Viscothermal wave propagation in a circular layer with a partially open and partially closed boundary

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Abstract

The so called *low reduced frequency model* has been shown to be both an accurate and a relatively simple description of wave propagation in narrow tubes or layers, under small signal conditions. In this paper, the low reduced frequency model will be applied on a circular layer between a fixed surface and a rigidly translating plate. The outer circumference of the layer is partially closed (Neumann boundary condition) and partially open (Dirichlet boundary condition). A *semi-analytical solution* for this problem is used to calculate the volume flow that is generated by the squeezing motion of the plate, and the resulting force on the plate. The volume flow per unit force is evaluated for several boundary conditions.

1 Introduction

Acoustic wave propagation under small signal conditions is usually described by the wave equation. This equation does not take thermal and viscous effects into account. This simplification is justified for wave propagation in relatively large spaces. In narrow layer (or tube) geometries however, the thermal and viscous effects can not be neglected. The wave equation is therefore not suitable to describe wave propagation in these geometries. The so called low reduced frequency model has been shown to be an accurate description of wave propagation, under these conditions [1]. This model is much simpler than the (linearized) Navier Stokes model from which it has been derived. The model has been used successfully to describe the behavior of folded solar panels during launch and the transmission loss of double wall panels, for example. The receiver of a hearing aid device, i.e. a tiny loudspeaker, is another application in which acoustic waves propagate in thin layers. It could be a design criterion to squeeze a certain amount of air out of a layer, using as less force as possible. The volume flow per unit force can be calculated from a semi-analytical solution of the low reduced frequency model. The effect of several changes in boundary conditions on the volume flow per unit force will be shown in this paper.

2 Theory

In this paper, the behavior of a circular layer of air will be modeled. This could be a layer in a hearing aid device. The layer is located between a fixed surface and a circular rigid plate that translates perpendicularly to this surface. The plate and fixed surface are parallel to each other. The gap is either open or closed at the outer circumference. Figure 1 gives a schematic overview of this system. The barriers close off the air layer so air can only escape through the openings. The polar coordinate system ($\bar{r}$,θ,$\bar{z}$) is displayed in this figure. The gap height h_0 is assumed to be very small compared to the wavelength of sound.

Due to the small height of the air layer it is expected that viscous effects can not be neglected. The low reduced frequency model is an accurate model for this situation and will be used to describe the air layer.

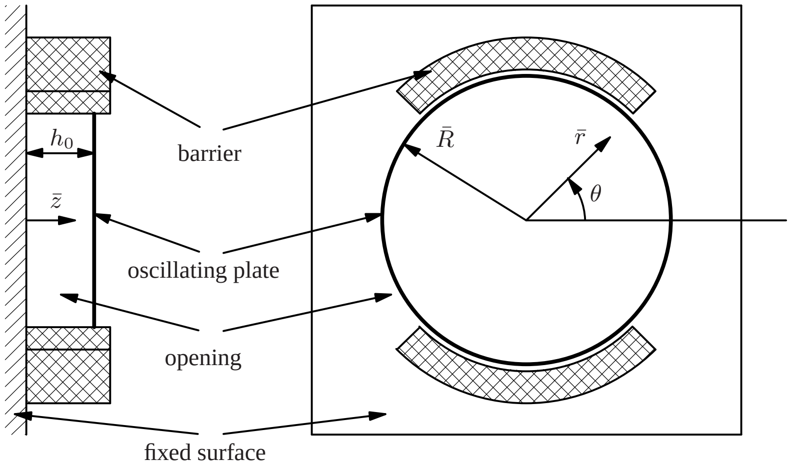

Figure 1: Plate translating near a fixed surface

An introduction to this model, the applied boundary conditions, and the procedure to formulate the semi-analytical solution will be presented in the next sections.

