ given by (4), for any subset of nodes $M\subseteq S(\boldsymbol{P}_\Gamma(i,j))$ containing nodes i and j, we may define

$$\kappa_i(M;j)=\sum_{r=0}^{n-2}(-1)^r\kappa^r_i(M;j), \tag{5}$$

where $\kappa_i^r(M;j)$ counts only the tuples $(i_0,...,i_{r+1})$ for which $i_0 = i$, $i_{r+1} = j$, and $i_h \in S_\Gamma(i_{h-1}) \cap M$, $h = 1,\ldots,r+1$. Remark that $\kappa_i(j) = \kappa_i(S(\boldsymbol{P}_\Gamma(i,j));j)$. The subset of $S(\boldsymbol{P}_\Gamma(i,j))$ composed by i, j, all proper subordinates $h \in F_\Gamma^*(i) \cap S(\boldsymbol{P}_\Gamma(i,j))$ and all proper intersection points in $S(\boldsymbol{P}_\Gamma(i,j))$ is called the *upper covering set* for $\boldsymbol{P}_\Gamma(i,j)$ and denoted $M_\Gamma(i,j)$. It turns out that on $F_\Gamma^*(i)$ and $S_\Gamma^1(i)$ the exact value of $\kappa_i(j)$ can be simply computed, while on $S_\Gamma^2(i)$ the computation of $\kappa_i(j)$ can be reduced to the enumeration only over the nodes from the upper covering set for $\boldsymbol{P}_\Gamma(i,j)$. For simplicity of notation we denote $\kappa_i(M_\Gamma(i,j);j)$ by $\kappa_i^M(j)$.

Theorem 2. *For every cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ the value t given by (3) admits the equivalent representation in the form*

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in F_\Gamma^*(i)} v(\bar{S}_\Gamma(j)) + \\ + \sum_{j\in S_\Gamma^1(i)} (\tilde{d}^i(j)-1)v(\bar{S}_\Gamma(j)) - \sum_{j\in S_\Gamma^2(i)} \kappa_i^M(j)v(\bar{S}_\Gamma(j)), \quad \textit{for all } i \in N. \tag{6}$$

If the consideration is restricted to only strongly cycle-free digraph games, then the above representation reduces to

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in F_\Gamma(i)} v(\bar{S}_\Gamma(j)), \quad \textit{for all } i \in N. \tag{7}$$

For rooted-forest digraph games defined by rooted forest digraph structures that are strongly cycle-free, the value given by (7) coincides with the tree value introduced first under the name of hierarchical outcome in (Demange, 2004), where it is also shown that under the mild condition of superadditivity it belongs to the core of the restricted game defined in (Myerson, 1977). More recently, the tree value for rooted-forest games was used as a basic element in the construction of the average tree solution for cycle-free undirected graph games in (Herings et al., 2008). In (Khmelnitskaya, 2010) it is shown that on the class of rooted-forest digraph games the tree value can be characterized via component efficiency and successor equivalence; moreover, it is shown that the class of rooted-forest digraph games is the maximal subclass in the class of strongly cycle-free digraph games where this axiomatization holds true. It is worth to recall that by definition for a rooted-tree digraph game every connected component is a tree. Hence, on the class of rooted-forest digraph games every connected component is productive and maximal-tree efficiency coincides with component efficiency.

From now on we refer to the value t given by (3), or equivalently by (6), as to the *root-tree value*, or simply the *tree value*, for cycle-free digraph games. The tree value assigns to every player the payoff equal to the worth of his full successors set minus the worths of all full successors sets of his proper subordinates plus or minus the worths of all full successors sets of any other of his successors that are subtracted or added more than once. Moreover, for any player $i \in N$ and his successor $j \in N$ that is not his proper subordinate, the coefficient $\kappa_i(j)$ indicates the number of overlappings of full successors sets of all proper subordinates of i at node j. In fact a player receives what he contributes when he joins his successors when only the full successors sets, that are the only efficient productive coalitions, are counted. Since a leaf has no successors, a leaf just gets his own worth. Besides, it is worth to note and not difficult to check that the right sides of both formulas (6) and (7) being considered with respect not to coalitional worths but to players in these coalitions contain only player i when taking into account all pluses and minuses.

The validity of the first statement of Theorem 2 follows directly from Theorem 1 and Lemma 1 below. The second statement follows easily from the first one. Indeed, in any strongly cycle-free digraph Γ all links are essential and $d^i(j) = 1$ for all $i \in N$, $j \in S_\Gamma(i)$. Whence it easily follows that $F^*_\Gamma(i) = F_\Gamma(i)$, $S^2_\Gamma(i) = \emptyset$, and $\tilde{d}^i(j) = d^i(j) = 1$ for all $j \in S^1_\Gamma(i)$.

Lemma 1. *For a given digraph Γ on N, the coefficients $\kappa_i(j)$, $i \in N$, $j \in S_\Gamma(i)$, defined by (4) satisfy the following properties:*

(i) *if a link $(k,l) \in \Gamma$ is inessential, then for all $i \in N$ and $j \in S_\Gamma(i)$, $\kappa_i(j)$ defined on Γ is equal to $\kappa_i(j)$ defined on $\Gamma \backslash (k,l)$;*
(ii) *$\kappa_i(j) = 1$ for all $i \in N$, $j \in F^*_\Gamma(i)$;*
(iii) *$\kappa_i(j) = -\tilde{d}^i(j) + 1$ for all $i \in N$, $j \in S^1_\Gamma(i)$;*
(iv) *$\kappa_i(j) = \kappa^M_i(j)$ for all $i \in N$ and $j \in S_\Gamma(i)$.*

Proof. (i). It is sufficient to prove the statement only in case when $k \in S_\Gamma(i)$ and $j \in S_\Gamma(l)$. Let $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$ be such that $\boldsymbol{p} \ni (k,l)$. By definition of an inessential link there exists $\boldsymbol{p}_0 \in \boldsymbol{P}_\Gamma(k,l)$ such that $\boldsymbol{p}_0 \neq (k,l)$. It is not difficult to see that the path $\boldsymbol{p}_1 = \boldsymbol{p} \backslash (k,l) \cup \boldsymbol{p}_0$ obtained from the path $\boldsymbol{p}$ by replacing the link (k,l) by the path $\boldsymbol{p}_0$ belongs to $\boldsymbol{P}_\Gamma(i,j)$, and moreover, all tuples $(i_0, ..., i_{r+1})$ in the definition of $\kappa_i(j)$ that belong to $\boldsymbol{p}$ also belong to $\boldsymbol{p}_1$. Whence it follows straightforwardly that deleting an inessential link does not change the value of $\kappa_i(j)$.

(ii). If $j \in F^*_\Gamma(i)$, then $\boldsymbol{P}_\Gamma(i,j)$ contains only the path $\boldsymbol{p} = (i,j)$ and the only tuple $(i_0, ..., i_{r+1})$ is (i,j) with $r = 0$. Wherefrom it follows that $\kappa_i(j) = 1$.

(iii). Let $j \in S^1_\Gamma(i)$. First consider the case when $\tilde{d}^i(j) = 1$. Then there exists $k \in S(\boldsymbol{P}_\Gamma(i,j))$, $k \neq i,j$, such that $k \in \boldsymbol{p}$ for all $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$. By definition, $\kappa^r_i(j)$ is equal to the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = i$, $i_{r+1} = j$, $i_h \in S_\Gamma(i_{h-1})$, $h = 1, \ldots, r+1$, or equivalently, $\kappa^r_i(j)$ is equal to the number of these tuples $(i_0, \ldots, i_{r+1})$ that do not contain k plus the number of these tuples $(i_0, \ldots, i_{r+1})$ that contain k. Notice that since $k \in \boldsymbol{p}$ for all $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i,j)$, for every r-tuple $(i_0, \ldots, i_{r+1})$ that does not contain k there exists a uniquely defined $(r+2)$-tuple composed by the same nodes plus the node k. From which together with equality (4) it follows that $\kappa_i(j) = 0$.

