Decentralized stabilization of linear time invariant systems subject to actuator saturation*

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Summary. We are concerned here with the stabilization of a linear time invariant system subject to actuator saturation via *decentralized control* while using linear time invariant dynamic controllers. When there exists no actuator saturation, i.e. when we consider just linear time invariant systems, it is known that *global stabilization* can be done via *decentralized control* while using linear time invariant dynamic controllers only if the so-called *decentralized fixed modes* of it are all in the open left half complex plane. On the other hand, it is known that for linear time invariant systems subject to actuator saturation, *semi-global stabilization* can be done via *centralized control* while using linear time invariant dynamic controllers if and only if the open-loop poles of the linearized model of the given system are in the closed left half complex plane. This chapter establishes that the necessary conditions for *semi-global stabilization* of linear time invariant systems subject to actuator saturation via *decentralized control* while using linear time invariant dynamic controllers, are indeed the above two conditions, namely (a) the *decentralized fixed modes* of the linearized model of the given system are in the open left half complex plane, and (b) the open-loop poles of the linearized model of the given system are in the closed left half complex plane. We conjecture that these two conditions are also sufficient in general. We prove the sufficiency for the case when the linearized model of the given system is open-loop conditionally stable with eigenvalues on the imaginary axis being distinct. Proving the sufficiency is still an open problem for the case when the linearized model of the given system has repeated eigenvalues on the imaginary axis.

1 Introduction

Non-classical information and control structure are two essential and distinguishing characteristics of large-scale systems. The research on decentralized control was formally initiated by Wang and Davison in their seminal paper [17] in 1973, and has been the subject of intense study during the 70's and 80's. Most recently there has been a renewed interest in decentralized control because of its fundamental role in the problem of coordinating the motion of multiple autonomous agents which by itself has attracted significant attention. Coordinating the motion of autonomous agents has many engineering applications besides having links to problems in biology, social behavior, statistical physics, and computer graphics. The engineering applications include unmanned aerial vehicles (UAVs), autonomous underwater vehicles (AUVs) and automated highway systems (AHS). A fundamental concept in the study of stabilization using decentralized feedback controllers is that of fixed modes. These are the poles of the system which cannot be shifted by just using any type of decentralized feedback controllers. The idea of fixed modes was introduced by Wang and Davison [17] who also show that decentralized stabilization is possible if and only if the fixed modes are stable. More definitive results are obtained by Corfmat and Morse [4] who present necessary and sufficient conditions under which spectrum assignment is possible in terms of the remnant polynomial of complementary subsystems. Since fixed modes constitute such an important concept in decentralized control, their characterization and determination has been the subject of many papers in the literature.

The majority of existing research in decentralized control makes a critical assumption that the interconnections between the subsystems of a given system are unknown but have known bounds. In this regard, tools borrowed from robust control theory and Lyapunov theory are used for the purpose of either synthesis or analysis of decentralized controllers [10, 14, 13]. For the case when the interconnections between the subsystems are known, the existing research is very sparse. In fact, in any case, beyond the decentralized stabilization, no results are yet available dealing with the fundamental control issues such as exact or almost disturbance decoupling, control for various performance objectives, etc.

From a different perspective, input saturation in any control scheme is a common phenomenon. Every physically conceivable actuator has bounds on its output. Valves can only be operated between fully open and fully closed states, pumps and compressors have a finite throughput capacity and tanks can only hold a certain volume. Ignoring such saturation effects in any control system design can be detrimental to the stability and performance of controlled systems. A classical example for the detrimental effect of neglecting *actuator constraints* is the Chernobyl unit 4 nuclear power plant disaster

* Support for this work was provided by the Office of Naval Research under grant number N000140310848.

in 1986 [16]. During the last decade and the present one, there has been an intense research activity in the area of control of linear plants with saturating actuators. Such intense research activity has been chronicled in special issues of journals and edited books (e.g. for recent literature see [8, 15, 2]). Fundamental fuel behind such a research activity has been to accentuate the industrial and thus the practical engineering relevance of modern control theory. In this regard, the primary focus of the research activity has been to take into account *a priori* the presence of saturation nonlinearities in any control system analysis and design. A number of control issues have been considered so far including internal, external, or internal plus external stabilization and output regulation among others. Although not all aspects of these issues have been completely resolved, it is fair to say that a good understanding of these issues exists at present. However, issues related to performance, robustness etc., are very poorly understood and still remain as challenging and complex problems for future research.

Having been involved deeply in the past with research on linear systems subject to constraints on its input and state variables, we are now ready to open up a new front line of research in decentralized control by bringing into picture the constraints of actuators. The focus of this chapter is to determine the necessary and sufficient conditions for decentralized stabilization of linear systems subject to constraints on actuators. Obviously, this is related to the seminal work of Wang and Davison [17] but goes beyond it by bringing into picture the input constraints on the top of decentralized constraint.

2 Problem formulation and preliminaries

Consider the linear time invariant systems subject to actuator saturation,

$$\Sigma : \begin{cases} \dot{x} = Ax + \sum_{i=1}^{\nu} B_i \operatorname{sat} u_i \\ y_i = C_i x, \qquad i = 1, \dots, \nu, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is a state, $u_i \in \mathbb{R}^{m_i}$, $i = 1, \dots, \nu$ are control inputs, $y_i \in \mathbb{R}^{p_i}$, $i = 1, \dots, \nu$ are measured outputs, and 'sat' denotes the standard saturation element with the property that for any vector u of arbitrary dimension, $\operatorname{sat}(u)$ is a vector of the same dimension as u, and moreover for any positive integer j less than or equal to the dimension of u, the j'th component of $\operatorname{sat} u$, denoted by $(\operatorname{sat} u)_j$, compared to the j'th component of u, denoted by $(u)_j$, has the property,

$$(\operatorname{sat} u)_j = \begin{cases} 1 & \text{if } 1 < (u)_j, \\ (u)_j & \text{if } -1 \le (u)_j \le 1, \\ -1 & \text{if } (u)_j < -1. \end{cases}$$

Here we are looking for ν controllers of the form,

$$\Sigma_i : \begin{cases} \dot{z}_i = K_i z_i + L_i y_i, \quad z_i \in \mathbb{R}^{s_i} \\ u_i = M_i z_i + N_i y_i. \end{cases} \tag{2}$$

The controller Σ_i is said to be i-th channel controller.

