A study of resonance tongues near a Chenciner bifurcation using MatcontM

Govaerts W. [1], Kuznetsov Yu.A. [2], Meijer H.G.E.[3] and Neirynck, N.[1]
[1] *Dept. of Applied Mathematics and Computer Science, Ghent University, Gent, Belgium*
[2] *Dept. of Mathematics, Utrecht University, Netherlands*
[3] *Department EEMCS, University of Twente, PO Box 217, 7500 AE Enschede, Netherlands*

Summary. MatcontM is a matlab toolbox for numerical analysis of bifurcations of fixed points and periodic orbits of maps. It computes codim 1 bifurcation curves and supports the computation of normal coefficients including branch switching from codim 2 points to secondary curves. Recently, the initialization and computation of connecting orbits was improved. Moreover, a graphical user interface was added enabling interactive control of all these computations. To further support these computations it allows to compute orbits of the map and its iterates and to represent them in 2D, 3D and numeric windows. We demonstrate the use of the toolbox in a study of Arnol'd tongues near a degenerate Neimark-Sacker (Chenciner) bifurcation. Here we illustrate the recent theory of [4] how resonance tongues interact with a quasi-periodic saddle-node bifurcation of invariant curves in maps. Using normal form coefficients we find evidence for one of their cases, but not the other. Actually, we find another unfolding, i.e. a third possibility. We also find a structure that resembles a quasi-periodic cusp bifurcation of invariant curves.

MatContM: Numerical Bifurcation Analysis of Maps

We consider dynamical systems generated by (nonlinear) smooth maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m \tag{1}$$

with state variable x and free parameters α. The understanding of the dynamics of the map (1) requires most often numerical bifurcation analysis with the use of a dedicated software package. For this purpose we have written MATCONTM, a MATLAB continuation toolbox available at http://sourceforge.net/projects/matcont, [1]. It computes bifurcations of fixed (periodic) points in one parameter and continues the bifurcation curve in two parameters and detects codim 2 bifurcation points, similar to other packages. In addition, critical normal form coefficients are computed where for fixed points with high iterates we rely on symbolic derivatives or if these are not available, on automatic differentiation [3]. Also, we grow 1D stable and unstable invariant manifolds of fixed points for n-dimensional maps, using generalizations of the standard algorithms for $n = 2$. For planar systems the search of intersection points is automated allowing easy initialization of continuation of homoclinic and heteroclinic orbits and also their tangencies. The most recent addition is a graphical user interface that allows interactive use of the various routines, see Figure 1.

Working with the command line version

The command line version contains a continuer that computes bifurcation curves. These curves are defined by a system of equations consisting of fixed point and bifurcation conditions. With one free parameter we have curves of fixed points and connecting orbits. With two free system parameters we can compute bifurcation curves of limit point, period-doubling and Neimark-Sacker as well as tangencies of homoclinic and heteroclinic manifolds. For the actual continuation we do the following steps.

We first select an initial point. This point may be obtained from analysis, simulations or previous continuations. We then initialize the continuer by setting certain variables. For instance, we set the map that we investigate, the system that defines the bifurcation curve, the use of derivatives, the iteration number of a fixed point and so on. Next, we specify continuer options such as tolerances and maximal stepsize. Then we call the continuer that computes the curve. Simultaneously, the continuer monitors the occurrence of zeros of test functions. Such a zero may correspond to a bifurcation of higher codimension. Finally, we can inspect the data and repeat this process.

The organization of the GUI

The GUI has been developed using the object-oriented syntax added in Matlab since version 7.6 (2008a). There are two important aspects of the GUI. First, there is a model that controls data and via the GUI the user can generate and manage such data. Figure 2 illustrates the implementation of this idea. Second, we have built the GUI on top of the command line (cl) version of MatcontM. That is, we keep the cl-version intact and call the continuer, the computational core, with correct settings from the GUI.