Let now $\tilde{d}^i(j) > 1$. Then $\boldsymbol{P}_\Gamma(i,j) = \bigcup_{h=1}^{q} \boldsymbol{P}_h$ with $q = \tilde{d}^i(j)$ and there exist nodes $k_h \neq i,j$, $h = 1, ..., q$ such that $k_h \in \boldsymbol{p}$ for all paths $\boldsymbol{p} \in \boldsymbol{P}_h$ and all paths $\boldsymbol{p}_h \in \boldsymbol{P}_h$ and $\boldsymbol{p}_l \in \boldsymbol{P}_l$, $h \neq l$, intersect only at i and j. We may split the computation of $\kappa_i(j)$ by the computation via groups of paths $\boldsymbol{P}_h$ excluding on each step the tuples that were counted at the previous steps, i.e.,

$$\kappa_i(j) = \kappa_i(S(\boldsymbol{P}_1); j) + [\kappa_i(S(\boldsymbol{P}_2); j) - \kappa_i(S(\boldsymbol{P}_1 \cap \boldsymbol{P}_2; j)] + \ldots$$

$$\ldots + [\kappa_i(S(\boldsymbol{P}_q); j) - \kappa_i(S(\bigcap_{h=1}^{q} \boldsymbol{P}_h); j)].$$

Applying the same argument as in the proof of the case when $\tilde{d}^i(j) = 1$, we obtain that for all $h = 1, \dots, q$,

$$\kappa_i(S(\boldsymbol{P}_h); j) = 0.$$

Since the paths from different groups $\boldsymbol{P}_h$ do not intersect between i and j, only tuple $(i_0, i_1) = (i, j)$ with $r = 0$ belongs to all $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i, j)$. Therefore, for all $h = 2, \dots, q$,

$$\kappa_i(S(\bigcap_{h=1}^{h} \boldsymbol{P}_h); j) = -1.$$

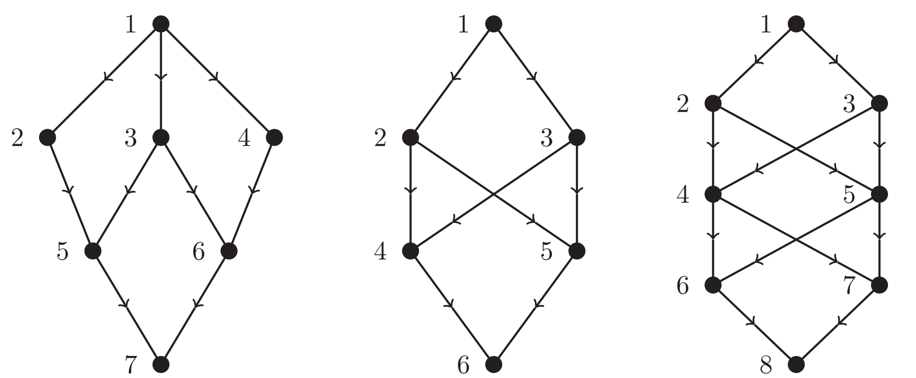
Then the validity of (iii) follows immediately from the last three equalities.

(iv). If $M_\Gamma(i, j) \neq S(\boldsymbol{P}_\Gamma(i, j))$, consider arbitrary $k \in S(\boldsymbol{P}_\Gamma(i, j)) \setminus M_\Gamma(i, j)$. We may split the computation of $\kappa_i(j)$ into two parts:

$$\kappa_i(j) = \kappa_i(j; k) + \kappa_i(S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}; j),$$

where $\kappa_i(j; k)$ is computed via tuples $(i_0, \dots, i_{r+1}) \ni k$ and $\kappa_i(S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}; j)$ is computed via tuples $(i_0, \dots, i_{r+1}) \subset S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}$, i.e., tuples $(i_0, \dots, i_{r+1}) \not\ni k$. By definition of a covering set, $M_\Gamma(i, j)$ contains predecessors of k, i.e., $M_\Gamma(i, j) \cap P_\Gamma(k) \neq \emptyset$. Moreover, since $k \notin M_\Gamma(i, j)$, i.e., k is neither a proper subordinate of i nor a proper intersection point in the subgraph $\Gamma|_{S(\boldsymbol{P}_\Gamma(i,j))}$, there exists $h \in M_\Gamma(i, j) \cap P_\Gamma(k)$ that belongs to all paths $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(i, j)$, $\boldsymbol{p} \ni k$. Applying the same argument as above in the proof of the statement (iii), now with respect to h, we obtain that $\kappa_i(j; k) = 0$. Thus $\kappa_i(j) = \kappa_i(S(\boldsymbol{P}_\Gamma(i, j)) \setminus \{k\}; j)$. Repeating the same reasoning successively with respect to all $k' \in S(\boldsymbol{P}_\Gamma(i, j)) \setminus \big(M_\Gamma(i, j) \cup \{k\}\big)$ we obtain $\kappa_i(j) = \kappa_i(M_\Gamma(i, j); j) \stackrel{def}{=} \kappa_i^M(j)$. ∎

Example 1. The examples of digraphs given in Figure 1 demonstrate the more complicated situation with the computation of coefficients $\kappa_i(j)$ when $j \in S_\Gamma^2(i)$ for some $i \in N$.



a) *b)* *c)*

Fig. 1.

Figure 1,a: $7 \in S_\Gamma^2(1)$, $d^1(7) = 2$, $\kappa_1(7) = 0$;
Figure 1,b: $6 \in S_\Gamma^2(1)$, $d^1(6) = 2$, $\kappa_1(6) = 1$.
Figure 1,c: $8 \in S_\Gamma^2(1)$, $d^1(8) = 2$, $\kappa_1(8) = -1$.

Example 2. Figure 2 provides an example of the tree value for a 10-person game with cycle-free but not strongly cycle-free digraph structure.

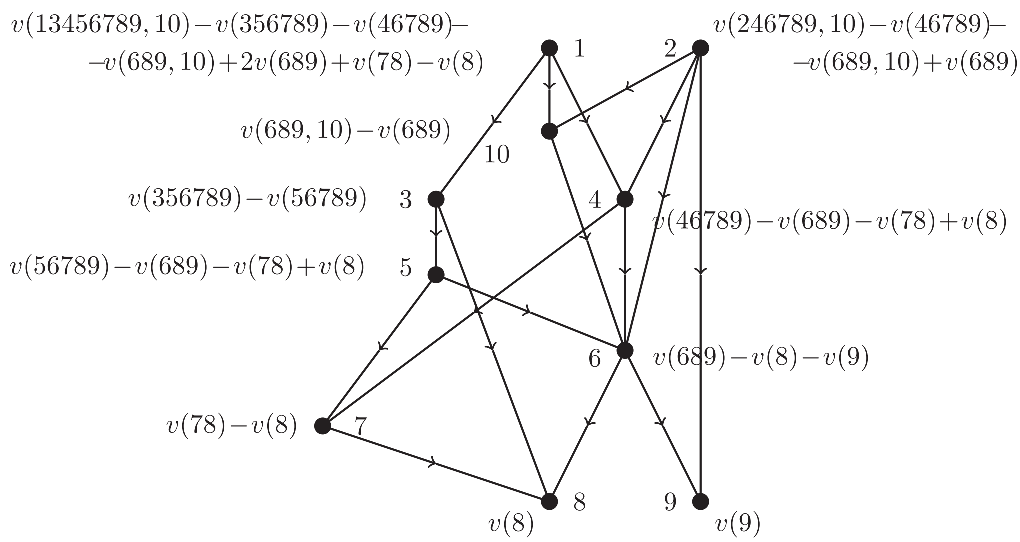


Fig. 2.