2.1 The low reduced frequency model

The low reduced frequency model is derived from the Navier stokes equations, see Beltman [1]. This model can be applied to an air layer between parallel plates. The equations of this model, written in dimensionless cylinder coordinates (r, θ, z), are:

$$iv^r = -\frac{1}{\gamma}\frac{\partial p}{\partial r} + \frac{1}{s^2}\frac{\partial^2 v^r}{\partial z^2}, \tag{1a}$$

$$iv^\theta = -\frac{1}{\gamma r}\frac{\partial p}{\partial \theta} + \frac{1}{s^2}\frac{\partial^2 v^\theta}{\partial z^2}, \tag{1b}$$

$$0 = \frac{\partial p}{\partial z}, \tag{1c}$$

$$0 = \frac{1}{r}\frac{\partial (rv^r)}{\partial r} + \frac{1}{r}\frac{\partial v^\theta}{\partial \theta} + \frac{1}{k}\frac{\partial v^z}{\partial z} + i\rho, \tag{1d}$$

$$p = \rho + T, \tag{1e}$$

$$iT = \frac{1}{(s\sigma)^2}\frac{\partial^2 T}{\partial z^2} + i\frac{\gamma - 1}{\gamma}p, \tag{1f}$$

where v^r, v^θ, and v^z denote the dimensionless velocities in the r, θ, and z directions; ρ, p, and T denote the dimensionless density, dimensionless pressure, and dimensionless temperature; and r, θ, and z are the dimensionless coordinates. The following relations were used to make the velocities, thermodynamic properties, and coordinates dimensionless:

$$\begin{aligned}
&\bar{v}^r = c_0 v^r e^{i\omega t}, && \bar{\rho} = \rho_0(1 + \rho e^{i\omega t}), && \bar{r} = \tfrac{c_0}{\omega} r,\\
&\bar{v}^\theta = c_0 v^\theta e^{i\omega t}, && \bar{p} = p_0(1 + p e^{i\omega t}), && \theta,\\
&\bar{v}^z = c_0 v^z e^{i\omega t}, && \bar{T} = T_0(1 + T e^{i\omega t}), && \bar{z} = h_0 z.
\end{aligned} \tag{2}$$

The barred variables are dimensionfull, c_0 is the speed of sound, ρ_0, p_0, and T_0 are the properties of the air in the layer at rest, ω is the frequency in radians per second, and h_0 is the thickness of the air layer.

The solution to the equations depends on the boundary conditions and the four dimensionless parameters:

$$k = \frac{h_0\omega}{c_0}, \qquad s = h_0\sqrt{\frac{\rho_0\omega}{\mu}}, \qquad \gamma = \frac{C_p}{C_v}, \qquad \sigma = \sqrt{\frac{\mu C_p}{\lambda}}, \tag{3}$$

with μ, C_p, C_v, and λ denoting viscosity, specific heat at constant pressure, specific heat at constant volume, and thermal conductivity. The square root of the Prandtl number σ and the ratio of specific heats γ are thermodynamic constants. The two other parameters are *the low reduced frequency* k and the *shear wave number* s. These two parameters depend on the layer thickness h_0 and the (angular) frequency ω and are therefore the most interesting from an engineering point of view.

2.2 Boundary conditions at the oscillating plate and the fixed surface

The boundary conditions that are applied at the upper and lower boundaries of the layer are the no-slip condition (the air at the boundaries has the same velocity as the surfaces) and the isothermal condition (air at the boundary has the same static temperature as the plates). The lower boundary is a fixed (static) surface located at $z = 0$. The upper boundary is a plate that translates rigidly around $z = 1$ (thus $\bar{z} = h_0$) according to:

$$\bar{h} = h_0(1 + he^{i\omega t}). \tag{4}$$

Thus the variable h is the dimensionless amplitude of the translation of the plate. Beltman [1] shows how these boundary conditions reduce the system of equations to a second order partial differential equation for the pressure that has a strong resemblance to the wave equation (in cylinder coordinates):

$$\frac{1}{r}\frac{\partial(r\frac{\partial p}{\partial r})}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \Gamma^2 p = n\Gamma^2 h, \tag{5a}$$