The data consists of a current system and a current curve. The current system specifies the map currently analyzed and special user functions to be monitored in addition of standard test functions. There is also a current diagram to which newly computed curves are added. The current curve consists of an initial point type and a curvetype. This data can be inspected using the Data browser. We have the folllowing natural hierarchy in the data:

- **System** This is the map that we currently analyze. A new system can be added or an existing one deleted.
- **Diagrams** Computed curves are stored in an active diagram, i.e. directory. The user can select one active diagram among multiple. Data can be transferred between diagrams as well.

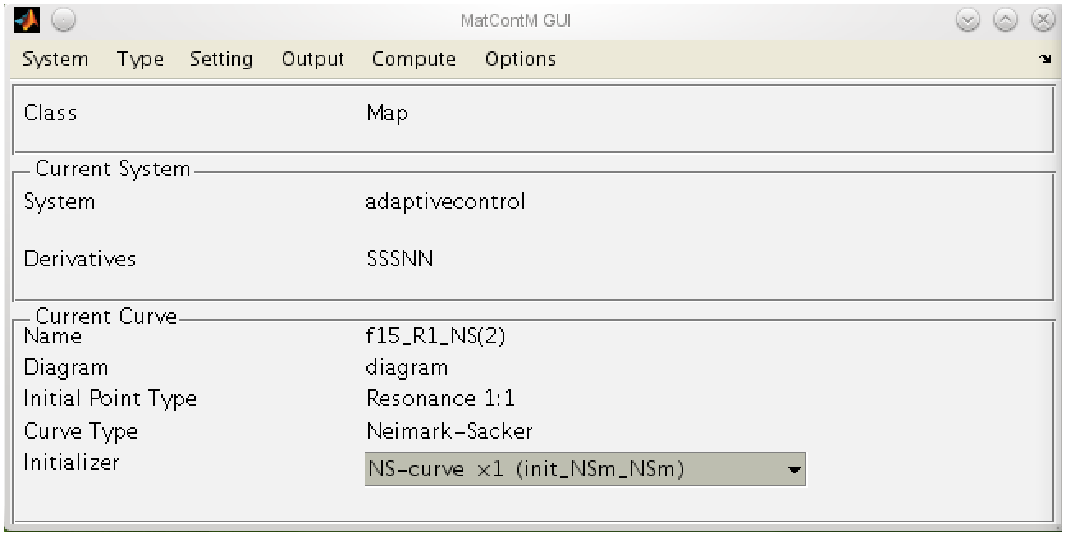

Figure 1: A screenshot of the GUI for MATCONTM. The main menu allows to select or edit the map, select initial point and curve type, set various continuation settings, start the continuation. Output can be generated graphically with 2D and 3D plots or numeric windows.

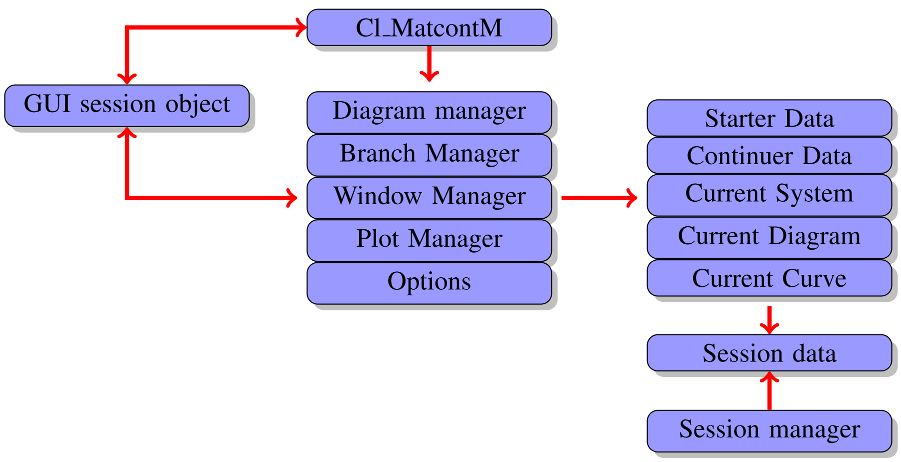

Figure 2: Model view control scheme: The model controls the data via several managers.