The tree value may be computed in two different ways, either by the recursive algorithm based on the recursive equality (2) or using the explicit formula representation (6).

We explain in detail the computation of $t_1(v, \Gamma)$ based on the explicit representation given by (6):
$\bar{S}_\Gamma(1) = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$.
$3, 4, 10 \in F^*_\Gamma(1) \Longrightarrow\ \kappa_1(3) = \kappa_1(4) = \kappa_1(10) = 1$.
$5, 6, 7, 9 \in S^1_\Gamma(1), d^1(5) = d^1(9) = 1,\ d^1(6) = 3,\ d^1(7) = 2 \Longrightarrow\ \kappa_1(5) = \kappa_1(9) = 0,\ \kappa_1(6) = -2,\ \kappa_1(7) = -1$.
$8 \in S^2_\Gamma(1)$:

$\boldsymbol{P}_\Gamma(1, 8) = \big\{\boldsymbol{p}_1 = (1, 3, 5, 7, 8),\ \boldsymbol{p}_2 = (1, 3, 5, 6, 8),\ \boldsymbol{p}_3 = (1, 10, 6, 8),$
$\boldsymbol{p}_4 = (1, 4, 7, 8),\ \boldsymbol{p}_5 = (1, 4, 6, 8),\ \boldsymbol{p}_6 = (1, 3, 8)\big\}$;
we eliminate the path $\boldsymbol{p}_6$ since it contains inessential link $(3, 8)$;
$M = \{1, 4, 5, 6, 7, 8, 10\}$ is a minimal covering set for $\boldsymbol{P}_\Gamma(1, 8)$;
$\kappa_1(\boldsymbol{p}_1; 8) = 0$;
$\boldsymbol{p}_2 \backslash \boldsymbol{p}_1$ contains tuples $(1, 6, 8)$ and $(1, 5, 6, 8) \Longrightarrow \kappa_1(\boldsymbol{p}_2 \backslash \boldsymbol{p}_1; 8) = 0$;
$\boldsymbol{p}_3 \backslash (\boldsymbol{p}_1 \cup \boldsymbol{p}_2)$ contains tuples $(1, 10, 8)$, $(1, 10, 6, 8)) \Longrightarrow \kappa_1(\boldsymbol{p}_3 \backslash (\boldsymbol{p}_1 \cup \boldsymbol{p}_2); 8) = 0$;
$\boldsymbol{p}_4 \backslash (\boldsymbol{p}_1 \cup \boldsymbol{p}_2 \cup \boldsymbol{p}_3)$ contains $(1, 4, 8)$, $(1, 4, 7, 8)) \Longrightarrow \kappa_1(\boldsymbol{p}_4 \backslash (\boldsymbol{p}_1 \cup \boldsymbol{p}_2 \cup \boldsymbol{p}_3); 8) = 0$;
$\boldsymbol{p}_5 \backslash (\boldsymbol{p}_1 \cup \boldsymbol{p}_2 \cup \boldsymbol{p}_3 \cup \boldsymbol{p}_4)$ contains $(1, 4, 6, 8)) \Longrightarrow \kappa_1(\boldsymbol{p}_5 \backslash (\boldsymbol{p}_1 \cup \boldsymbol{p}_2 \cup \boldsymbol{p}_3 \cup \boldsymbol{p}_4); 8) = 1$;
$\Longrightarrow\ \kappa_1(8) = 1$.

$t_1(v, \Gamma) = v(13456789, 10) - v(356789) - v(46789) - v(689, 10) + 2v(689) + v(78) - v(8)$.

Example 3. Figure 3 gives an example of the tree value for a 10-person game with strongly cycle-free digraph structure.

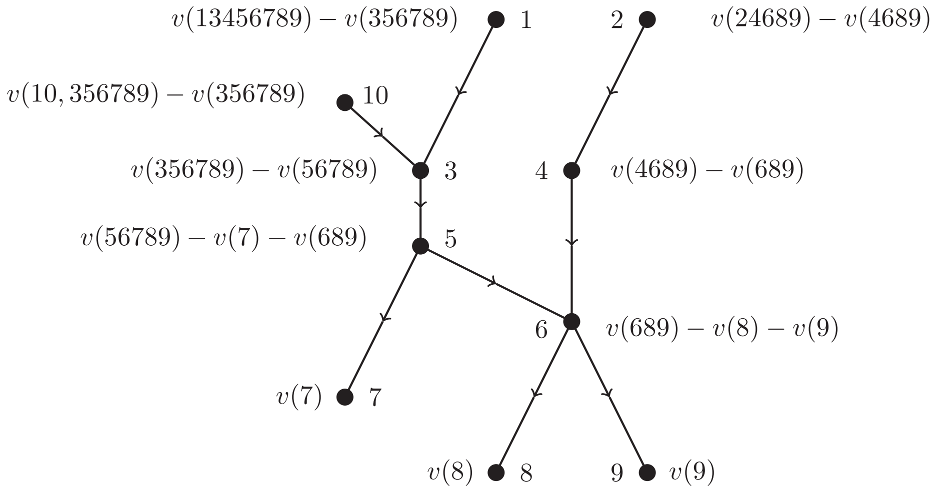


Fig. 3.

It turns out that the tree value not only meets FTE but FTE alone uniquely defines the tree value on the class of cycle-free digraph games.

Theorem 3. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ the tree value is the unique value that satisfies FTE.*

Proof. Since the tree value satisfies FTE, to prove the theorem it is enough to show that the tree value is the unique value that meets FTE on $\mathcal{G}_N^\Gamma$. Let a value ξ on $\mathcal{G}_N^\Gamma$ satisfy axiom FTE. Then, because of FTE, (1) holds for every $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$. Every digraph Γ under consideration is cycle-free, i.e., no player in N appears to be a successor of itself. Hence, due to the arbitrariness of game $\langle v, \Gamma \rangle$, the n equalities in (1) are independent. Therefore, we have a system of n independent linear equalities with respect to n variables $\xi_j(v, \Gamma)$ which uniquely determines the value $\xi(v, \Gamma)$ that in this case coincides with $t(v, \Gamma)$. ∎

Corollary 2. FTE on the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ implies not only MTE but SE as well.

3.2. Overall efficiency and stability

In this subsection we consider efficiency and stability of the tree value. First we derive for the tree value the total payoff for any t-connected coalition.

Theorem 4. *In a cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, for any t-connected coalition $S \in C_t^\Gamma(N)$ it holds that*

$$\sum_{i \in S} t_i(v, \Gamma) = \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \\ - \sum_{i \in S \setminus R_\Gamma(S)} \big(\kappa_S(i) - 1\big) v(\bar{S}_\Gamma(i)) - \sum_{i \in \bar{S}_\Gamma(S) \setminus S} \kappa_S(i) v(\bar{S}_\Gamma(i)), \tag{8}$$

where

$$\bar{S}_\Gamma(S) = \bigcup_{i \in R_\Gamma(S)} \bar{S}_\Gamma(i),$$

$$\kappa_S(i) = \sum_{j \in \bar{P}_\Gamma(i) \cap \bar{S}_\Gamma(S)} \kappa_j(i), \quad \textit{for all } i \in \bar{S}_\Gamma(S),$$

while $\kappa_S(i) = 1$ *when* $d_N(i) = 1$, *where for any* t*-connected coalition* $S \in C_t^\Gamma(N)$, *for all* $i \in \bar{S}_\Gamma(S)$, $d_S(i)$ *is the in-degree of* i *in the subgraph* $\Gamma|_{\bar{S}_\Gamma(S)}$, *i.e.,*

$$d_S(i) = |O_\Gamma(i) \cap \bar{S}_\Gamma(S)|,$$

in particular, $d_N(i) = d_\Gamma(i)$ *for all* $i \in N$.