with:

$$\begin{aligned} \Gamma &= \sqrt{\tfrac{\gamma}{nB}}, & n &= \left(1 + \tfrac{\gamma-1}{\gamma} D\right)^{-1}, \\ B &= 2\left(\tfrac{\cosh(s\sqrt{i})-1}{s\sqrt{i}\sinh(s\sqrt{i})}\right) - 1, & D &= 2\left(\tfrac{\cosh(s\sigma\sqrt{i})-1}{s\sigma\sqrt{i}\sinh(s\sigma\sqrt{i})}\right) - 1. \end{aligned} \tag{5b}$$

The parameter Γ is called 'propagation constant' and n 'polytropic constant'. Viscous effects are applied trough the parameter B and thermal effects through the parameter D. These parameters change if the boundary conditions change (to adiabatic boundary conditions for instance). The differential equation shows that the pressure is independent of the z coordinate. This is a direct consequence of (1c). The problem is now reduced to a two dimensional problem.

The particle velocities in the propagation directions can be expressed in terms of the pressure p (for the other properties see [1]):

$$\hat{v}^r = -\frac{iB}{\gamma}\frac{\partial p}{\partial r}, \tag{6a}$$

$$\hat{v}^\theta = -\frac{iB}{\gamma r}\frac{\partial p}{\partial \theta}, \tag{6b}$$

in which $\hat{v}^r$ and $\hat{v}^\theta$ are the particle velocities averaged over the layer thickness:

$$\hat{v}^r = \int_0^1 v^r \, \mathrm{d}z, \tag{6c}$$

$$\hat{v}^\theta = \int_0^1 v^\theta \, \mathrm{d}z \tag{6d}$$

The shape of the velocity profile across the layer is of little interest for this paper. See [1] for more details.

2.3 Boundary conditions at the circular boundary of the layer

In the previous section we found a second order partial differential equation for the pressure (5). This equation can be solved after formulating a boundary condition for the every point of the boundary. The boundary is either closed off by a barrier, or open. At the open boundary locations $\partial\Omega_D$ we demand the pressure to equal zero (Dirichlet boundary condition). At the closed boundary locations $\partial\Omega_N$ we demand the radial particle velocity to be zero. Equation (6a) shows that this corresponds to demanding the radial (normal) derivative of the pressure to be zero (Neumann boundary condition). The boundary conditions are therefore:

$$p = 0 \qquad \text{at } \partial\Omega_D, \tag{7a}$$

$$\frac{\partial p}{\partial r} = 0 \qquad \text{at } \partial\Omega_N. \tag{7b}$$

The boundary (outer circumference) is located at:

$$R = \frac{\omega}{c_0}\bar{R}. \tag{8}$$

The differential equation of the pressure (5) can be solved by separation of variables This results in the following general solution:

$$p(r,\theta) = \sum_{m=0}^{\infty}\Big(\big(C_m^s \sin(m\theta) + C_m^c \cos(m\theta)\big)\big(C_m^{\mathrm{I}}\mathrm{I}_m(\Gamma r) + C_m^{\mathrm{K}}\mathrm{K}_m(\Gamma r)\big)\Big) - hn. \tag{9}$$

The C_m's are constants of which C_0^s has no influence on the solution and can be set to zero. The functions I_m and K_m are the modified bessel functions of the first and second kind. The functions $\mathrm{K}_m(\Gamma r)$ go to infinity at $r = 0$ and will therefore be omitted from the solution. If the plate would be annular, and not contain $r = 0$, these functions do need to be taken into account. This paper will only evaluate systems with barriers that are symmetric around the line $\theta = 0$. This allows for the sine terms to be omitted. What remains of the solution is:

$$p(r,\theta) = \sum_{m=0}^{\infty} C_m \cos(m\theta)\mathrm{I}_m(\Gamma r) - hn. \tag{10}$$

The constants C_m in this solution can be determined by evaluating the boundary conditions (7). A numerical scheme will be used to do this.