Before we state the problem we study in this chapter, we would like to recall the concept of semi-global stabilization via decentralized control.

Definition 1. *Consider a system Σ of the form (1). Then, we say that Σ is semi-globally stabilizable via decentralized control if there exists nonnegative integers $s_1, \ldots, s_\nu$ such that for any given collection of compact sets $\mathcal{W} \subset \mathbb{R}^n$ and $\mathcal{S}_i \subset \mathbb{R}^{s_i}$, $i = 1, \ldots \nu$, there exist a decentralized set of controllers ν controllers Σ_i, $i = 1, \ldots \nu$, of the form (2) such that the origin of the resulting closed-loop system is asymptotically stable and the domain of attraction includes $\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$.*

The problem we would like to study in this chapter can be stated as follows:

Problem 1. Consider a system Σ of the form (1). Develop the necessary and sufficient conditions such that Σ is semi-globally stabilizable via decentralized control.

Remark 1. For the case when $\nu = 1$, the above decentralized control problem retorts to centralized semi-global stabilization of linear time invariant systems subject to actuator saturation. Such a problem has been studied in depth by Saberi and his coworkers. By now it is well known that such a centralized semi-global stabilization problem is solvable by a linear time invariant dynamic controller if and only if the linearized model of the given system is stabilizable and detectable and all the open-loop poles of linearized model are in the closed left half complex plane.

3 Review of decentralized stabilization of linear time invariant systems

Before we proceed to consider the conditions for the solvability of Problem 1, it is prudent to review the necessary and sufficient conditions for the global decentralized stabilization of linearized model of the given system Σ. To do so, we first write the linearized model of the given system Σ of (1) as,

$$\bar{\Sigma} : \begin{cases} \dot{x} = Ax + \sum_{i=1}^{\nu} B_i u_i \\ y_i = C_i x, \qquad i = 1, \ldots, \nu. \end{cases} \tag{3}$$

The classical decentralized global stabilization problem or more general decentralized pole placement problem for the linearized model $\bar{\Sigma}$ can be stated as follows: Find linear time invariant dynamic controllers Σ_i, $i = 1, \ldots \nu$, of the form (2) such that the poles of the closed-loop system comprising $\bar{\Sigma}$ and the controllers Σ_i, $i = 1, \ldots \nu$, has pre-specified poles in the open left half complex plane.

It is easy to observe that, if (A, B_i) and (A, C_i) are respectively controllable and observable pairs for some i, the above decentralized pole placement problem can be solved trivially.

Wang and Davison in [17] considered the general decentralized pole placement problem for the linearized model $\bar{\Sigma}$. Before we state their result, we need to recall the important concept of decentralized fixed modes as was introduced by Wang and Davison:

Definition 2. *Consider a system $\bar{\Sigma}$ of the form* (3)*. Then, λ is called a decentralized fixed mode of the system $\bar{\Sigma}$ if for all matrices $K_1, \ldots, K_\nu$ we have that λ is an eigenvalue of*

$$A_K := A + \sum_{i=1}^{\nu} B_i K_i C_i.$$

Wang and Davison proved in [17] that there exist dynamic controllers Σ_i, $i = 1, \ldots \nu$, of the form (2) such that the poles of the closed-loop system comprising $\bar{\Sigma}$ and the controllers Σ_i, $i = 1, \ldots \nu$ are at pre-specified locations in the open left half complex plane provided that the decentralized fixed modes of $\bar{\Sigma}$ are themselves in the open left half complex plane and the set of pre-specified locations in the open left half complex plane includes the set of decentralized fixed modes of $\bar{\Sigma}$. This obviously implies that *the decentralized stabilization of the linear time invariant system $\bar{\Sigma}$ is possible if and only if the decentralized fixed modes of it are all in open left half complex plane.*

The above result implies that the decentralized fixed modes of $\bar{\Sigma}$ play a crucial role in decentralized stabilization of linear time invariant systems. As such it is important to know how to compute such fixed modes. One of the easiest procedure to do so is as follows: Since $K_i = 0$, $i = 1, \ldots \nu$, are admissible, in this case A_K retorts to A, and hence in view of Definition 2, the decentralized fixed modes are naturally a subset of the eigenvalues of A. Thus the first step is to compute the eigenvalues of A. Second, it can be shown that if K_i, $i = 1, \ldots \nu$, are randomly chosen, then with probability one the decentralized fixed modes are common eigenvalues of A and A_K. Since algorithms are well developed to determine the eigenvalues of a matrix, the computation of decentralized fixed modes is quite straightforward.

After the introduction of the concept of decentralized fixed modes, there has been quite some research on interpretations of this concept. The crucial step in understanding the decentralized fixed modes was its connection to complementary systems as introduced by Corfmat and Morse in the paper [4]. The paper [1] by Anderson and Clements used the ideas of Corfmat and Morse to yield the following characterization of decentralized fixed modes:

Lemma 1. *Consider the system $\bar{\Sigma}$ of* (3)*. We define,*

$$B = \begin{pmatrix} B_1 & \cdots & B_\nu \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ \vdots \\ C_\nu \end{pmatrix}.$$

Then λ is a decentralized fixed mode if and only if at least one of the following three conditions is satisfied:

- *λ is an uncontrollable eigenvalue of (A, B).*
- *λ is an unobservable eigenvalue of (C, A).*
- *There exists a partition of the integers $\{1, 2, \dots, \nu\}$ into two disjoint sets $\{i_1, \dots, i_\alpha\}$ and $\{j_1, \dots, j_{\nu-\alpha}\}$ where $0 < \alpha < \nu$ for which we have*

$$\operatorname{rank} \begin{pmatrix} \lambda I - A & B_{i_1} & \cdots & B_{i_\alpha} \\ C_{j_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{j_{\nu-\alpha}} & 0 & \cdots & 0 \end{pmatrix} < n.$$

Basically the decentralized fixed modes are therefore common blocking zeros of all complementary systems which are, moreover, either unobservable or uncontrollable for each complementary system. For a detailed investigation of blocking zeros we refer to the paper [3]. Other attempts to characterize the decentralized fixed modes can be found in for instance [12, 6, 7].