If the consideration is restricted to only strongly cycle-free digraph games, then for any t*-connected coalition* $S \in C_t^\Gamma(N)$ *it holds that*

$$\sum_{i \in S} t_i(v, \Gamma) = \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \\ - \sum_{i \in S \setminus R_\Gamma(S)} \big(d_S(i) - 1\big) v(\bar{S}_\Gamma(i)) - \sum_{i \in R_\Gamma(\bar{S}_\Gamma(S) \setminus S)} d_S(i) v(\bar{S}_\Gamma(i)). \tag{9}$$

Proof. Let $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ be a cycle-free digraph game and let S be any t-connected coalition $S \in C_t^\Gamma(N)$. Then it holds that

$$\sum_{i \in S} t_i(v, \Gamma) \overset{(3)}{=} \sum_{i \in S} \big(v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(j)) \big) =$$

$$= \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \sum_{i \in S \setminus R_\Gamma(S)} \Big(\sum_{j \in S_\Gamma(i)} (\kappa_i(j) - 1) v(\bar{S}_\Gamma(i)) \Big) - \sum_{i \in \bar{S}_\Gamma(S) \setminus S} \Big(\sum_{j \in S_\Gamma(i)} \kappa_i(j) v(\bar{S}_\Gamma(i)) \Big).$$

Since for all $i, j \in S$ with $j \in S_\Gamma(i)$ every path from i to j belongs to S, (8) follows straightforwardly from the last equality.

Next, if $d_N(i) = 1$, then due to Lemma 1 for all $j \in \big(\bar{P}_\Gamma(i) \cap \bar{S}_\Gamma(S)\big) \setminus F_\Gamma(i)$, $d^j(i) = 0$ and therefore $\kappa_j(i) = 0$, and for $j \in F_\Gamma(i) \cap \bar{S}_\Gamma(S)$, $\kappa_j(i) = 1$.

In case Γ is a strongly cycle-free digraph, it holds that

$$\sum_{i \in S} t_i(v, \Gamma) \overset{(7)}{=} \sum_{i \in S} \big(v(\bar{S}_\Gamma(i)) - \sum_{j \in F_\Gamma(i)} v(\bar{S}_\Gamma(j)) \big) =$$

$$= \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \sum_{i \in S \setminus R_\Gamma(S)} \big(d_S(i) - 1\big) v(\bar{S}_\Gamma(i)) - \sum_{\substack{j \in F_\Gamma(i) \\ i \in S,\, j \notin S}} d_S(j) v(\bar{S}_\Gamma(j)).$$

To complete the proof of (9) it suffices to notice that, since Γ a strongly cycle-free digraph, every subordinate $j \in F_\Gamma(i)$ of $i \in S$ that does not belong to S is a root in $\bar{S}_\Gamma(S) \setminus S$. ∎

From Theorem 4 it follows that for any cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ the overall efficiency is given by

$$\sum_{i \in N} t_i(v, \Gamma) = \sum_{i \in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i \in N \setminus R_\Gamma(N)} \big(\kappa_N(i) - 1\big) v(\bar{S}_\Gamma(i)), \tag{10}$$

while if the consideration is restricted to only strongly cycle-free digraph games, (10) reduces to

$$\sum_{i\in N} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i\in N\setminus R_\Gamma(N)} \big(d_\Gamma(i)-1\big) v(\bar{S}_\Gamma(i)). \tag{11}$$

To support these expressions we recall the Myerson model in of a game with undirected cooperation structure (Myerson, 1977), in which the component efficiency entails the equality

$$\sum_{i\in N} \xi_i(v,\Gamma) = \sum_{C\in N/\Gamma} v(C). \tag{12}$$

While the right-side expression in (12) is composed by connected components that are the only efficient productive elements in the Myerson's model, the building bricks in (10) and (11) are the full successors sets which are the only efficient productive coalitions under the assumption of tree connectedness. Observe also that for strongly cycle-free rooted-forest digraph games (11) reduces to (12),

$$\sum_{i\in N} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) = \sum_{C\in N/\Gamma} v(C).$$

For a cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, we define the *t-core* $C^t(v,\Gamma)$ as the set of component efficient payoff vectors that are not dominated by any t-connected coalition,

$$C^t(v,\Gamma) = \{x \in \mathbb{R}^N \mid x(C) = v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_t^\Gamma(N)\}, \tag{13}$$

while the *weak t-core* $\tilde{C}^t(v,\Gamma)$ is the set of weakly component efficient payoff vectors that are not dominated by any t-connected coalition,

$$\tilde{C}^t(v,\Gamma) = \{x \in \mathbb{R}^N \mid x(C) \leq v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_t^\Gamma(N)\}. \tag{14}$$

Theorem 5. *The tree value on the subclass of superadditive rooted-forest digraph games is t-stable.*

Proof. Let $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ be a superadditive rooted-forest digraph game arbitrarily chosen. We show that the tree value $t(v,\Gamma)$ belongs to the core $C^t(v,\Gamma)$. Consider arbitrary $C \in N/\Gamma$, then C is a tree. Let $i \in C$ be a root in Γ, then $C = \bar{S}_\Gamma(i)$ because of the rooted-forest structure of Γ. Due to the full-tree efficiency of the tree value, it holds that

$$\sum_{j\in \bar{S}_\Gamma(i)} t_j(v,\Gamma) \overset{FTE}{=} v(\bar{S}_\Gamma(i)),$$

wherefrom it follows that

$$\sum_{j\in C} t_j(v,\Gamma) = v(C).$$

Let now $S \in C_t^\Gamma(N)$. Because of the rooted-forest structure of Γ, it holds that $d_N(i) = 1$ for all $i \in N\setminus R_\Gamma(N)$. Wherefrom it follows that $\Gamma|_S$ contains exactly one root, say, node i, $\Gamma|_S$ is a subtree, and $S \subseteq \bar{S}_\Gamma(i)$. Moreover, since Γ is strongly cycle-free, $\Gamma|_{\bar{S}_\Gamma(i)}$ is a full subtree, and because of the tree structure of $\Gamma|_S$, $\Gamma|_{\bar{S}_\Gamma(i)\setminus S}$ consists of a collection (might be empty) of disconnected full subtrees, i.e., $\Gamma|_{\bar{S}_\Gamma(i)\setminus S} = \bigcup_{k=1}^q T_k$ where $T_k \bigcap T_l = \emptyset$, $k \neq l$, and $q = |[\bar{S}_\Gamma(i)\setminus S]/\Gamma|$ is the number of components in $\bar{S}_\Gamma(i)\setminus S$. Hence,

$$\bar{S}_\Gamma(i) = S \cup \bigcup_{k=1}^{q} T_k.$$

Applying again the full-tree efficiency of the tree value, we obtain that

$$\sum_{j \in \bar{S}_\Gamma(i)} t_j(v, \Gamma) \overset{FTE}{=} v(\bar{S}_\Gamma(i)),$$

and

$$\sum_{j \in T_k} t_j(v, \Gamma) \overset{FTE}{=} v(T_k), \qquad \text{for all } \; k = 1, \ldots, q.$$

From the superadditivity of v and the last three equalities, it follows that

$$\sum_{j \in S} t_j(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{k=1}^{q} v(T_k) \geq v(S). \qquad \blacksquare$$

Remark 1. The statement of Theorem 5 can also be obtained as a corollary of the stability result proved in (Demange, 2004). Indeed, in a rooted forest every connected component has a tree structure and, therefore, is t-connected. Whence, for any rooted-forest digraph game the t-core coincides with the core of the Myerson restricted game.