2.4 Numerical calculation of the constants

The constants C_m from equation (10) can be calculated by creating a weak formulation with with cosines as weighing functions. Equation (10) was obtained by demanding barrier locations that are symmetric to the line $\theta = 0$. Wijnant [2] gives a more extensive (and slightly different) solution for the non-symmetric problem.

A linear system of equations is created with following the following recipe:

- The solution (10) is substituted into the boundary conditions (7)
- the r-derivative is written out
- The equations are multiplied with the weighing functions $\sum_{w=0}^{N}\cos(w\theta)$
- The equations are integrated along the boundaries on which these are valid
- The series is truncated up to $m = N$

- The order of the summations and the integration is reversed

The resulting matrix vector equation is:

$$\sum_{w=0}^{N}\sum_{m=0}^{N} C_m \mathrm{I}_m(\Gamma R) \int_{\partial\Omega_D} \cos(m\theta)\cos(w\theta)\,\mathrm{d}\theta = \sum_{w=0}^{N} \int_{\partial\Omega_D} hn\cos(w\theta)\,\mathrm{d}\theta, \tag{11a}$$

$$\sum_{w=0}^{N}\sum_{m=0}^{N} C_m \mathrm{I}_m(\Gamma R)\left(\frac{m}{R} + \Gamma\frac{\mathrm{I}_{m+1}(\Gamma R)}{\mathrm{I}_m(\Gamma R)}\right) \int_{\partial\Omega_N} \cos(m\theta)\cos(w\theta)\,\mathrm{d}\theta = \mathbf{0}. \tag{11b}$$

This equation gives a matrix of size $2N$ by N. A numerical computer program can easily find the least squares solution for this system. It is numerically much less problematic to take $C_m \mathrm{I}_m(\Gamma R)$ as the vector of unknowns instead of just C_m.

3 Results

The found solution can be verified by looking at the values of the pressure and the radial velocity at the boundary, see figure 2. The boundary conditions are satisfied judging from figure 2. The spikes in the velocity tend to get narrower, but higher when N increases. Note that the low reduced frequency model only accounts for viscothermal effects in the direction across the narrow layer, not in the propagation directions.

This semi-analytic solution has been verified by means of finite element calculations (also based on the low reduced frequency model) by Wijnant [2]. Van Blijderveen [3] has verified this model with experiments.