The above discussion focuses on developing the necessary and sufficient condition under which stabilization of a linear time invariant system by a set of decentralized controllers is possible. The next issue that needs to be discussed pertains to how does one construct systematically the set of decentralized controllers that stabilize a given system assuming that it is possible to do so. In this regard, it is important to recognize that implicit in the proof of pole placement result of Wang and Davison [17] is a constructive algorithm. This algorithm requires as a first step the (possibly random) selection of K_i, $i = 1, \dots \nu$, such that all the eigenvalues of

$$A_K = A + \sum_{i=1}^{\nu} B_i K_i C_i$$

are distinct from those of A except for the decentralized fixed modes. Then, dynamic feedback is successively employed to arrive at a dynamic controller Σ_i, $i = 1, \dots \nu$, placing the poles of resulting closed-loop system that are both controllable and observable eigenvalues of the pairs (A, B_i) and (A, C_i) respectively. Also, Corfmat and Morse [4] have studied the decentralized feedback control problem from the point of view of determining a more complete characterization of conditions for stabilizability and pole placement as well as constructing a set of stabilizing decentralized controllers. Their basic approach is to determine conditions under which a system $\bar{\Sigma}$ of the form (3) can be made controllable and observable from the input and output variables of a given controller by static feedback applied by the other controllers. Then dynamic compensation can be employed at this controller in a standard way to place the poles of the closed-loop system.

It is not hard to see that a necessary condition to make $\bar{\Sigma}$ controllable and observable from a single controller is that none of the transfer functions

$$C_i(sI - A)^{-1}B_j$$

vanish identically for all $i = 1, \ldots \nu$, and $j = 1, \ldots \nu$. A system satisfying this condition is termed strongly connected. If a system is not strongly connected, the given system can be decomposed into strongly connected subsystems and each subsystem can be made then controllable and observable from one of its controllers.

As outlined in an early survey paper by Sandell et al [11], as a practical design method, the Corfmat and Morse method suffers some defects. At first, it can be noted that even if all the modes of a large scale system can be made controllable and observable from a single controller (or a few controllers if the given system is not strongly connected), some of the modes may be very weakly controllable and observable. Thus, impractically large gains may be required to place all the poles from a single controller. Second, it is unclear that the approach uses the designer's available degrees of freedom in the best way. Essentially, the approach seems to require that all the disturbances in the system propagate to a single output, where they can be observed and compensated for by the control signals at an adjacent input. Finally, concentration of all the complexity of the control structure at a single (or few) controllers may be undesirable.

As pointed out once again in [11], the constructive approach of Wang and Davison also suffers similar drawbacks as mentioned above. Although there is no explicit attempt in their approach to make all of the strongly connected subsystems controllable and observable from a single controller, the generic outcome of the first step of their approach will be precisely this situation.

After the early first phase of work of Wang and Davison [17] as well as Corfmat and Morse [4], there has been a lot of second phase of work (see [14, 13] and references there in) on how to construct the set of decentralized controllers for a large scale system. These researchers view the given large scale system such as $\bar{\Sigma}$ of (3) as consisting of ν interconnected subsystems, the i-th subsystem being controlled by the i-th controller Σ_i. Then, the research in decentralized control is dominated by the point of view of considering the interconnections between the subsystems essentially as disturbances, and then using robust control theory to design strongly robust subsystems in such a way that the effect of such disturbances is minimal. Essentially, the framework of viewing the interconnections as disturbances is fundamentally flawed. Such work belongs to the field of centralized robust control theory. In our opinion, the decentralized control is still in its infancy, and is a very complex and open field.

4 Main results

In this section, we will present the necessary and sufficient conditions for semi-global stabilizability of linear time invariant systems with actuator saturation by utilizing a set of decentralized linear time invariant dynamic controllers.

We have the following theorem that pertains to necessary conditions, the proof of which is given in Section 5.

Theorem 1. *Consider the system* Σ *given by* (1). *There exists nonnegative integers* $s_1, \ldots, s_\nu$ *such that for any given collection of compact sets* $\mathcal{W} \subset \mathbb{R}^n$ *and* $\mathcal{S}_i \subset \mathbb{R}^{s_i}$, $i = 1, \ldots \nu$, *there exist* ν *controllers of the form* (2) *such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes* $\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$ *only if*

- *All decentralized fixed modes of* $\bar{\Sigma}$ *given by* (3) *are in the open left half complex plane, and*
- *All eigenvalues of* A *are in the closed left half plane.*

The following theorem says that besides decentralized fixed modes being in the open left half complex plane, a sufficient condition for semi-global stabilizability of (1) when the set of controllers given by (2) are utilized is that all the eigenvalues of A be in the closed left half plane with those eigenvalues on the imaginary axis having algebraic multiplicity equal to one.

Theorem 2. *Consider the system* Σ *given by* (1). *There exists nonnegative integers* $s_1, \ldots, s_\nu$ *such that for any given collection of compact sets* $\mathcal{W} \subset \mathbb{R}^n$ *and* $\mathcal{S}_i \subset \mathbb{R}^{s_i}$, $i = 1, \ldots \nu$, *there exist* ν *controllers of the form* (2) *such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes* $\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$ *if*

- *All decentralized fixed modes of* $\bar{\Sigma}$ *given by* (3) *are in the open left half complex plane, and*
- *All eigenvalues of* A *are in the closed left half plane with those eigenvalues on the imaginary axis having algebraic multiplicity equal to one.*

The above theorem is proved in Section 6. Our work done up to now convinces us to state the following conjecture that the necessary conditions given in Theorem 1 are also sufficient for semi-global stabilizability of decentralized linear systems with actuator saturation.

Conjecture 1. Consider the system Σ given by (1). There exists nonnegative integers $s_1, \ldots, s_\nu$ such that for any given collection of compact sets $\mathcal{W} \subset \mathbb{R}^n$ and $\mathcal{S}_i \subset \mathbb{R}^{s_i}$, $i = 1, \ldots \nu$, there exist ν controllers of the form (2) such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes $\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$ if and only if

- All decentralized fixed modes of $\bar{\Sigma}$ given by (3) are in the open left half complex plane, and
- All eigenvalues of A are in the closed left half plane.