- **Curves** A diagram consists of computed curves. The automatic naming scheme adopted is as follows: dn_PT_CT(x). Here d stands for the direction along the curve (forward or backward), n is the iteration number, PT is the initial point type, CT is the curve type and x is the number of the curve among all curves of its particular type. A curve can be loaded to start the continuation of a similar curve. We can also rename or delete curves.

- **(Special) Points** A curve actually consists of a set of (ordered) points. We can browse these by loading them into the MATLAB array editor or selecting special points. These special points correspond to bifurcations of higher codimension along the curve. The first and last point are also always given. We can check the settings of the starter and continuer by inspecting the Starterdata and Continuerdata, respectively.

This data is managed through several *functions*, which we describe below.

- **Startdata** This function controls the initial point, i.e. the state variables and parameters, and the iteration number. Here we select active parameters and adjust settings for derivatives. This is also called **Starter**.

- **Continuer** This function controls the general parameters of the continuation, such as minimal and maximal step-sizes, the number of points to be computed, when to decide that the computation of a bifurcation point was successful, etc.

- **Branchmanager** This function monitors the selection of the initializer currently chosen. There are certain codim 2 bifurcations from which branches with higher iteration number (2, 3 or 4 times the original period) can be started. Therefore, the correct iteration number is managed by this function as well.

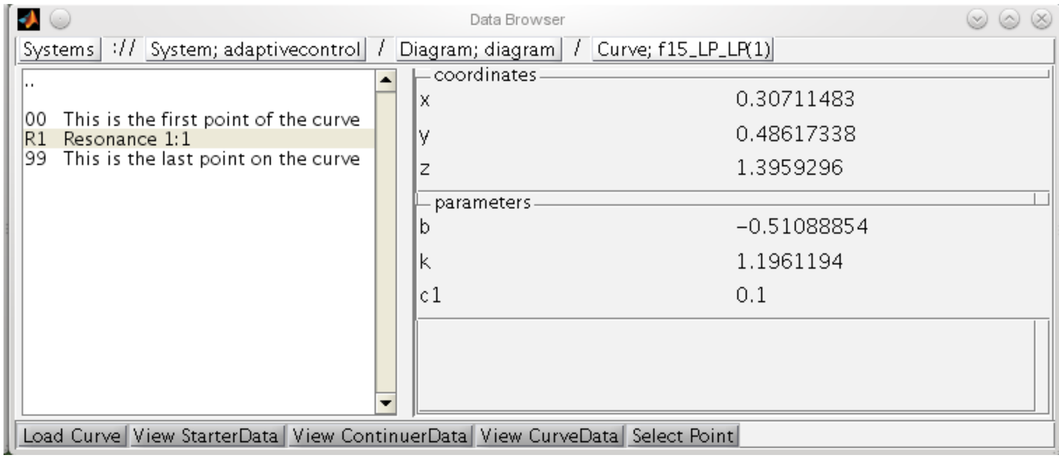

Figure 3: A screenshot of the data browser for MATCONTM. One can specify new systems or edit existing ones. The main menu allows to select or edit the map, select initial point and curve type, set various continuation settings, start the continuation. Output can be generated graphically with 2D and 3D plots or numeric windows.

- **Curvemanager** This function decides how a computed curve is named and where it is saved. It further decides (using **Options**) whether old curves are deleted. The user can move curves to different diagrams, rename or delete them.
- **Windowmanager** This function controls which windows are open, where they are placed on the screen and which data are represented.
- **Plotmanager** This function interacts between the sessiondata and the windows, using the **Options**.
- **Options** The user can specify preferences on the runtime options. One can pause the computation after each point, at special points only or never. Computed points are printed in a dialog window after a number of steps that can be specified. The maximum number of computed curves of a particular type can be specified. This limits the size of the archive.