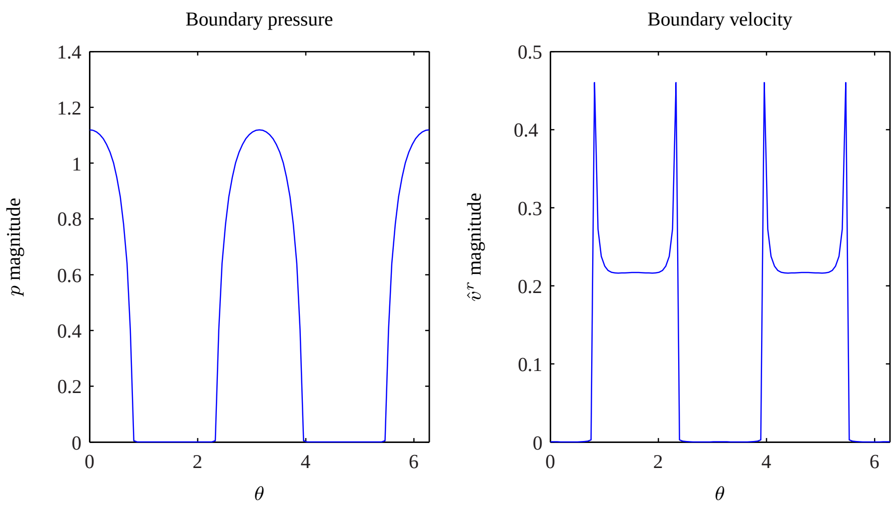

Figure 2: Boundary

The pressure profile (figure 3) can also be plotted. Wijnant shows pressure profiles for many different values of the shear wave number s and the dimensionless radius R in his paper [2]. Figures 2 and 3 were obtained with $R = 1$, $s = 1$, $h = 1$, $\sigma = 0.844$, $\gamma = 1.401$, $N = 200$ (which is high), and two barriers that close off 50% of the circumference. The constants resulting from solving equation (11) were corrected with Lanczos sigma factors. These factors are intended for use with Fourier transforms and are also useful in this (Fourier-like) application.

Once the constants C_m are known, other interesting properties can be calculated from the solution. The force

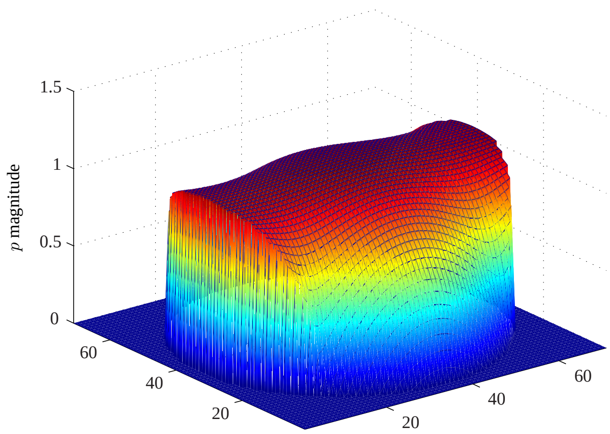

Figure 3: Pressure profile

of the air layer on the plate can be calculated by integrating the pressure over the area of the plate:

$$F = \int_0^R \int_0^{2\pi} pr\,\mathrm{d}\theta\,\mathrm{d}r = \frac{2\pi R}{\Gamma} C_0 \mathrm{I}_1(\Gamma R) - \pi R^2 hn. \tag{12}$$

The constant C_0 is the only constant that the force depends on. However, a sufficient number of variables must be calculated to get an accurate value for C_0.

The volume flow can be calculated by integrating the normal (radial) velocity along the circular boundary:

$$Q = \int_0^{2\pi} \hat{v}^r R\,\mathrm{d}\theta = -\frac{2iB\pi R\Gamma}{\gamma} C_0 \mathrm{I}_1(\Gamma R). \tag{13}$$

Thus de volume flow depends on no other constant than C_0 as well.

The realized volume flow per unit force could be an interesting value for some applications, like the hearing aid receiver:

$$\frac{Q}{F} = -\frac{iB\Gamma^2}{\gamma - \frac{\gamma\Gamma Rhn}{2C_0\mathrm{I}_1(\Gamma R)}}. \tag{14}$$

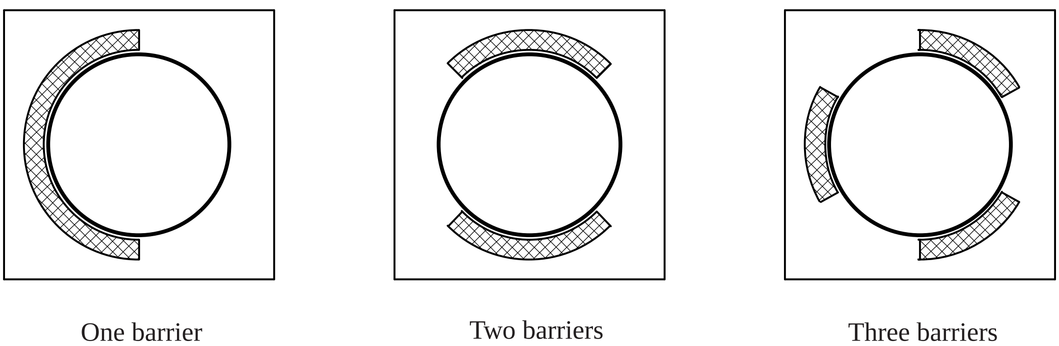

Figure 4: Translating plates with different boundary conditions

Different boundary configurations will be considered in this paper. Figure 4 shows symmetric barrier placements with one, two, and three barriers with an open/closed ratio of 50%. The pressure profiles under these