5 Proof of Theorem 1

We prove Theorem 1 in this section. Assume that decentralized semi-global stabilization of the given system Σ of (1) is possible. Then, the decentralized stabilization of the linearized model $\bar{\Sigma}$ of Σ as given in (3) is possible. By the result of Wang and Davison [17], this implies that it is necessary to have all the decentralized fixed modes of $\bar{\Sigma}$ in the open left half complex plane. However, we have a simple alternate proof of this fact as follows: Since the linearized model needs to be asymptotically stable, there exists ν linear controllers achieving locally an asymptotically stable system. We define the following matrices for these ν controllers of the form (2):

$$K = \begin{pmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & K_\nu \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_\nu \end{pmatrix},$$
$$M = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_\nu \end{pmatrix}, \quad N = \begin{pmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & N_\nu \end{pmatrix}.$$

For any λ with $\operatorname{Re}\lambda \geq 0$ there exists a δ such that $(\lambda+\delta)I - K$ is invertible and the closed loop system when replacing K by $K-\delta I$ is still asymptotically stable. But then the linearized model of the closed loop system cannot have a pole at λ which implies that we must have that

$$\det\left(\lambda I - A - B\left[M(\lambda I - (K-\delta I))^{-1}L + N\right]C\right) \neq 0.$$

Hence the block diagonal matrix

$$S = M(\lambda I - (K-\delta I))^{-1}L + N$$

has the property that

$$\det(\lambda I - A - BSC) \neq 0,$$

and thus λ is not a fixed mode of the system. Since this argument is valid for any λ in the closed right half plane this implies that all the fixed modes must be in the open left half plane. This proves the necessity of the first item of Theorem 1.

To prove the necessity of the second item of Theorem 1, assume that λ is an eigenvalue of A in the open right half plane with corresponding left eigenvector p, i.e. $pA = \lambda p$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} px(t) = \lambda px(t) + v(t)$$

where

$$v(t) := \sum_{i=1}^{\nu} pB_i \operatorname{sat} u_i(t).$$

There clearly exists an $\tilde{M} > 0$ such that $\|v(t)\| \leq \tilde{M}$ for all $t > 0$. But then

$$|px(t)| > \left|e^{\lambda t}\right| \left(|px(0)| - \frac{\tilde{M}}{\operatorname{Re}\lambda} \right) + \frac{\tilde{M}}{\operatorname{Re}\lambda}$$

which does not converge to zero since $\operatorname{Re}\lambda \geq 0$, provided the initial condition is such that

$$|px(0)| > \frac{\tilde{M}}{\operatorname{Re}\lambda}.$$

Note that this is valid for all controllers and therefore we can clearly not achieve semi-global stability.

6 Preliminary lemmas and proof of Theorem 2

We will use two lemmas. The first lemma given below is a well-known classical result from Lyapunov theory.

Lemma 2. *Consider a matrix $A \in \mathbb{R}^{n\times n}$, and assume that it has all its eigenvalues in the closed left half plane with those eigenvalues on the imaginary axis having a geometric multiplicity equal to the algebraic multiplicity. Then, there exists a matrix $P > 0$ such that*

$$A'P + PA \leq 0. \tag{4}$$

Another useful tool is the following continuity result related to (4).

Lemma 3. *Assume that we have a sequence of matrices $A_\delta \in \mathbb{R}^{n\times n}$ parameterized by δ and a matrix $A \in \mathbb{R}^{n\times n}$ such that $A_\delta \to A$ as $\delta \to 0$. Assume that A has all its eigenvalues in the closed left half plane, and that there are p distinct eigenvalues of A on the imaginary axis (i.e. there are p eigenvalues of A on the imaginary axis each with algebraic multiplicity equal to 1). Moreover, assume that A_δ also has all its eigenvalues in the closed left half plane. Let $P > 0$ be such that (4) is satisfied. Then there exists for small $\delta > 0$ a family of matrices $P_\delta > 0$ such that*

$$A_\delta' P_\delta + P_\delta A_\delta \leq 0$$

and $P_\delta \to P$ as $\delta \to 0$.

Proof. We first observe that there exists a matrix S such that

$$S^{-1}AS = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix},$$

and such that all the eigenvalues of A_{11} are on the imaginary axis while A_{22} has all its eigenvalues in the open left half plane. Since A_{11} and A_{22} have no common eigenvalues and $A_\delta \to A$, there exists a parameterized matrix S_δ such that for δ sufficiently small

$$S_\delta^{-1}A_\delta S_\delta = \begin{pmatrix} A_{11,\delta} & 0 \\ 0 & A_{22,\delta} \end{pmatrix}$$

where $S_\delta \to S$, $A_{11,\delta} \to A_{11}$ and $A_{22,\delta} \to A_{22}$ as $\delta \to 0$. This follows from classical results on the sensitivity of invariant subspaces (see for instance [9, 5]).

Given is a matrix $P > 0$ such that $A'P + PA \leq 0$. Let us define

$$\bar{P} = S'PS = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}'_{12} & \bar{P}_{22} \end{pmatrix}.$$

Obviously, we note that

$$\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} + \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \leq 0. \tag{5}$$

Next, given an eigenvector x_1 such that $A_{11}x_1 = \lambda x_1$ with $\operatorname{Re}\lambda = 0$, we have

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}^* \left[\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} + \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Using (5), the above implies that

$$\left[\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} + \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Since all the eigenvalues on the imaginary axis of $A_{11} \in \mathbb{R}^{v\times v}$ are distinct we find that the eigenvectors of A_{11} span $\mathbb{R}^v$ and hence

$$\left[\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} + \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \right] \begin{pmatrix} I \\ 0 \end{pmatrix} = 0.$$

This leads to

$$\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} + \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \leq 0.$$

This immediately implies that $A'_{11}\bar{P}_{12} + \bar{P}_{12}A_{22} = 0$ and since A_{11} and A_{22} have no eigenvalues in common we find that $\bar{P}_{12} = 0$. Thus, we have

$$A'_{11}\bar{P}_{11} + \bar{P}_{11}A_{11} = 0 \text{ and } A'_{22}\bar{P}_{22} + \bar{P}_{22}A_{22} = V \leq 0.$$

Next, since A_{22} has all its eigenvalues in the open left half plane, there exists a parameterized matrix $P_{22,\delta}$ for δ small enough such that

$$A'_{22,\delta}\bar{P}_{22,\delta} + \bar{P}_{22,\delta}A_{22,\delta} = V \leq 0$$

while $\bar{P}_{22,\delta} \to P_{22}$ as $\delta \to 0$.