New data can be generated by an actual continuation. This proceeds similar as when using the cl-version but automated. First, the continuer options are generated and some sanity checks are performed. We need to check whether we extend the current curve or start a new curve. Command line data structures are set up. Next we call the initializer and then the continuer. During computation there is a simple output screen displaying the computed points and other messages. These messages are start, end or failure of the continuation and special points together with their normal form coefficients. The information of the current point can also be monitored using a numeric window. Finally, we can plot the computed data during the continuation as described above using the Plotmanager.
We note that MatContM also allows simulation, i.e. the computation of orbits of the map and its iterates and their representation in 2D, 3D and numerical windows. This is useful both to find starting data for continuation and for the validation of the results obtained by continuation.

Resonance bubbles near a Chenciner bifurcation: theory

We use the software here to understand the precise bifurcation structure near a degenerate Neimark-Sacker (NS), i.e. a Chenciner (CH), bifurcation. It is known that away from the CH point one can expect either stable or unstable invariant curves to emanate from the NS bifurcation, see Figure 4(left). The boundary between the parameter region with two invariant curves and that with only a fixed point has a delicate bifurcation structure. These invariant curves may have irrational enough, i.e. Diophantine, rotation number and then collide in a quasi-periodic saddle-node bifurcation of invariant curves. This happens on a set of positive measure but excludes a set of parameters that is the collection of diamond shaped regions. Inside these diamonds, the rotation number is rational, i.e. there is q-periodic behavior inside an Arnol'd tongue in parameter space. When the Arnol'd tongue passes through the diamond (in parameter space) a delicate bifurcation structure arises as proposed by [4]. Their unfolding is shown schematically with local bifurcations only in Figure 4(middle and right).

Resonance bubbles near a Chenciner bifurcation: example

We want to see the structure described above in the following map [6]

$$F : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ bx + k + zy \\ z - (bx + k + zy - 1)ky/(c + y^2) \end{pmatrix} \qquad (2)$$

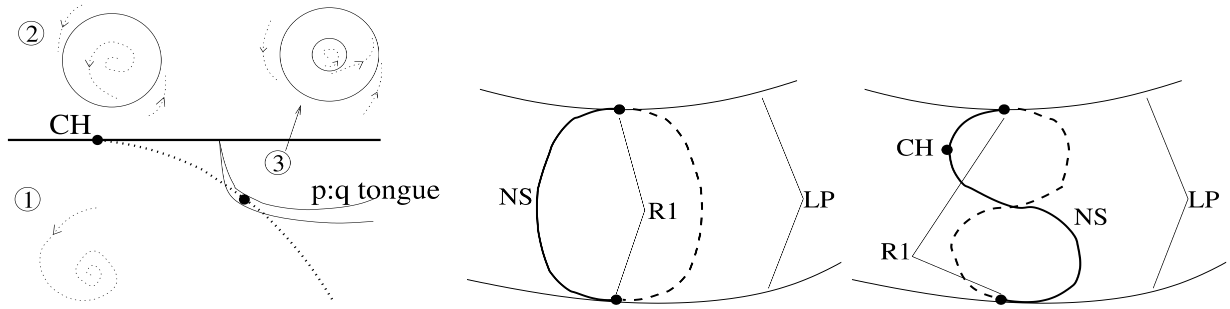

Figure 4: Unfolding near a Chenciner bifurcation with sub- and super-critical NS bifurcations on either side. Along a parabolic curve a stable and unstable invariant curve collide. This happens as a quasi-periodic saddle-node bifurcation or through a resonance bubble. Such a resonance occurs inside an Arnol'd tongue demarcated by two limit point curves. According to [4], this resonance bubble connects two R1 points via a NS curve that can either be simple (middle) or contain a figure eight (right). Labels: LP=limit point and R1=1:1 resonance.

with $c = 0.1$ and k, b free parameters. A partial bifurcation diagram is shown in figure 5(left), other details may be found in [6]. Arnol'd tongues of period q ($4 < q < 101$) are shown. Initial points were computed by simulation and checked for periodicity. Then the outer boundaries of the tongue were computed with continuation. Along these boundaries we detected R1 points of fixed points of period q.