However, the following examples show that for t-stability of a superadditive digraph game the requirement on the digraph to be a rooted forest is non-reducible. In Example 4 the tree value of a superadditive cycle-free but not strongly cycle-free digraph game violates individual rationality and, therefore, does not meet the second constraint of the weak t-core, while in Example 5 the tree value of a superadditive strongly cycle-free game in which the graph contains two roots violates weak efficiency.

Example 4. Consider a 4-person cycle-free superadditive digraph game $\langle v, \Gamma \rangle$ with $v(24) = v(34) = v(234) = v(N) = 1, v(S) = 0$ otherwise, and Γ given in Figure 4.

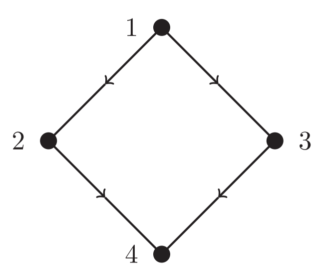


Fig. 4.

Then $t(v, \Gamma) = (-1, 1, 1, 0)$, whence $t_1(v, \Gamma) = -1 < 0 = v(1)$. Remark that every singleton coalition, in particular $S = \{1\}$, is t-connected.

Example 5. Consider a 3-person cycle-free superadditive digraph game $\langle v, \Gamma\rangle$ with $v(12) = v(13) = v(N) = 1, v(S) = 0$ otherwise, and Γ given in Figure 5.

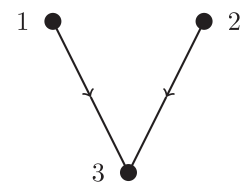


Fig. 5.

Then $t(v, \Gamma) = (1, 1, 0)$, whence $t_1(v, \Gamma) + t_2(v, \Gamma) + t_3(v, \Gamma) = 2 > 1 = v(N)$.

A cycle-free digraph game $\langle v, \Gamma\rangle$ is *t-convex*, if for all t-connected coalitions $T, Q \subset C_t^\Gamma(N)$ such that T is a full t-connected set, Q is a full successors set, and $T \cup Q \in C_t^\Gamma(N)$, it holds that

$$v(T) + v(Q) \leq v(T \cup Q) + v(T \cap Q). \tag{15}$$

Theorem 6. *The tree value on the subclass of t-convex strongly cycle-free digraph games is weakly efficient.*

Proof. Let $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ be any t-convex strongly cycle-free digraph game. Assume that Γ is connected, otherwise we apply the same argument to any component $C \in N/\Gamma$. If there is only one root in Γ, it holds that $\sum_{i=1}^n t_i(v, \Gamma) = v(N)$ and the tree value is even efficient. So, suppose that there are q different roots $r_1, \ldots, r_q$ in Γ for some $q \geq 2$. Since Γ is connected, the roots in Γ can be ordered in such a way that

$$\bigcup_{h=1}^{j-1} \bar{S}_\Gamma(r_h) \cap \bar{S}_\Gamma(r_j) \neq \emptyset, \qquad \text{for } j = 2, ..., q.$$

For $j = 1, ..., q$ let $T_j = \bigcup_{h=1}^{j} \bar{S}_\Gamma(r_h)$. Then from the strongly cycle-freeness of Γ it follows that for $j = 2, ..., q$ there exists a unique $i_j \in N$ such that

$$T_{j-1} \cap \bar{S}_\Gamma(r_j) = \bar{S}_\Gamma(i_j).$$

By t-convexity of the digraph game $\langle v, \Gamma\rangle$ it holds that

$$v(T_{j-1}) + v(\bar{S}_\Gamma(r_j)) \leq v(T_j) + v(\bar{S}_\Gamma(i_j)), \qquad \text{for } j = 2, ..., q.$$

Since $T_1 = \bar{S}_\Gamma(r_1)$ and $T_q = N$, then applying the last inequality successively $q-1$ times we obtain

$$\sum_{j=1}^{q} v(\bar{S}_\Gamma(r_j)) \leq v(N) + \sum_{j=2}^{q} v(\bar{S}_\Gamma(i_j)).$$

Hence,

$$v(N) \geq \sum_{j=1}^{q} v(\bar{S}_\Gamma(r_j)) - \sum_{j=2}^{q} v(\bar{S}_\Gamma(i_j)).$$

Since Γ is strongly cycle-free, for any $i \in N \backslash R_\Gamma(N)$, node i has $d_\Gamma(i)$ different roots as predecessor, which implies that the term $v(\bar{S}_\Gamma(i))$ appears precisely $d_\Gamma(i) - 1$ times. Therefore,

$$v(N) \geq \sum_{i \in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i \in N \backslash R_\Gamma(N)} \big(d_\Gamma(i) - 1\big) v(\bar{S}_\Gamma(i)). \qquad \blacksquare$$

The following example of a convex strongly cycle-free digraph game shows that even under the assumption of convexity of a given digraph game, which is stronger than t-convexity, one or more constraints for not being dominated in the definition of the week t-core might be violated by the tree value, and therefore, the tree value is not weakly t-stable.

Example 6. Consider a 5-person cycle-free convex digraph game $\langle v, \Gamma \rangle$ with $v(N) = 10$, $v(123) = v(1234) = v(1235) = 3$, $v(1345) = v(2345) = 2$, $v(S) = 0$ otherwise, and the strongly cycle-free digraph Γ given in Figure 6.

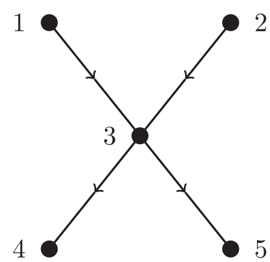


Fig. 6.

Then $t(v, \Gamma) = (1, 1, 0, 0, 0)$, whence, the total payoff of t-connected coalition $S = \{1, 2, 3\}$ $t_1(v, \Gamma) + t_2(v, \Gamma) + t_3(v, \Gamma) = 2 < 3 = v(123)$.

From (10) it follows that for a cycle-free (for simplicity connected) digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ a necessary and sufficient condition for the weak efficiency of the tree value is that

$$\sum_{i \in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) \leq v(N) + \sum_{i \in N \backslash R_\Gamma(N)} \big(\kappa_N(i) - 1\big) v(\bar{S}_\Gamma(i)). \tag{16}$$

Since $N = \bigcup_{i \in R_\Gamma(N)} \bar{S}_\Gamma(i)$, the grand coalition equals the union of the successors sets of all roots in the graph Γ. In case there is only one root in Γ, condition (16) is redundant, because the left side is then equal to $v(N)$. In case there is more than one root in Γ, the different successors sets of the roots of Γ will intersect each other and for any $i \in N \backslash R_\Gamma(N)$ the number $\kappa_N(i) - 1$ is the number of times that the successors set $\bar{S}_\Gamma(i)$ of node i equals the intersection of successors sets of the roots of Γ. Therefore, condition (16) is a kind of convexity condition for the grand coalition saying that the sum of the worths of the successors sets of all the roots of the graph should be less than or equal to the worth of the grand coalition (their union) plus the total worths of their intersections. In a firm where any full successors set of a root is a division within the firm and subdivisions that are intersections of several divisions are shared by these divisions, in (16) the left-side minus the sum in the right-side can be economically interpreted as the total worths of the divisions when they do not cooperate, while $v(N)$ is the worth of the firm when the divisions do cooperate. To have weak efficiency the latter value should be at least equal to the former value. Remark that $v(N)$ minus the total payoff at the tree value can be interpreted as the net profit of the firm (or the synergy effect from cooperation) that can be given to its shareholders.