Let $A_{11} = W\Lambda_A W^{-1}$ with Λ_A a diagonal matrix. Because the eigenvectors of A_{11} are distinct and $A_{11,\delta} \to A_{11}$, for δ small enough the eigenvectors of $A_{11,\delta}$ depend continuously on δ and hence there exists a parameterized matrix W_δ such that $W_\delta \to W$ while $A_{11,\delta} = W_\delta \Lambda_{A_\delta} W_\delta^{-1}$ with Λ_{A_δ} diagonal. The matrix $\bar{P}_{11}$ satisfies

$$A^*_{11}\bar{P}_{11} + \bar{P}_{11}A_{11} = 0$$

This implies that $\Lambda_P = W^*\bar{P}_{11}W$ satisfies

$$\Lambda^*_A\Lambda_P + \Lambda_P\Lambda_A = 0.$$

The above equation then shows that Λ_P is a diagonal matrix. We know that

$$\Lambda_{A_\delta} \to \Lambda_A.$$

We know that Λ_{A_δ} is a diagonal matrix whose diagonal elements have real part less than or equal to zero while Λ_P is a positive-definite diagonal matrix. Using this, it can be verified that we have

$$\Lambda^*_{A_\delta}\Lambda_P + \Lambda_P\Lambda_{A_\delta} \leq 0.$$

We choose $\bar{P}_{11,\delta}$ as

$$\bar{P}_{11,\delta} = (W^*_\delta)^{-1}\Lambda_P W_\delta^{-1}.$$

Obviously, our choice of $\bar{P}_{11,\delta}$ satisfies

$$A^*_{11,\delta}\bar{P}_{11,\delta} + \bar{P}_{11,\delta}A_{11,\delta} \leq 0.$$

We observe that $\bar{P}_{11,\delta} \to \bar{P}_{11}$ as $\delta \to 0$. But then

$$P_\delta = (S_\delta^{-1})' \begin{pmatrix} \bar{P}_{11,\delta} & 0 \\ 0 & \bar{P}_{22,\delta} \end{pmatrix} S_\delta^{-1}$$

satisfies the conditions of the lemma. This completes the proof of Lemma 3.

We proceed now with the proof of Theorem 2. Our proof is constructive and involves a sequential design. We present a recursive algorithm which at each step applies a decentralized feedback law which stabilizes at least one eigenvalue on the imaginary axis while preserving the stability of the stable modes of the system in such a way that the magnitude of each decentralized feedback control is guaranteed never to exceed $1/n$. Therefore, after at most n steps the combination of these decentralized feedback laws will asymptotically stabilize the system without ever violating the magnitude constraints of each of the inputs. The basic steps of the algorithm are as formalized below:

Algorithm:

- **Step 0 (Initialization):** We first initialize our algorithm at step 0. To do so, let $A_0 := A$, $B_{0,i} := B_i$, $C_{0,i} := C_i$, $n_{i,0} := 0$, $N^0_{i,\varepsilon} := 0$, $i = 1, \ldots, \nu$ and $x_0 := x$. Moreover, define $P^\varepsilon_0 := \varepsilon P$, where $P > 0$ is a matrix such that

$$A'P + PA \leq 0.$$

Since all the eigenvalues of A on the imaginary axis have multiplicity 1, we know from Lemma 2 that such a matrix P exists.

- **Step k:**
For the system Σ given by (3), we have to design ν parameterized decentralized feedback control laws,

$$\Sigma^{k,\varepsilon}_i : \begin{cases} \dot{p}^k_i = K^k_{i,\varepsilon} p^k_i + L^k_{i,\varepsilon} y_i, & p^k_i \in \mathbb{R}^{n_{i,k}} \\ u_i = M_{i,\varepsilon} p^k_i + N^k_{i,\varepsilon} y_i + v^k_i \end{cases} \tag{6}$$

in case $n_{i,k} > 0$, and otherwise

$$\Sigma^{k,\varepsilon}_i : \{ \; u_i = N^k_{i,\varepsilon} y_i + v^k_i, \tag{7}$$

for $i = 1, \ldots, \nu$. The closed-loop system comprising the above decentralized feedback control laws and the system Σ of (1) can be written as

$$\Sigma^{k,\varepsilon}_{cl} : \begin{cases} \dot{x}_k = A^\varepsilon_k x_k + \sum_{i=1}^{\nu} B_{k,i} v^k_i \\ y_i = C_{k,i} x_k, \qquad i = 1, \ldots, \nu, \end{cases} \tag{8}$$

where $x_k \in \mathbb{R}^{n_k}$ with $n_k = n + \sum_{i=1}^{\nu} n_{i,k}$ is given by

$$x_k = \begin{pmatrix} x \\ p^k_1 \\ \vdots \\ p^k_\nu \end{pmatrix}. \tag{9}$$

In view of (9), we can rewrite u_i as

$$u_i = F^k_{i,\varepsilon} x_k + v^k_i$$

for some appropriate matrix $F^k_{i,\varepsilon}$.
The above decentralized feedback control laws given by either (6) or (7) are to be designed in such a way that they satisfy the following properties:

1) The matrix A_k^ε has all its eigenvalues in the closed-left half plane, and those eigenvalues of A_k^ε which are on the imaginary axis are distinct.
2) The number of eigenvalues of A_k^ε on the imaginary axis must at least be one less than the number of eigenvalues of A_{k-1}^ε on the imaginary axis (i.e. at each step of our recursive algorithm we design a decentralized feedback law which stabilizes at least one more eigenvalue on the imaginary axis while preserving the stability of the stable modes of the system designed until then).
3) There exists a family of matrices P_k^ε such that $P_k^\varepsilon \to 0$ as $\varepsilon \to 0$ while for $v_i^k = 0$, $i = 1, \dots, \nu$, the closed-loop system $\Sigma_{cl}^{k,\varepsilon}$ of (8) is such that

$$x_k(t)' P_k^\varepsilon x_k(t)$$

is non-increasing in t for all initial conditions, i.e.

$$(A_k^\varepsilon)' P_k^\varepsilon + P_k^\varepsilon A_k^\varepsilon \leq 0. \tag{10}$$

Moreover, there exists an ε^* such that for all $\varepsilon \in (0, \varepsilon^*]$ we have

$$\|u_i(t)\| \leq \frac{k}{n} \tag{11}$$

for all states with $x_k(t)' P_k^\varepsilon x_k(t) \leq n - k + 1$.