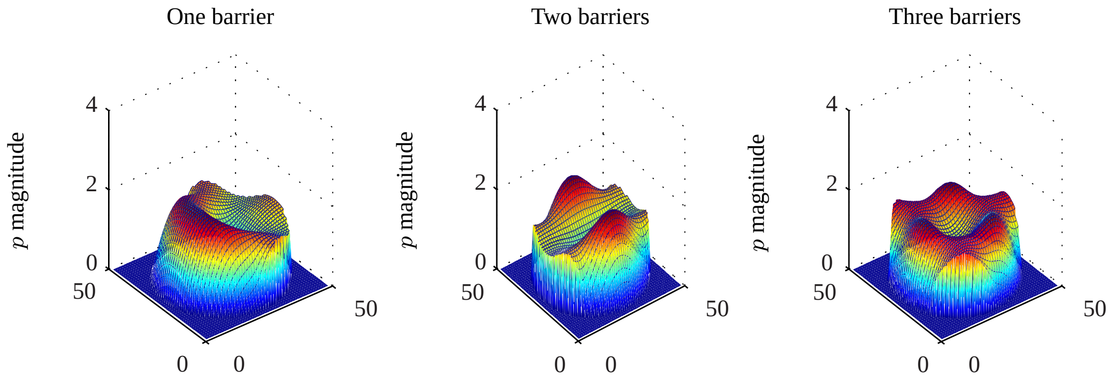

Figure 5: Pressure profiles

plates are plotted in figure 5 (for $s = 5$, $R = 5$, $\gamma = 1.40$, and $\sigma = 0.844$). As can be seen, the three profiles are completely different.

Equation (14) can be used to calculate the volume flow per unit force. Figure 6 shows $\frac{Q}{F}$ for these three configurations for different dimensionless radii and shear wave numbers. It shows that that the flow is lower for small shear wave numbers (highly viscous behavior). At $s = 0.1$, the volume flow per unit force is not dependent on the number of boundaries (for the given range of the dimensionless radius). For a shear wave number of $s = 1$, and a dimensionless radius less than 1, more flow is generated in the configuration with three boundaries. For $s = 10$ it can be seen how the dimensionless radii on which resonances occur change with barrier placement. Note that the dimensionless radius depends on the frequency.

Another possibility is to vary the open/closed ratio. Figure 7 shows $\frac{Q}{F}$ for different open/closed ratios in a configuration with three boundaries. As could be expected, the volume flow is higher in more open configurations for all shear wave numbers. The angle (phase) is hardly influenced by the open/closed ratio.

Figures 6 and 7 were obtained by using $\gamma = 1.40$, $\sigma = 0.844$, and $N = 100$. All figures show the dimensionless values. These can be made dimensionfull by:

$$\bar{F} = p_0 \left(\frac{c_0}{\omega}\right)^2 F, \tag{15}$$

$$\bar{Q} = \frac{h_0 c_0^2}{\omega} Q, \tag{16}$$

$$\frac{\bar{Q}}{\bar{F}} = \frac{h_0 \omega}{p_0} \frac{Q}{F}. \tag{17}$$

Thus the frequency ω is needed to calculate the shear wave number, the dimensionless radius, and to make the obtained results dimensionfull. To get a frequency response of a physical system, equation (11) must be solved for each frequency point.

4 Conclusions

The low reduced frequency model is used to model a circular layer between a rigidly translating plate and a fixed surface. The outer circumference of this layer is partially open and partially closed. The model can be solved analytically up to a series expansion with unknown constants. These constants can be calculated numerically. Only the first constant is needed to calculate the force of the layer on the plate, and the volume flow. However, the first constant can only be calculated accurately, if a sufficient number of constants is taken into account. The volume flow per unit force has been calculated for a few different boundary configurations.