3.3. Sink connectedness

We consider now another scenario of controlling cooperation in case of directed communication and assume that in any coalition each player may be controlled only by his successors and that nobody accepts that his former superior becomes his equal partner if a coalition forms. This entails the assumption that the only productive coalitions are the so-called *sink connected*, or simply *s-connected*, being the connected coalitions $S \in C^{\Gamma(N)}$ that meet also the condition that for every leaf $i \in L_\Gamma(S)$ it holds that $i \notin P_\Gamma(j)$ for another leaf $j \in L_\Gamma(S)$. Similar to the case of tree connectedness, every s-connected coalition inherits the subordination of players prescribed by Γ in N, every component $C \in N/\Gamma$ is s-connected, and any full predecessors set in Γ is s-connected. We say that an s-connected coalition is *full s-connected*, if it together with its leaves contains all predecessors of these leaves. Observe that a full s-connected coalition is the union of several full predecessors sets. For a cycle-free digraph Γ on N and a coalition $S \subseteq N$, let $C_s^\Gamma(S)$ denote the set of all s-connected subcoalitions of S, $[S/\Gamma]^s$ the set of maximally s-connected subcoalitions of S, called the *s-connected components of* S, and $[S/\Gamma]_i^s$ the s-connected component of S containing player $i \in S$.

For efficiency of a value we require that each leaf of the given communication digraph together with all his predecessors realizes the total worth they possess. This generates the first property a value must satisfy, what we call maximal-sink efficiency.

A value ξ on $\mathcal{G}_N^\Gamma$ is *maximal-sink efficient* (MSE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j \in \bar{P}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{P}_\Gamma(i)), \qquad \text{for all } i \in L_\Gamma(N).$$

The next property, called the predecessor equivalence, says that if a directed link is broken, each member of the full predecessors set of the origin of this link still receives the same payoff.

A value ξ on $\mathcal{G}_N^\Gamma$ is *predecessor equivalent* (PE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that for all $(i,j) \in \Gamma$

$$\xi_k(v, \Gamma \backslash (i,j)) = \xi_k(v,\Gamma), \qquad \text{for all } k \in \bar{P}_\Gamma(i).$$

Along with MSE we consider a stronger efficiency property, what we call full-sink efficiency, that requires that every full predecessors set realizes its worth.

A value ξ on $\mathcal{G}_N^\Gamma$ is *full-sink efficient* (FSE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j \in \bar{P}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{P}_\Gamma(i)), \qquad \text{for all } i \in N.$$

It is easy to see that the assumption of sink connectedness in digraph Γ is equivalent to the assumption of tree connectedness in the digraph $\tilde{\Gamma}$ composed by the same set of links as Γ but with the opposite orientation. Moreover, each of axioms MSE, FSE and PE with respect to any cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ is equivalent to the corresponding MTE, FTE or SE axiom with respect to the digraph game $\langle v, \widetilde{\Gamma}\rangle$. In case of sink connectedness the last two observations allow to obtain the following results straightforwardly from the results proved above in Subsections 3.1. and 3.2. under the assumption of tree connectedness.

Proposition 2. *On the class of cycle-free digraph games $\mathcal{G}_N^{\Gamma}$ MSE and PE together imply FSE.*

MSE and PE uniquely define a value on the class of cycle-free digraph games.

Theorem 7. *On the class of cycle-free digraph games $\mathcal{G}_N^{\Gamma}$ there is a unique value s that satisfies MSE and PE. For every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, the value $s(v, \Gamma)$ satisfies the following conditions:*
(i) it obeys the recursive equality

$$s_i(v, \Gamma) = v(\bar{P}_{\Gamma}(i)) - \sum_{j \in P_{\Gamma}(i)} s_j(v, \Gamma), \qquad \text{for all } i \in N; \tag{17}$$

(ii) it admits the explicit representation in the form

$$s_i(v, \Gamma) = v(\bar{P}_{\Gamma}(i)) - \sum_{j \in P_{\Gamma}(i)} \tilde{\kappa}_i(j) v(\bar{P}_{\Gamma}(j)), \qquad \text{for all } i \in N, \tag{18}$$

where for all $i \in N$, $j \in P_{\Gamma}(i)$,

$$\tilde{\kappa}_i(j) = \sum_{r=0}^{n-2} (-1)^r \tilde{\kappa}_i^r(j), \tag{19}$$

and $\tilde{\kappa}_i^r(j)$ is the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = j$, $i_{r+1} = i$, $i_h \in P_{\Gamma}(i_{h-1})$, $h = 1, \ldots, r+1$.

Before stating the next theorem providing simpler explicit representation of the value s we introduce some additional notions and notation. Let

$$P_{\Gamma}^1(i) = \Big\{ j \in P_{\Gamma}(i) \backslash O_{\Gamma}^*(i) \mid \boldsymbol{P}_{\Gamma}(j,i) = \bigcup_{h=1}^{q} \boldsymbol{P}_h, \boldsymbol{P}_h \cap \boldsymbol{P}_l = \emptyset, h \neq l \colon$$
$$\forall h = 1, ..., q, \exists k_h \in S(\boldsymbol{P}_h) \backslash \{j, i\} \colon$$
$$k_h \in \boldsymbol{p}, \forall \boldsymbol{p} \in \boldsymbol{P}_h \text{ and } \boldsymbol{p}_h \cap \boldsymbol{p}_l = \{j, i\}, \forall \boldsymbol{p}_h \in \boldsymbol{P}_h, \forall \boldsymbol{p}_l \in \boldsymbol{P}_l, h \neq l \Big\};$$

and

$$P_{\Gamma}^2(i) = P_{\Gamma}(i) \setminus \big(O_{\Gamma}^*(i) \cup P_{\Gamma}^1(i)\big).$$

Both sets $P_{\Gamma}^1(i)$ and $P_{\Gamma}^2(i)$ are composed by predecessors of i that are not proper superiors of i. $P_{\Gamma}^1(i)$ consists of any such j for which all paths from j to i can be partitioned into a number of separate groups, might be only one group, such that all paths in the same group have at least one common node different from

j and i and paths from different groups do not intersect between j and i. Notice that all $j \in P_\Gamma(i) \setminus O^*_\Gamma(i)$ with $d_i(j) = 1$ belong to $P^1_\Gamma(i)$ since the unique proper subordinate of j belongs to all paths $\boldsymbol{p} \in \boldsymbol{P}_\Gamma(j,i)$; in particular, it holds that $j \in P^1_\Gamma(i)$, when there is only one path from j to i, i.e., when $|\boldsymbol{P}_\Gamma(j,i)| = 1$. From here besides it follows that for all $j \in P^2_\Gamma(i)$, $d_i(j) > 1$. For every $j \in P^1_\Gamma(i)$ we define the *proper out-degree* $\tilde{d}_i(j)$ of j with respect to i as the number of groups $\boldsymbol{P}_h$, $h = 1, ..., q$, in the partition of $\boldsymbol{P}_\Gamma(j,i)$. The subset of $\tilde{M}_\Gamma(j,i) \subseteq S(\boldsymbol{P}_\Gamma(j,i))$, $j \in P_\Gamma(i)$, composed by j, i, all proper intersection points in $S(\boldsymbol{P}_\Gamma(j,i))$ and all proper superiors $h \in O^*_\Gamma(i) \cap S(\boldsymbol{P}_\Gamma(j,i))$ we call the *lower covering set* for $\boldsymbol{P}_\Gamma(j,i)$. Similarly to the definition of $\tilde{\kappa}_i(j)$ given by (19) we define

$$\tilde{\kappa}^M_i(j) = \sum_{r=0}^{n-2} (-1)^r \tilde{\kappa}^{r,M}_i(j),$$

where $\tilde{\kappa}^{r,M}_i(j)$ counts only the tuples $(i_0, ..., i_{r+1})$ for which $i_0 = j$, $i_{r+1} = i$, and $i_h \in P_\Gamma(i_{h-1}) \cap \tilde{M}_\Gamma(j,i)$, $h = 1, \ldots, r+1$.