It is easy to verify that all of the above conditions are true for $k = 0$.

- **Terminal step:**
There exists a value for k, say $\ell \leq n$, such that the matrix A_ℓ^ε has all its eigenvalues in the open-left half plane. We set $v_i^\ell = 0$ for $i = 1, \dots, \ell$. The decentralized control laws $\Sigma_i^{\ell,\varepsilon}$, $i = 1, \dots, \ell$ as given by (6) or (7), all together, represent a decentralized semi-global state feedback law for the given system Σ of (1). More precisely, for any given compact sets $\mathcal{W} \subset \mathbb{R}^n$, and $\mathcal{S}_i \subset \mathbb{R}^{n_{i,\ell}}$ for $i = 1, \dots \nu$, there exists an ε^* such that the origin of the closed-loop system comprising the given system Σ of (1) and the decentralized control laws $\Sigma_i^{\ell,\varepsilon}$, $i = 1, \dots, \ell$ as given by (6) or (7) is exponentially stable for any $0 < \varepsilon < \varepsilon^*$, and the compact set $\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$ is within the domain of attraction. Moreover, for all the initial conditions within $\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$, the said closed-loop system behaves like a linear dynamic system, that is the saturation is not activated implying that $\|u_i\| < 1$ for all $i = 1, \dots, \nu$.
The fact that the decentralized control laws $\Sigma_i^{\ell,\varepsilon}$, $i = 1, \dots, \ell$ as given by (6) or (7) are semi-globally stabilizing follows from the property 3) as given in step k of the above algorithm. To be explicit, we observe that, for an ε sufficiently small, the set

$$\Omega_1^\varepsilon := \{x_\ell \in \mathbb{R}^{n_\ell} | x_\ell' P_\ell^\varepsilon x_\ell \leq 1\}$$

is inside the domain of attraction of the equilibrium point of the closed-loop system comprising the given system Σ of (1) and the decentralized

control laws $\Sigma_i^{\ell,\varepsilon}$, $i = 1, \ldots, \ell$ as given by (6) or (7). This follows from the fact that for all the initial conditions within Ω_1^ε, it is obvious from (11) that $\|u_i\| \leq 1$ for all $i = 1, \ldots, \nu$. This implies that the said closed-loop system behaves like a linear dynamic system, that is the saturation is not activated. Moreover, this linear dynamic system is asymptotically stable since A_ℓ^ε has all its eigenvalues in the open left half plane, and hence the state converges to zero asymptotically.
Next, since $P_\ell^\varepsilon \to 0$ as $\varepsilon \to 0$, for an ε sufficiently small, we have that the compact set $\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$ is inside Ω_1^ε. This concludes that the decentralized control laws $\Sigma_i^{\ell,\varepsilon}$, $i = 1, \ldots, \ell$ as given by (6) or (7) are semi-globally stabilizing.

This completes the description of our recursive algorithm to design the decentralized feedback control laws having the properties as given in Theorem 2.

It remains to prove that the above recursive algorithm succeeds in designing the decentralized feedback control laws having the properties as given in Theorem 2. In order to do so, we assume that the design of decentralized feedback control laws as described in step k can be done, and then prove that the corresponding design in step $k+1$ can be done. We proceed now to prove this.

After step k we have for the system Σ of (1), ν feedback control laws of the form (6) or (7) such that the system (8) obtained after applying these feedbacks has the properties 1), 2) and 3). We consider the closed-loop system $\Sigma_{cl}^{k,\varepsilon}$ of (8). Let λ be an eigenvalue on the imaginary axis of A_k^ε. We know that decentralized feedback laws do not change the fixed modes and therefore, since λ was not a fixed mode of the original system (1), it is not a fixed mode of the system (8) obtained after applying ν feedback laws either. Hence there exists a $\bar{K}_i$ such that

$$A_k^\varepsilon + \sum_{i=1}^{\nu} B_{k,i} \bar{K}_i C_{k,i}$$

has no eigenvalue at λ. Therefore,

$$A_k^\varepsilon + \delta \sum_{i=1}^{\nu} B_{k,i} \bar{K}_i C_{k,i}$$

has no eigenvalue at λ for almost all $\delta > 0$ (the determinant of $\lambda I - A_k^\varepsilon - \delta \sum_{i=1}^{\nu} B_{k,i} \bar{K}_i C_{k,i}$ is a polynomial in δ and is nonzero for $\delta = 1$ and therefore the determinant has a finite number of zeros).

Let j be the largest integer such that

$$A_k^{\varepsilon,\delta} = A_k^\varepsilon + \delta \sum_{i=1}^{j} B_{k,i} \bar{K}_i C_{k,i}$$

has λ as an eigenvalue and the same number of eigenvalues on the imaginary axis as A_k^ε for $\delta > 0$ small enough. This implies that $A_k^{\varepsilon,\delta}$ still has all its eigenvalues in the closed left half plane for δ small enough.