Some of these R1 points come in pairs, one on each boundary of the tongue. This occurs for with $-.53 < b < -.45$. These pairs correspond to resonances near a quasi-periodic saddle-node bifurcation and their rough unfolding is sketched above. This occurs for with $-.53 < b < -.45$. MATCONTM automatically computes the normal form coefficient s yielding the local stability of the bifurcating invariant curve. For a simple resonance bubble, the sign of s on both sides of the tongue is equal, while in the other case it is not. First, we inspected the sign of s for each R1 point in a bubble. These coincided for all $q < 100$. Second, we checked the geometry of resonance bubbles. None of the NS-curves showed self-intersections. Summarizing, we find only the evidence for the simple unfolding. The situation is however more complicated. We detect two Chenciner bifurcations for some resonance bubbles of high periods, see Figure 5(right), yielding a third possibility for unfolding a bubble.

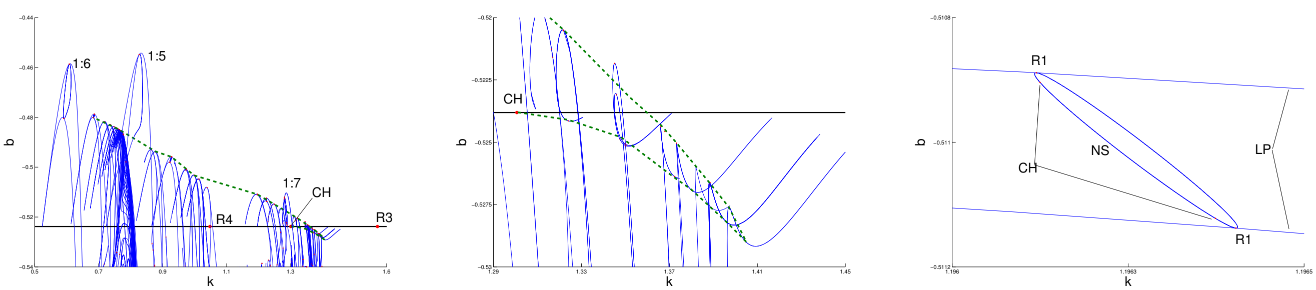

Figure 5: Left: Partial bifurcation diagram of map (2). The black line indicates the NS bifurcation of the fixed point $(x, y, z) = (1, 1, 1 - b - k)$ with strong resonances (R3 and R4) and a Chenciner bifurcation (near $b = -.5238, k = 1.30$) indicated by red dots. Some weak resonance tongues are indicated. All other red dots correspond to R1 points of fixed point with period>4. The green dashed curve indicates for which parameters approximately, two invariant curves collide. Middle: some tongues loop around and display two resonance bubbles. Right: Resonance bubble within an Arnol'd tongue with period 15 with two Chenciner points.

We take the resonance bubbles as an indication where the quasi-periodic saddle-node bifurcation occurs, which we have plotted with a green dashed curve. Theoretically this curve should depart quadratically from the Chenciner point which it appears to do indeed. Continuing we see that it turns around suggestive of a quasi-periodic cusp bifurcation. Indeed, also the Arnol'd tongues make a turn and show two resonance bubbles, see Figure 5(middle). We note however that very close to this point we have also found a homoclinic structure of a period 3 orbit starting at the R3 point. The full bifurcation diagram is still under investigation. One feature that follows from the double folding of the Arnol'd tongues is that three global invariant curves coexist for a small parameter region, with or without resonances. One emanates from the supercritical branch of the Neimark-Sacker bifurcation curve. It has small amplitude and is stable. There is a large stable invariant curve and one unstable in between, see Figure 6.