Theorem 8. *For every cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}^\Gamma_N$ the value s given by (18) admits the equivalent representation in the form*

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j \in O^*_\Gamma(i)} v(\bar{P}_\Gamma(j)) + \\ + \sum_{j \in P^1_\Gamma(i)} (\tilde{d}_i(j) - 1) v(\bar{P}_\Gamma(j)) - \sum_{j \in P^2_\Gamma(i)} \tilde{\kappa}^M_i(j) v(\bar{P}_\Gamma(j)), \ \ \textit{for all } i \in N. \qquad (20)$$

If the consideration is restricted to only strongly cycle-free digraph games, then the above representation reduces to

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j \in O_\Gamma(i)} v(\bar{P}_\Gamma(j)), \quad \textit{for all } i \in N. \qquad (21)$$

For sink-forest digraph games defined by sink forest digraph structures that are strongly cycle-free, the value given by (21) coincides with the sink value introduced in (Khmelnitskaya, 2010). By that reason from now on we refer to the value s given by (18), or equivalently by (20), as to the *sink-tree value*, or simply the *sink value*, for cycle-free digraph games.

The sink value assigns to every player the payoff equal to the worth of his full predecessors set minus the worths of all full predecessors sets of his proper superiors plus or minus the worths of all full predecessors sets of any other of his predecessors that are subtracted or added more than once. Moreover, for any player $i \in N$ and his predecessor $j \in N$ that is not his proper superior, the coefficient $\tilde{\kappa}_i(j)$ indicates the number of overlappings of full predecessors sets of all proper superiors of i at node j. In fact a player receives what he contributes when he joins his predecessors when only the full predecessors sets, that are the only efficient productive coalitions under given assumptions, are counted. Since a root has no predecessors, a root just gets his own worth. Furthermore, it is not difficult to check that the right-sides of both formulas (20) and (21) being considered with respect not to coalitional worths but to players in these coalitions contain only player i when taking into account all pluses and minuses. Besides, according to (17) the sink value assigns to every player the worth of his full predecessors set minus the total payoff to his predecessors. Wherefrom we obtain a simple recursive algorithm for computing the sink value going downstream from the roots of the given digraph.

Example 7. Figure 7 provides an example of the sink value for a 10-person game with cycle-free but not strongly cycle-free digraph structure.

The sink value may be computed in two different ways, either by the recursive algorithm based on the recursive equality (17), or using the explicit formula representation (20).

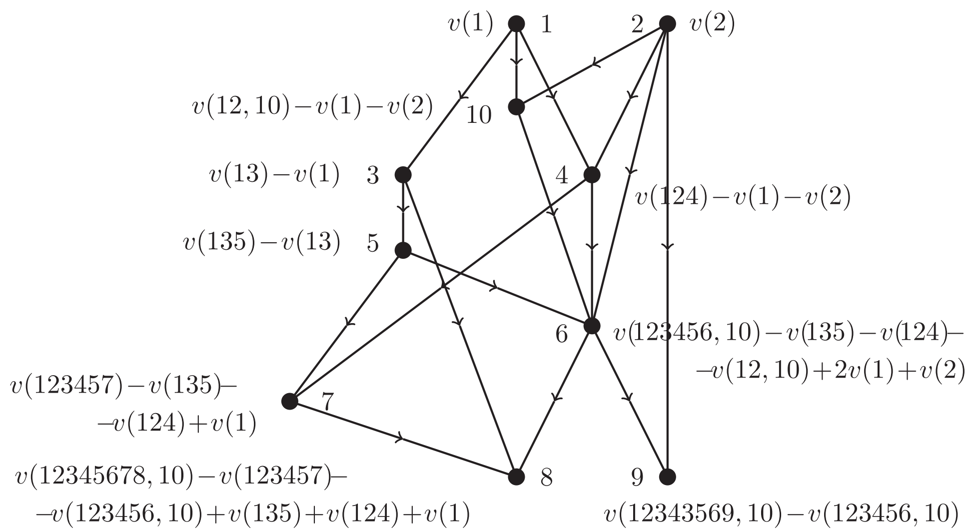


Fig. 7.

The sink value not only meets FSE but FSE alone uniquely defines the sink value on the class of cycle-free digraph games.

Theorem 9. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ the sink value is the unique value that satisfies FSE.*

Corollary 3. FSE on the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ implies not only MSE but PE as well.

The next theorem derives the total sink value payoff for any s-connected coalition.

Theorem 10. *In a cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for any s-connected coalition $S \in C_s^\Gamma(N)$ it holds that*

$$\sum_{i\in S} s_i(v,\Gamma) = \sum_{i\in L_\Gamma(S)} v(\bar{P}_\Gamma(i)) - \sum_{i\in S\setminus L_\Gamma(S)} \big(\tilde{\kappa}_S(i)-1\big) v(\bar{P}_\Gamma(i)) - \sum_{i\in \bar{P}_\Gamma(S)\setminus S} \tilde{\kappa}_S(i) v(\bar{P}_\Gamma(i)),$$

where

$$\bar{P}_\Gamma(S) = \bigcup_{i\in L_\Gamma(S)} \bar{P}_\Gamma(i),$$

$$\tilde{\kappa}_S(i) = \sum_{j \in \bar{S}_\Gamma(i) \cap \bar{P}_\Gamma(S)} \tilde{\kappa}_j(i), \quad \textit{for all } i \in \bar{P}_\Gamma(S),$$

while $\tilde{\kappa}_S(i) = 1$ *when* $\tilde{d}_N(i) = 1$*, where for any s-connected coalition* $S \in C_s^\Gamma(N)$*, for all* $i \in \bar{P}_\Gamma(S)$*,* $\tilde{d}_S(i)$ *is the out-degree of* i *in the subgraph* $\Gamma|_{\bar{P}_\Gamma(S)}$*, i.e.,*

$$\tilde{d}_S(i) = |F_\Gamma(i) \cap \bar{P}_\Gamma(S)|,$$

in particular, $\tilde{d}_N(i) = \tilde{d}_\Gamma(i)$ *for all* $i \in N$*.*

If the consideration is restricted to only strongly cycle-free digraph games, then for any s-connected coalition $S \in C_s^\Gamma(N)$ *it holds that*

$$\sum_{i \in S} s_i(v,\Gamma) = \sum_{i \in L_\Gamma(S)} v(\bar{P}_\Gamma(i)) - \sum_{i \in S \setminus L_\Gamma(S)} \big(\tilde{d}_S(i) - 1\big) v(\bar{P}_\Gamma(i)) - \sum_{i \in L_\Gamma(\bar{P}_\Gamma(S) \setminus S)} \tilde{d}_S(i) v(\bar{P}_\Gamma(i)).$$