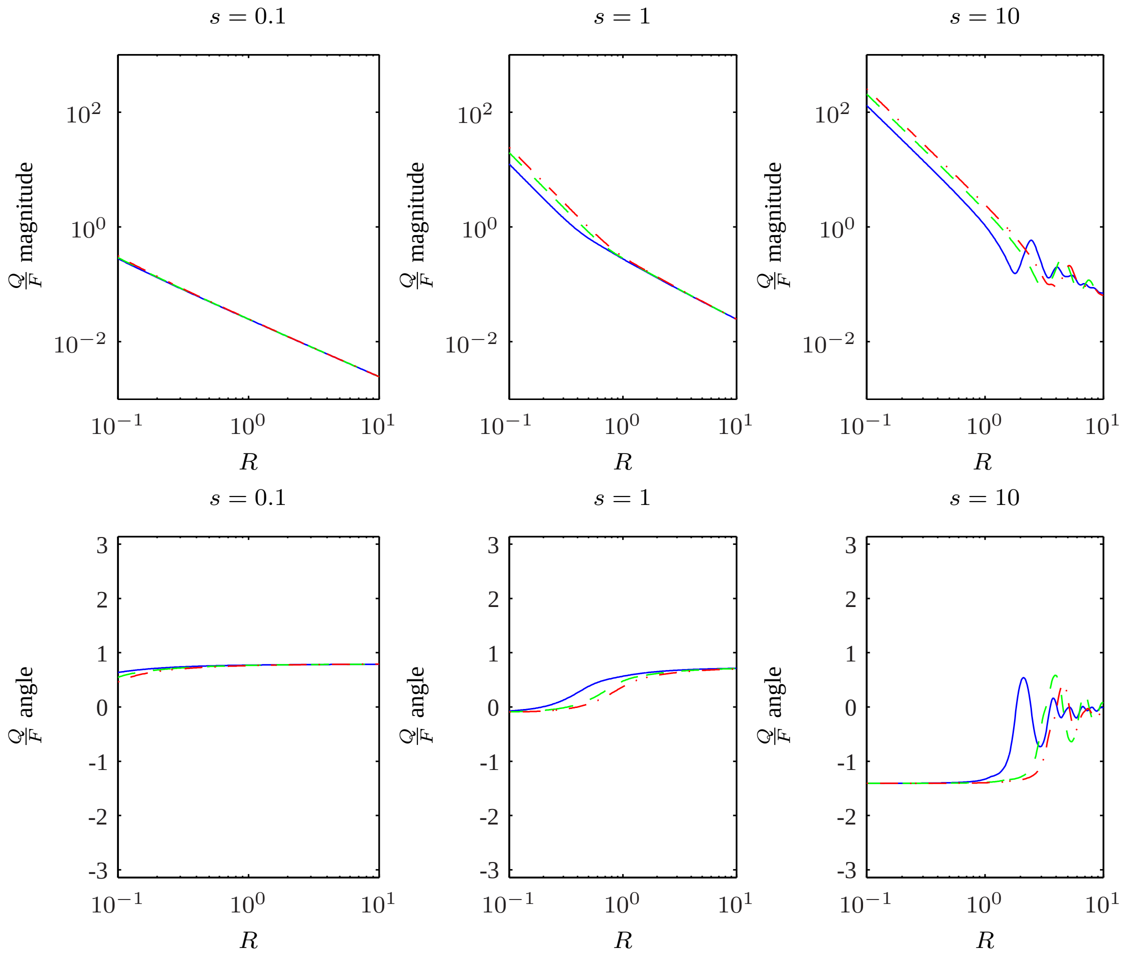

Figure 6: The effect of the number of barriers (50% open). Dimensionless volume flow per unit force versus dimensionless radius for different shear wave numbers: one barrier (blue, solid), two barriers (green, dash), and three barriers (red, dash-dot)