We know that (10) is satisfied and hence using Lemma 3 we find that there exists a $\bar{P}_k^{\varepsilon,\delta}$ such that

$$(A_k^{\varepsilon,\delta})'\bar{P}_k^{\varepsilon,\delta} + \bar{P}_k^{\varepsilon,\delta}A_k^{\varepsilon,\delta} \leq 0$$

while $\bar{P}_k^{\varepsilon,\delta} \to P_k^\varepsilon$ as $\delta \to 0$. Hence for δ small enough

$$x_k'\bar{P}_k^{\varepsilon,\delta}x_k \leq n - k + \tfrac{1}{2} \implies x_k'P_k^\varepsilon x_k \leq n - k + 1 \tag{12}$$

and for δ small enough we have that

$$\|\delta\bar{K}_i x_k\| \leq \tfrac{1}{2n} \text{ for all } x_k \text{ with } x_k'P_k^\varepsilon x_k \leq n - k + 1. \tag{13}$$

For each ε choose $\delta = \delta_\varepsilon$ small enough such that the above two properties (12) and (13) are satisfied. We define $K_i^\varepsilon = \delta_\varepsilon\bar{K}_i$, $\bar{P}_k^\varepsilon = \bar{P}_k^{\varepsilon,\delta_\varepsilon}$ and

$$\bar{A}_k^\varepsilon := A_k^\varepsilon + \sum_{i=1}^{j} B_{k,i}K_i^\varepsilon C_{k,i}.$$

By the definition of j, we know that

$$A_k^\varepsilon + \sum_{i=1}^{j+1} B_{k,i}K_i^\varepsilon C_{k,i} \tag{14}$$

either no longer has λ as an eigenvalue while λ is an eigenvalue of $\bar{A}_k^\varepsilon$ or this matrix (14) has less eigenvalues on the imaginary axis than $\bar{A}_k^\varepsilon$. In either case we can conclude that

$$(\bar{A}_k^\varepsilon, B_{k,j+1}, C_{k,j+1})$$

has a stabilizable and detectable eigenvalue on the imaginary axis. Choose V such that

$$VV' = I \quad \text{and} \quad \ker V = \langle \ker C_{k,j+1} \mid \bar{A}_k^\varepsilon \rangle.$$

We choose the following decentralized feedback law,

$$v_i^k = K_i^\varepsilon x_k + v_i^{k+1}, \qquad i = 1, \dots, j\,, \tag{15}$$

$$\begin{aligned} \dot{p} &= A_s^\varepsilon p + VB_{k,j+1}v_{j+1}^k + K(C_{k,j+1}V'p - y_{j+1}) \\ v_{j+1}^k &= F_\rho p + v_{j+1}^{k+1} \end{aligned} \tag{16}$$

$$v_i^k = v_i^{k+1}, \qquad i = j+2, \dots, \nu. \tag{17}$$

Equations (15), (16), and (17) together represent our decentralized feedback control laws at step $k+1$. Here $p \in \mathbb{R}^s$ and A_s^ε is such that $A_s^\varepsilon V = V \bar{A}_k^\varepsilon$ while K is chosen such that $A_s^\varepsilon + KC_{k,j+1}V'$ has all its eigenvalues in the open left half plane while $A_s^\varepsilon + KC_{k,j+1}V'$ and $\bar{A}_k^\varepsilon$ have no eigenvalues in common. Moreover, for all ρ the matrix $\bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V$ has at least one eigenvalue less on the imaginary axis than A_k^ε does, and still has all its eigenvalues in the closed left half plane while $F_\rho \to 0$ as $\rho \downarrow 0$. Rewriting the resulting system in a new basis consisting of x_k and $p - Vx_k$ results in

$$\begin{aligned} \dot{\bar{x}}_{k+1} &= \begin{pmatrix} \bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V & B_{k,j+1}F_\rho \\ 0 & A_s^\varepsilon + KC_{k,j+1}V' \end{pmatrix} \bar{x}_{k+1} + \sum_{i=1}^{\nu} \bar{B}_{k+1,i} v_i^{k+1} \\ y_i &= \bar{C}_{k+1,i}\bar{x}_{k+1}, \qquad i = 1, \ldots, \nu, \end{aligned} \tag{18}$$

where

$$\bar{B}_{k+1,i} = \begin{pmatrix} B_{k,i} \\ -VB_{k,i} \end{pmatrix}, \quad \bar{C}_{k+1,i} = \begin{pmatrix} C_{k,i} & 0 \end{pmatrix}$$

for $i \neq j+1$ while

$$\bar{B}_{k+1,j+1} = \begin{pmatrix} B_{k,j+1} \\ 0 \end{pmatrix}, \quad \bar{C}_{k+1,j+1} = \begin{pmatrix} C_{k,j+1} & 0 \\ V & I \end{pmatrix}$$

and

$$\bar{x}_{k+1} = \begin{pmatrix} x_k \\ p - Vx_k \end{pmatrix}.$$

Obviously, the above feedback laws (15), (16), and (17) satisfy at step $k+1$ the properties 1), and 2) as mentioned in step k. What remains to show is that they also satisfy property 3). Moreover, we need to write the control laws (15), (16), and (17) in the form of (6) or (7) for step $k+1$. In what follows we focus on these aspects.

For any ε there exists a $R_k^\varepsilon > 0$ with

$$(A_s^\varepsilon + KC_{k,j+1}V')'R_k^\varepsilon + R_k^\varepsilon(A_s^\varepsilon + KC_{k,j+1}V') < 0$$

such that $R_k^\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Since $F_\rho \to 0$ as $\rho \to 0$, for each ε we have for ρ small enough

$$\|F_\rho e\| < \tfrac{1}{2n} \text{ for all } e \text{ such that } e'R_k^\varepsilon e \leq n - k + \tfrac{1}{2} \tag{19}$$

where $e = p - Vx_k$.

We have that $\bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V$ is a perturbation of $\bar{A}_k^\varepsilon$, but it has at least one eigenvalue less on the imaginary axis than $\bar{A}_k^\varepsilon$ while having all its eigenvalues in the closed left half plane. But then, using Lemma 3, for ρ small there exists a matrix $\bar{P}_\rho^\varepsilon > 0$ such that

$$(\bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V)'\bar{P}_\rho^\varepsilon + \bar{P}_\rho^\varepsilon(\bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V) \leq 0$$

with $\bar{P}_\rho^\varepsilon \to \bar{P}_k^\varepsilon$ as $\rho \to 0$.