References

[1] W. Govaerts, R. Khoshsiar Ghaziani, Yu.A. Kuznetsov and H.G.E. Meijer, Numerical methods for two-parameter local bifurcation analysis of maps, *SISC* 29: 2644-2667, 2007.

[2] W. Govaerts, R. Khoshsiar Ghaziani, Yu.A. Kuznetsov, and H.G.E. Meijer, Numerical continuation of connecting orbits of maps, *Journal of Difference Equations and Applications*, 15 (8): 849-875, 2009.

[3] J.D. Pryce, R. Khoshsiar Ghaziani, V. De Witte, W. Govaerts, Computation of normal form coefficients of cycle bifurcations of maps by algorithmic differentiation, *Mathematics and Computers in Simulation* 81: 109-119, 2010.

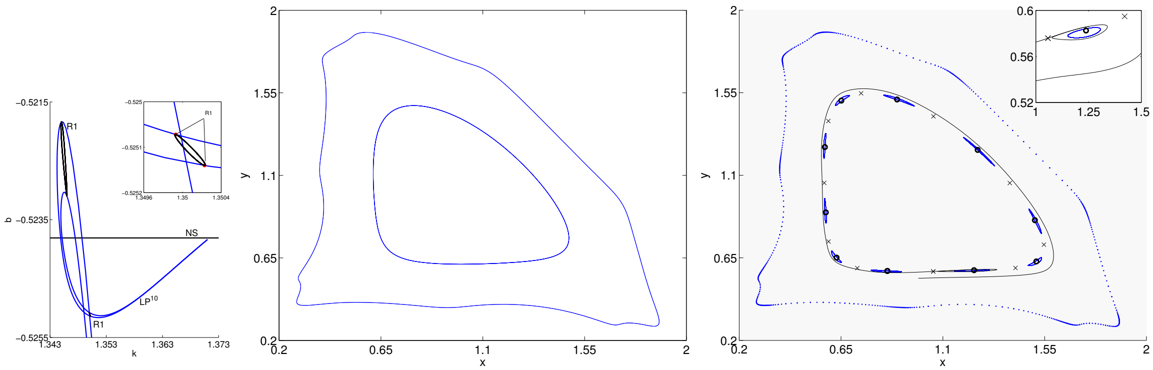

Figure 6: Left: Bifurcation diagram showing an Arnol'd tongue of period 10 (LP^{10}-curves blue) emanating from the Neimark-Sacker curve (black). The tongue makes two turns accompanied by the occurrence of a resonance bubble with (closed) NS-curves of period 10 (black). The inset shows an enlargement of the lower bubble. Middle: Phase portrait of (2) at $(b, k) = (-.525, 1.35)$. Two stable invariant curves, there is an unstable one in between. Right: Phase portrait of (2) at $(b, k) = (-.525115, 1.35015)$. The outer invariant curve is close to 1:10 phase-locking. The two inner invariant curves have broken up in a resonance structure with saddles (x) and foci (o). In this bubble there is a supercritical Neimark-Sacker resulting in 10 small invariant curves that are periodically visited under iteration. The black lines indicate the unstable manifold of the saddle (for clarity only one is shown). The inset shows the small invariant curve and the unstable manifold more clearly.

[4] Baesens C. and MacKay R.S., Resonances for weak coupling of the unfolding of a saddle-node periodic orbit with an oscillator, *Nonlinearity* 20: 1283-1298, 2007

[5] Chenciner A, Bifurcations de points fixes elliptiques: III. Orbites périodiques de petites périodes et élimination résonnante des couples de courbes invariantes *Publ. Math. IHES* 66: 5-91, 1989

[6] Frouzakis C.E., Adomaitis R.A. and Kevrekidis I.G., Resonance phenomena in an adaptively-controlled system, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 1:, 83-106, 1991