References

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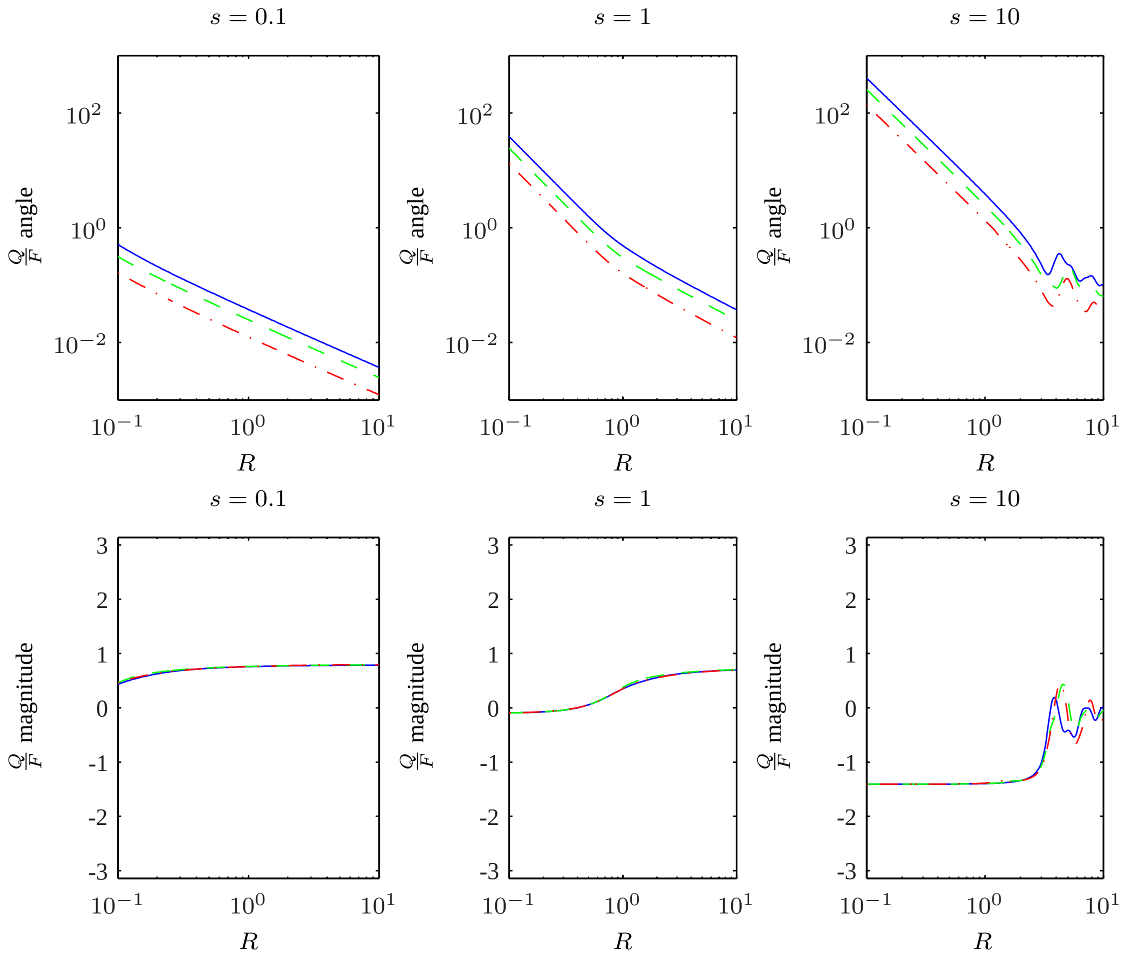

Figure 7: The effect of the open/closed ratio for a setup with three barriers. Dimensionless volume flow per unit force versus dimensionless radius for different shear wave numbers: 75% open (blue, solid), 50% open (green, dash), and 25% open (red, dash-dot)