Finally because $\bar{A}_k^\varepsilon$ and $A_s^\varepsilon + KC_{k,j+1}V'$ have disjoint eigenvalues we note that for ρ small enough we get that $\bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V$ and $A_s^\varepsilon + KC_{k,j+1}V'$ have disjoint eigenvalues since $F_\rho \to 0$ as $\rho \downarrow 0$. But then there exists a $W_{\varepsilon,\rho}$ such that

$$B_{k,j+1}F_\rho + (\bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V)W_{\varepsilon,\rho} - W_{\varepsilon,\rho}(A_s^\varepsilon + KC_{k,j+1}V') = 0$$

while $W_{\varepsilon,\rho} \to 0$ as $\rho \downarrow 0$. Note that this implies that

$$\bar{P}_{k+1}^{\varepsilon,\rho} = \begin{pmatrix} I & 0 \\ -W_{\varepsilon,\rho}' & I \end{pmatrix} \begin{pmatrix} \bar{P}_\rho^\varepsilon & 0 \\ 0 & R_k^\varepsilon \end{pmatrix} \begin{pmatrix} I & -W_{\varepsilon,\rho} \\ 0 & I \end{pmatrix}$$

has the property that:

$$(\bar{A}_{k+1}^{\varepsilon,\rho})'\bar{P}_{k+1}^{\varepsilon,\rho} + \bar{P}_{k+1}^{\varepsilon,\rho}\bar{A}_{k+1}^{\varepsilon,\rho} \leq 0 \tag{20}$$

for

$$\bar{A}_{k+1}^{\varepsilon,\rho} = \begin{pmatrix} \bar{A}_k^\varepsilon + B_{k,j+1}F_\rho V & B_{k,j+1}F_\rho \\ 0 & A_s^\varepsilon + KC_{k,j+1}V' \end{pmatrix}$$

and

$$\lim_{\rho\downarrow 0} \bar{P}_{k+1}^{\varepsilon,\rho} = \begin{pmatrix} \bar{P}_k^\varepsilon & 0 \\ 0 & R_k^\varepsilon \end{pmatrix}.$$

We consider $\bar{x}_{k+1}$ such that

$$\bar{x}_{k+1}'\bar{P}_{k+1}^{\varepsilon,\rho}\bar{x}_{k+1} \leq n - k. \tag{21}$$

Then we can choose ρ small enough such that

$$x_k'\bar{P}_k^\varepsilon x_k \leq n - k + \tfrac{1}{2} \text{ and } (p - Vx_k)'R_k^\varepsilon(p - Vx_k) \leq n - k + \tfrac{1}{2}. \tag{22}$$

We choose for each ε a $\rho = \rho_\varepsilon$ such that (19) is satisfied while (21) implies that (22) is satisfied and finally

$$\|F_\rho V x_k\| < \tfrac{1}{2n} \text{ for all } x_k \text{ such that } x_k'\bar{P}_k^\varepsilon x_k \leq n - k + \tfrac{1}{2}.$$

In order to establish that the bounds on the inputs are satisfied in step $k+1$, we set $v_i^{k+1} = 0$ for $i = 1, \ldots \nu$. Then, we obtain for $i = 1, \ldots, j$,

$$\|u_i(t)\| = \|F_{i,\varepsilon}^k x_k + K_i^\varepsilon x_k\| \leq \tfrac{k}{n} + \tfrac{1}{2n} \leq \tfrac{k+1}{n}$$

for all $\bar{x}_{k+1}$ such that (21) is satisfied. For $i = j + 1$ we obtain,

$$\|u_i(t)\| = \|F_{i,\varepsilon}^k x_k + F_{\rho_\varepsilon} p\|$$
$$= \|F_{i,\varepsilon}^k x_k - F_{\rho_\varepsilon} V x_k + F_{\rho_\varepsilon}(p - V x_k)\| \leq \tfrac{k}{n} + \tfrac{1}{2n} + \tfrac{1}{2n} = \tfrac{k+1}{n}.$$

Finally for $i = j+2, \ldots, \nu$, we obtain

$$\|u_i(t)\| = \|F_{i,\varepsilon}^k x_k\| \leq \tfrac{k}{n} \leq \tfrac{k+1}{n}.$$

We now focus on rewriting the decentralized control laws (15), (16), and (17) in the form of (6) or (7) for step $k+1$. We set $n_{i,k+1} = n_{i,k}$ for $i \neq j+1$ while for $i = j+1$ we set $n_{i,k+1} = n_{i,k} + s$ and

$$p_i^{k+1} = \begin{pmatrix} p_i^k \\ p \end{pmatrix} \text{ or } p_i^{k+1} = p$$

in case $n_{i,k} > 0$ or $n_{i,k} = 0$ respectively. We can then rewrite the system (18) in terms of the state x_{k+1} (defined by (9)) instead of $\bar{x}_{k+1}$ which requires a basis transformation T_{k+1}, i.e. $\bar{x}_{k+1} = T_{k+1} x_{k+1}$. We define

$$P_{k+1}^{\varepsilon} = T_{k+1}' \bar{P}_{k+1}^{\varepsilon,\rho_\varepsilon} T_{k+1}$$

and obviously, for $i = 1, \ldots, \nu$, we can write the relationship between y_i, v_i^{k+1} and u_i in the form (6) or (7) depending on whether $n_{i,k+1} = 0$ or not. We can now rewrite the control laws (15), (16), and (17) in the form

$$\Sigma_i^{k+1,\varepsilon} : \begin{cases} \dot{p}_i^{k+1} = K_{i,\varepsilon}^{k+1} p_i^{k+1} + L_{i,\varepsilon}^{k+1} y_i, & p_i^{k+1} \in \mathbb{R}^{n_{i,k+1}} \\ u_i \;\; = M_{i,\varepsilon} p_i^{k+1} + N_{i,\varepsilon}^{k+1} y_i + v_i^{k+1} \end{cases} \tag{23}$$

in case $n_{i,k+1} > 0$, and otherwise

$$\Sigma_i^{k+1,\varepsilon} : \{ \; u_i = N_{i,\varepsilon}^{k+1} y_i + v_i^{k+1}, \tag{24}$$

for $i = 1, \ldots, \nu$. It is then clear that properties 1), 2) and 3) are satisfied in step $k+1$.

This concludes the proof of Theorem 2.

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