For any cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ the overall efficiency is given by

$$\sum_{i \in N} s_i(v,\Gamma) = \sum_{i \in L_\Gamma(N)} v(\bar{P}_\Gamma(i)) - \sum_{i \in N \setminus L_\Gamma(N)} \big(\tilde{\kappa}_N(i) - 1\big) v(\bar{P}_\Gamma(i)),$$

while if the consideration is restricted to only strongly cycle-free digraph games, the last equality reduces to

$$\sum_{i \in N} s_i(v,\Gamma) = \sum_{i \in L_\Gamma(N)} v(\bar{P}_\Gamma(i)) - \sum_{i \in N \setminus L_\Gamma(N)} \big(\tilde{d}_\Gamma(i) - 1\big) v(\bar{P}_\Gamma(i)).$$

For a cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, the *s-core* $C^s(v,\Gamma)$ is defined as the set of component efficient payoff vectors that are not dominated by any s-connected coalition,

$$C^s(v,\Gamma) = \{x \in \mathbb{R}^N \mid x(C) = v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_s^\Gamma(N)\},$$

while the *weak s-core* $\tilde{C}^s(v,\Gamma)$ as the set of weakly component efficient payoff vectors that are not dominated by any s-connected coalition,

$$\tilde{C}^s(v,\Gamma) = \{x \in \mathbb{R}^N \mid x(C) \leq v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_s^\Gamma(N)\}.$$

Theorem 11. *The sink value on the subclass of superadditive sink-forest digraph games is s-stable.*

A cycle-free digraph game $\langle v, \Gamma \rangle$ is *s-convex*, if for all s-connected coalitions $T, Q \subset C_s^\Gamma(N)$ such that T is a full s-connected set, Q is a full predecessors set, and $T \cup Q \in C_s^\Gamma(N)$, it holds that

$$v(T) + v(Q) \leq v(T \cup Q) + v(T \cap Q).$$

Theorem 12. *The sink value on the subclass of s-convex strongly cycle-free digraph games is weakly efficient.*

4. Sharing a river with multiple sources, a delta and possible islands

In (Ambec and Sprumont, 2002) the problem of optimal water allocation for a given river with certain capacity over the agents (cities, countries) located along the river is approached from the game theoretic point of view. Their model assumes that between each pair of neighboring agents there is an additional inflow of water. Each agent, in principal, can use all the inflow between itself and its upstream neighbor, however, this allocation in general is not optimal in respect to total welfare. To obtain a more profitable allocation it is allowed to allocate more water to downstream agents which in turn can compensate the extra water obtained by side-payments to upstream ones. The problem of optimal water allocation is approached as the problem of optimal welfare distribution. In (van den Brink et al., 2007) it is shown that the Ambec-Sprumont river game model can be naturally embedded into the framework of a graph game with line-graph cooperation structure. In (Khmelnitskaya, 2010) the line-graph river model is extended to the rooted-tree and sink-tree digraph model of a river with a delta or with multiple sources, respectively. We extend the line-graph, rooted-tree or sink-tree model of a river to the cycle-free digraph model of a river with both multiple sources and a delta, and also possible islands along the river bed as well.

Let N be a set players (users of water) located along the river from upstream to downstream. Let $e_{ki} \geq 0$, $i \in N$, $k \in O(i)$, be the inflow of water in front of the most upstream player(s) (in this case $k = 0$) or the inflow of water entering the river between neighboring players in front of player i. Figure 8 provides a schematic representation of the model.

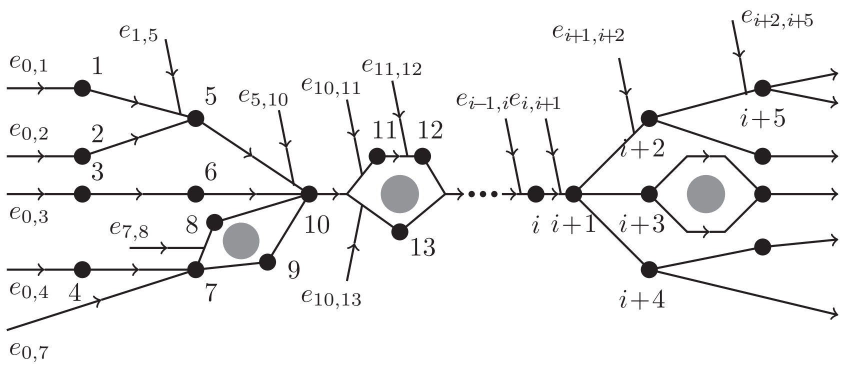


Fig. 8.

A river with multiple sources, a delta, and several islands along the river bed

Following (Ambec and Sprumont, 2002) it is assumed that each player $i \in N$ has a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$ where t_i is a monetary compensation to player i, x_i is the amount of water allocated to player i, and $b^i\colon \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function providing benefit $b^i(x_i)$ to player i when he consumes the amount x_i of water. Moreover, in case of a river with a delta it is also assumed that if a splitting of the river into branches happens to occur after a certain player, then this player takes, besides his own quota, also the responsibility to split the rest of the water flow to the branches such to guarantee the realization of the water distribution plan x^* to his successors.

The superadditive river game $v \in \mathcal{G}_N$ introduced under the same assumptions in (Khmelnitskaya, 2010) for a river with multiple sources or a delta defined as:

for any connected coalition $S \subseteq N$, $v(S) = \sum_{i \in S} b^i(x_i^S)$, where $x^S \in \mathrm{I\!R}^s$ solves

$$\max_{x \in \mathrm{I\!R}^s_+} \sum_{i \in S} b^i(x_i) \;\; s.t. \;\; \begin{cases} \displaystyle\sum_{j \in \bar{P}_\Gamma(i)} x_j \leq \sum_{j \in \bar{P}_\Gamma(i)} \sum_{k \in O(j)} e_{kj}, \\ \displaystyle\sum_{j \in P_\Gamma(i) \cup \bar{B}_\Gamma(i)} x_j \leq \sum_{j \in P_\Gamma(i) \cup \bar{B}_\Gamma(i)} \sum_{k \in O(j)} e_{kj}, \end{cases} \quad \forall i \in S,$$

and for any disconnected coalition $S \subset N$, $v(S) = \sum_{T \in C^\Gamma(S)} v(T)$,

suits to the case of a river with both multiple sources and a delta, and also possible islands along the river bed as well. The tree and sink values proposed above can be applied for the solution of the river game in the general case.

References

Ambec, S. and Y. Sprumont (2002). *Sharing a river*, Journal of Economic Theory, **107**, 453–462.

Demange, G. (2004). *On group stability in hierarchies and networks*, Journal of Political Economy, **112**, 754–778.

Gillies, D. B. (1953). *Some Theorems on n-Person Games*, Ph.D. thesis, Princeton University.

Herings, P. J. J., G. van der Laan, and A. J. J. Talman (2008). *The average tree solution for cycle-free graph games*, Games and Economic Behavior, **62**, 77–92.

Khmelnitskaya, A. B. (2010). *Values for rooted-tree and sink-tree digraphs games and sharing a river*, Theory and Decision, **69**, 657–669.

Myerson, R. B. (1977). *Graphs and cooperation in games*, Mathematics of Operations Research, **2**, 225–229.

van den Brink, R., G. van der Laan, and V. Vasil'ev (2007). *Component efficient solutions in line-graph games with applications*, Economic Theory, **33**, 349–364.